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Research Article

Couple of the Variational Iteration Method and Legendre Wavelets for Nonlinear Partial Differential Equations

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This paper develops a modified variational iteration method coupled with the Legendre wavelets, which can be used for the efficient numerical solution of nonlinear partial differential equations (PDEs). The approximate solutions of PDEs are calculated in the form of a series whose components are computed by applying a recursive relation. Block pulse functions are used to calculate the Legendre wavelets coefficient matrices of the nonlinear terms. The main advantage of the new method is that it can avoid solving the nonlinear algebraic system and symbolic computation. Furthermore, the developed vector-matrix form makes it computationally efficient. The results show that the proposed method is very effective and easy to implement.

1. Introduction

Nonlinear phenomena are of fundamental importance in applied mathematics and physics and thus have attracted much attention. It is well known that most engineering problems are nonlinear, and it is very difficult to achieve the solution analytically or numerically. The analytical methods commonly used to solve them are very restricted, while the numerical techniques involving discretization of the variables on the other hand give rise to rounding off errors. Considerable attention has been paid to developing an efficient and fast convergent method. Recently, several approximate methods are introduced to find the numerical solutions of nonlinear PDEs, such as Adomian's decomposition method (ADM) [1–6], homotopy perturbation method (HPM) [7–12], homotopy analysis method (HAM) [13, 14], variational iteration method (VIM) [15–23], and wavelets method [24–29].

The variational iteration method (VIM) proposed by He [15–23] has been shown to be very efficient for handling a wide class of physical problems [16–18, 30–41]. If the exact solution of the nonlinear PDEs exists, the VIM gives rapidly convergent successive approximations; otherwise, a few approximations can be used for numerical purposes. In order to improve the efficiency of these algorithms, several modifications, such as variational iteration method using He's Polynomials [42–48] or using Adomian's Polynomials [49–54], have been developed and successfully applied to various engineering problems. However, since the variational iteration method provides the solution as a sequence of iterates, its successive iterations may be very complex, so that the resulting integrations in its iterative relation may be impossible to perform analytically.

In recent years, wavelets have found their way into many different fields of science and engineering. Various wavelets [24–29] have been used for studying problems with greater computational complexity and proved to be powerful tools to explore a new direction in solving differential equations. Unlike the variational iteration method that requires symbolic computations, the wavelets method converts the PDE into algebraic equations by the operational matrices, which can be solved by an iterative procedure. It is worthy to mention here that the method based on operational matrices of an orthogonal function for solving differential equations is computer oriented. The problem with this approach is that the algebraic equations may be singular and nonlinear.

Recently, some efficient modifications of ADM (using [55, 56]) and VIM or HAM [57] (using Legendre polynomials) are presented to approximate nonhomogeneous terms in

nonlinear differential equations. Motivated and inspired by the ongoing research in these areas, we implement Legendre wavelets within the framework of VIM to facilitate the computational work of the method while still keeping the accuracy. The remainder of the paper is organized as follows. Section 2 introduces the VIM. In Section 3, we describe the basic formulation of Legendre wavelets and the operational matrix required for our subsequent development. In Section 4, we propose a new variational iteration method using Legendre wavelets (VIMLW). In order to demonstrate the validity and applicability of VIMLW, four examples are given in Section 5. Finally, a brief summary is presented.

2. Variational Iteration Method

This section introduces the basic ideas of variational iteration method (VIM). Here a description of the method [15–23] is given to handle the general nonlinear problem:

$$L(u) + N(u) = g(t), \tag{1}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a known analytic function. According to He's VIM, we can construct a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\left\{L\left(u_n(\tau)\right) + N\left(\widetilde{u}_n(\tau)\right) - g(\tau)\right\}, \quad n \geq 0, \tag{2}$$

where λ is a general Lagrange multiplier which can be optimally identified via variational theory and $\widetilde{u}_n$ is a restricted variation which means $\delta\widetilde{u}_n = 0$. Therefore, the Lagrange multiplier λ should be first determined via integration by parts. The successive approximation $u_n(t)$ $(n \geq 0)$ of the solution $u(t)$ will be readily obtained by using the obtained Lagrange multiplier and any selective function u_0. The zeroth approximation u_0 may select any function that just meets, at least, the initial and boundary conditions. With λ determined, several approximations $u_n(t)$, $n \geq 0$, follow immediately. Consequently, the exact solution may be obtained as

$$u(t) = \lim_{n\to\infty} u_n(t). \tag{3}$$

The VIM depends on the proper selection of the initial approximation $u_0(t)$. Finally, we approximate the solution of the initial value problem (1) by the nth-order term $u_n(t)$. It has been validated that VIM is capable of effectively, easily, and accurately solving a large class of nonlinear problems.

3. Legendre Wavelets

3.1. Legendre Wavelets. Legendre wavelets $\psi_{nm}(t) = \psi(k, \widehat{n}, m, t)$ have four arguments: k is any positive integer, $\widehat{n} = 2n - 1$ $(n = 1, 2, 3, \ldots, 2^{k-1})$, m is the order for Legendre polynomials, and t is the normalized time. They are defined on the interval $[0, 1)$ as follows:

$$\psi_{nm}(t) = \begin{cases} \sqrt{m+1/2}\,2^{(k/2)} L_m\left(2^k t - \widehat{n}\right), & \text{for } \dfrac{\widehat{n}-1}{2^k} \leq t \leq \dfrac{\widehat{n}+1}{2^k}, \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

where $m = 0, 1, 2, \ldots, M-1$, $n = 1, 2, \ldots, 2^{k-1}$. The coefficient $\sqrt{m+1/2}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$, and the translation parameter $b = \widehat{n}2^{-k}$. Here, $L_m(t)$ are the well-known Legendre polynomials of order m defined on the interval $[-1, 1]$.

A function $f(t)$ defined over $[0, 1)$ may be expanded by Legendre wavelet series as

$$f(t) = \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} c_{nm}\psi_{nm}(t), \tag{5}$$

with

$$c_{nm} = \left\langle f(t), \psi_{nm}(t)\right\rangle, \tag{6}$$

in (6); $\langle\cdot,\cdot\rangle$ denotes the inner product.

If the infinite series in (5) is truncated, then it can be written as

$$f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t), \tag{7}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C(t) = \left[c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}\right]^T, \tag{8}$$

$$\Psi(t) = \left[\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1M-1}(t), \ldots, \psi_{2^{k-1}0}(t), \ldots, \psi_{2^{k-1}M-1}(t)\right]^T. \tag{9}$$

A two-dimensional function $f(x,t)$ defined over $[0, 1) \times [0, 1)$ may be expanded by Legendre wavelet series as

$$f(x,t) = \sum_{i=1}^{2^kM}\sum_{j=1}^{2^kM} c_{ij}\psi_i(x)\psi_j(t) = \Psi^T(x)C\Psi(t), \tag{10}$$

with

$$c_{ij} = \int_0^1 f(x,t)\psi_i(x)\,dx \int_0^1 f(x,t)\psi_j(t)\,dt. \tag{11}$$

Equation (10) can be written into the discrete form (in matrix form) by

$$f(x,t) = \Psi^T(x)C\Psi(t), \tag{12}$$

where C is a $2^{k-1}M \times 2^{k-1}M$ matrix given by

$$C = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,2^{k-1}M} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,2^{k-1}M} \\ \vdots & \vdots & \ddots & \vdots \\ c_{2^{k-1}M,0} & c_{2^{k-1}M,1} & \cdots & c_{2^{k-1}M,2^{k-1}M} \end{bmatrix}. \tag{13}$$

The integration and derivative operation matrices of the Legendre wavelets have been derived in [58, 59].

The integration of the vector $\Psi(t)$ defined in (9) can be obtained as

$$\int_0^t \Psi(s)\, ds \cong P\Psi(t), \tag{14}$$

where P is a $2^{k-1}M \times 2^{k-1}M$ matrix given by [58].

The derivative of the vector $\Psi(t)$ can be expressed by

$$\frac{d\Psi(t)}{dt} = D\Psi(t), \tag{15}$$

where D is the $2^{k-1}M \times 2^{k-1}M$ operational matrix of derivative given by [59].

The integration of $u(x,t) = \Psi^T(x)C\Psi(t)$ with respect to variable t can be expressed as

$$\begin{aligned} \int_0^t u(x,\tau)\, d\tau &= \int_0^t \left(\Psi^T(x)\, C\Psi(\tau)\right) d\tau \\ &= \Psi^T(x)\, C\left(\int_0^t \Psi(\tau)\, d\tau\right) = \Psi^T(x)\, CP\Psi(t). \end{aligned} \tag{16}$$

Similarly, the integration of $u(x,t) = \Psi^T(x)C\Psi(t)$ with respect to variable x can be expressed as

$$\begin{aligned} \int_0^x u(\tau,t)\, d\tau &= \int_0^x \left(\Psi^T(\tau)\, C\Psi(t)\right) d\tau \\ &= \left(\int_0^x \Psi^T(\tau)\, d\tau\right) C\Psi(t) = \Psi^T(x)\, P^T C\Psi(t). \end{aligned} \tag{17}$$

The derivative of $u(x,t) = \Psi^T(x)C\Psi(t)$ with respect to variable t can be expressed as

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{\partial\left(\Psi^T(x)\, C\Psi(t)\right)}{\partial t} = \Psi^T(x)\, C\frac{\partial\Psi(t)}{\partial t} \\ &= \Psi^T(x)\, CD\Psi(t). \end{aligned} \tag{18}$$

Similarly, the derivative of $u(x,t) = \Psi^T(x)C\Psi(t)$ with respect to variable x can be expressed as

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x} &= \frac{\partial\left(\Psi^T(x)\, C\Psi(t)\right)}{\partial x} = \frac{\partial\Psi^T(x)}{\partial x} C\Psi(t) \\ &= \Psi^T(x)\, D^T C\Psi(t). \end{aligned} \tag{19}$$

3.2. Block Pulse Functions. The block pulse functions (BPFs) form a complete set of orthogonal functions that are defined on the interval $[0, b)$ by

$$b_i(t) = \begin{cases} 1, & \dfrac{i-1}{m}b \le t < \dfrac{i}{m}b \\ 0, & \text{elsewhere} \end{cases} \tag{20}$$

for $i = 1, 2, \ldots, m$. It is also known that for arbitrary absolutely integrable function $f(t)$ on $[0, b)$ can be expanded in block pulse functions:

$$f(t) \simeq \xi^T B_m(t), \tag{21}$$

in which

$$\begin{aligned} \xi^T &= [f_1, f_2, \ldots, f_m], \\ B_m(t) &= [b_1(t), b_2(t), \ldots, b_m(t)], \end{aligned} \tag{22}$$

where f_i are the coefficients of the block pulse function given by

$$f_i = \frac{m}{b}\int_0^b f(t)\, b_i(t)\, dt = \frac{m}{b}\int_{((i-1)/m)b}^{(i/m)b} f(t)\, b_i(t)\, dt. \tag{23}$$

The elementary properties of BPFs are as follows.

(1) Disjointness: the BPFs are disjoined with each other in the interval $t \in [0, T)$:

$$b_i(t)\, b_j(t) = \delta_{ij} b_i(t) \tag{24}$$

for $i, j = 1, 2, \ldots, m$.

(2) Orthogonality: the BPFs are orthogonal with each other in the interval $t \in [0, T)$:

$$\int_0^T b_i(t)\, b_j(t)\, dt = h\delta_{ij} \tag{25}$$

for $i, j = 1, 2, \ldots, m$.

(3) Completeness: the BPFs set is complete when m approaches infinity. This means that for every $f \in L^2([0, T))$, when m approaches to the infinity, Parseval's identity holds:

$$\int_0^T f^2(t)\, dt = \sum_{i=1}^{\infty} f_i^2 \left\| b_i(t) \right\|^2, \tag{26}$$

where

$$f_i = \frac{1}{h}\int_0^T f(t)\, b_i(t)\, dt. \tag{27}$$

Definition 1. Let A and B be two matrices of $m \times m$, then $A \otimes B = (a_{ij} \times b_{ij})_{mm}$.

Lemma 2. *Assuming that $f(t)$ and $g(t)$ are two absolutely integrable functions, which can be expanded in block pulse function as $f(t) = FB(t)$, and $g(t) = GB(t)$ respectively, then one has*

$$f(t)\, g(t) = FB(t)\, B^T(t)\, G^T = HB(t), \tag{28}$$

where $H = F \otimes G$.

Proof. According to the disjointness property of BPFS in (16), we have

$$\begin{aligned} &FB(t)\, B^T(t)\, G^T \\ &\quad = \left[f_{11} g_{11} \phi_1(t) \;\; f_{12} g_{12} \phi_2(t) \;\; \cdots \;\; f_{1m} g_{1m} \phi_2(t) \right] \\ &\quad = HB(t). \end{aligned} \tag{29}$$

□

Lemma 3. *Let $f(x,t)$ and $g(x,t)$ be two absolutely integrable functions, which can be expanded in block pulse function as $f(x,t) = B^T(x)FB(t)$ and $g(x,t) = B^T(x)GB(t)$, respectively, one has*

$$f(x,t)\, g(x,t) = B^T(x)\, HB(t), \tag{30}$$

where $H = F \otimes G$.

3.3. Nonlinear Term Approximation. The Legendre wavelets can be expanded into m-set of block pulse functions as

$$\Psi(t) = \Phi_{m\times m} B_m(t). \tag{31}$$

Taking the collocation points as follow,

$$t_i = \frac{i - 1/2}{2^{k-1}M}, \quad i = 1, 2, \ldots, 2^{k-1}M. \tag{32}$$

The m-square Legendre matrix $\Phi_{m\times m}$ is defined as

$$\Phi_{m\times m} \triangleq \left[\Psi(t_1) \;\; \Psi(t_2) \;\; \cdots \;\; \Psi(t_{2^{k-1}M}) \right]. \tag{33}$$

The operational matrix of product of Legendre wavelets can be obtained by using the properties of BPFs. Let $f(x,t)$ and $g(x,t)$ be two absolutely integrable functions, which can be expanded in Legendre wavelets as $f(x,t) = \Psi^T(x)F\Psi(t)$ and $g(x,t) = \Psi^T(x)G\Psi(t)$, respectively.

From (31), we have

$$\begin{aligned} f(x,t) &= \Psi^T(x)\, F\Psi(t) = B^T(x)\, \Phi_{mm}^T F \Phi_{mm} B(t), \\ g(x,t) &= \Psi^T(x)\, G\Psi(t) = B^T(x)\, \Phi_{mm}^T G \Phi_{mm} B(t), \end{aligned} \tag{34}$$

and let $F_b = \Phi_{mm}^T F \Phi_{mm}$, $G_b = \Phi_{mm}^T G \Phi_{mm}$, $H_b = F_b \otimes G_b$.

By employing Lemma 3, we get

$$\begin{aligned} f(x,t)\, g(x,t) &= B^T(x)\, H_b B(t) \\ &= B^T(x)\, \Phi_{mm}^T \mathrm{inv}\left(\Phi_{mm}^T\right) H_b \mathrm{inv}\left(\Phi_{mm}\right) \\ &\quad \times \Phi_{mm} B(t) \\ &= \Psi^T(x)\, H\Psi(t), \end{aligned} \tag{35}$$

where $H = \mathrm{inv}(\Phi_{mm}^T) H_b \mathrm{inv}(\Phi_{mm})$.

4. Variational Iteration Method Using Legendre Wavelets

In this section, we present a new modification of variational iteration method using Legendre wavelets (called VIMLW). This algorithm can be implemented for solving nonlinear PDEs effectively.

To deduce the basic relations of our proposed algorithm, consider the following forms of initial value problems:

$$L[u(x,t)] + N[u(x,t)] = g(x,t), \quad x \in [0,1], \; t > 0, \tag{36}$$

where L and N are linear operator and nonlinear operator, respectively, and $g(x,t)$ is a known analytic function, subject to the initial condition $u(x,0)$. It should be noted here that $L[u(x,t)]$ contains the term $\partial^m u/\partial t^m$, where m is a positive integer.

According to the traditional VIM, we can construct the correction functional for (36) as

$$\begin{aligned} u_{k+1}(x,t) = u_k(x,t) + \int_0^t \lambda \big[&L\left(u_k(x,\tau)\right) + N\left(u_k(x,\tau)\right) \\ &- g(x,\tau) \big] d\tau. \end{aligned} \tag{37}$$

The Lagrange multiplier of (37) is

$$\lambda(t,\tau) = -\frac{(t-\tau)^{m-1}}{(m-1)!} = \frac{(-1)^m (\tau - t)^{m-1}}{(m-1)!}. \tag{38}$$

In order to improve the performance of VIM, we introduce Legendre wavelets to approximate $u_k(x,t)$ and the nonhomogeneous term $g(x,t)$ as

$$u_k(x,t) = \Psi^T(x)\, C_k \Psi(t), \qquad g(x,t) = \Psi^T(x)\, G\Psi(t). \tag{39}$$

Now for the nonlinear part, by nonlinear term approximation described in Section 3.3, we have

$$N\left[u_k(x,t)\right] = \Psi^T(x)\, N_k \Psi(t), \tag{40}$$

where N is matrix of order $2^{k-1}M \times 2^{k'-1}M'$.

For the linear part, we have

$$L\left[u_k(x,t)\right] = \Psi^T(x)\, L_k \Psi(t), \tag{41}$$

where L is a matrix of order $2^{k-1}M \times 2^{k'-1}M'$.

Then the iteration formula (37) can be constructed as

$$\begin{aligned} \Psi^T(x)\, C_{k+1} \Psi(t) = \Psi^T(x)\, C_k \Psi(t) \\ + \int_0^t \lambda \Psi^T(x) \left[L_k + N_k - G \right] \Psi(\tau)\, d\tau. \end{aligned} \tag{42}$$

If λ is constant, we have

$$C_{k+1} = C_k + \lambda \left[L_k + N_k - G \right] P. \tag{43}$$

Table 1: Numerical values when $t = 0.25, 0.50, 0.75$, and 1.0 for (52).

t	0.25				0.50			
x	0.25	0.50	0.75	1.00	0.25	0.50	0.75	1.00
VIM	0.20002	0.40004	0.60006	0.80007	0.16702	0.33405	0.50107	0.66810
VIMLW	0.20002	0.40004	0.60006	0.80006	0.16702	0.33405	0.50107	0.66809
Exact	0.20000	0.40000	0.60000	0.80000	0.16667	0.33333	0.50000	0.66667
t	0.75				1.00			
x	0.25	0.50	0.75	1.00	0.25	0.50	0.75	1.00
VIM	0.14461	0.28921	0.43382	0.57842	0.12989	0.25977	0.38966	0.51955
VIMLW	0.14460	0.28921	0.43382	0.57841	0.12989	0.25977	0.38966	0.51953
Exact	0.14286	0.28571	0.42857	0.57143	0.12500	0.25000	0.37500	0.50000

When λ is a function of τ, the Legendre wavelets are used to approximate $\lambda(\tau)$ as

$$\lambda(t,\tau) = \Psi^T(t) S \Psi(\tau). \tag{44}$$

Substituting (44) into (42), we have

$$\begin{aligned}\Psi^T(x) C_{k+1} \Psi(t) &= \Psi^T(x) C_k \Psi(t) \\ &\quad + \int_0^t \Psi^T(x) \left[L_k + N_k - G\right] \Psi(\tau) \Psi^T(\tau) S^T \Psi(t)\, d\tau.\end{aligned} \tag{45}$$

Since

$$\Psi(\tau) = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1m} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m1} & \phi_{m2} & \cdots & \phi_{mm} \end{bmatrix} \begin{bmatrix} B_1(\tau) \\ B_2(\tau) \\ \vdots \\ B_m(\tau) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m \phi_{1i} B_i(\tau) \\ \sum_{i=1}^m \phi_{2i} B_i(\tau) \\ \vdots \\ \sum_{i=1}^m \phi_{mi} B_i(\tau) \end{bmatrix}, \tag{46}$$

we get

$$\begin{aligned}\Psi(\tau)\Psi^T(\tau) &= \begin{bmatrix} \sum_{i=1}^m \phi_{1i} B_i(t) \\ \sum_{i=1}^m \phi_{2i} B_i(t) \\ \vdots \\ \sum_{i=1}^m \phi_{mi} B_i(t) \end{bmatrix} \\ &\quad \times \begin{bmatrix} \sum_{i=1}^m \phi_{1i} B_i(t) & \sum_{i=1}^m \phi_{2i} B_i(t) & \cdots & \sum_{i=1}^m \phi_{mi} B_i(t) \end{bmatrix} \\ &= \sum_{j=1}^m \left(\sum_{i=1}^m \phi_{ji} B_i(t) \right)^2.\end{aligned} \tag{47}$$

According to the property of block pulse functions, we obtain

$$\begin{aligned}&\Psi(\tau)\Psi^T(\tau) \\ &\quad = \sum_{j=1}^m \sum_{i=1}^m \phi_{ji}^2 B_i(t) \\ &\quad = \begin{bmatrix} \phi_{11}^2 & \phi_{12}^2 & \cdots & \phi_{1m}^2 \\ \phi_{21}^2 & \phi_{22}^2 & \cdots & \phi_{2m}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m1}^2 & \phi_{m2}^2 & \cdots & \phi_{mm}^2 \end{bmatrix} \begin{bmatrix} B_1(t) \\ B_2(t) \\ \vdots \\ B_m(t) \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \\ &\quad = \begin{bmatrix} \phi_{11}^2 & \phi_{12}^2 & \cdots & \phi_{1m}^2 \\ \phi_{21}^2 & \phi_{22}^2 & \cdots & \phi_{2m}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m1}^2 & \phi_{m2}^2 & \cdots & \phi_{mm}^2 \end{bmatrix} \operatorname{inv}(\Phi_{m\times m}) \Psi(t) \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \\ &\quad = H \Psi(t) \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix},\end{aligned} \tag{48}$$

where

$$H = \begin{bmatrix} \phi_{11}^2 & \phi_{12}^2 & \cdots & \phi_{1m}^2 \\ \phi_{21}^2 & \phi_{22}^2 & \cdots & \phi_{2m}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m1}^2 & \phi_{m2}^2 & \cdots & \phi_{mm}^2 \end{bmatrix} \operatorname{inv}(\Phi_{m\times m}). \tag{49}$$

Substituting (48) into (45), we have

$$\begin{aligned}\Psi^T(x) C_{k+1} \Psi(t) &= \Psi^T(x) C_k \Psi(t) + \int_0^t \Psi^T(x) \left[L_k + N_k - G\right] \\ &\quad \times H \Psi(\tau) \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} S^T \Psi(t)\, d\tau\end{aligned}$$

Table 2: Numerical values when $t = 0.25, 0.50, 0.75$, and 1.0 for (54).

t			0.25				0.50	
x	0.25	0.50	0.75	1.00	0.25	0.50	0.75	1.00
VIM	0.05001	0.20002	0.45004	0.80007	0.04176	0.16702	0.37581	0.66810
VIMLW	0.05000	0.20002	0.45004	0.80021	0.04168	0.16699	0.37572	0.66831
Exact	0.05000	0.20000	0.45000	0.80000	0.04167	0.16667	0.37500	0.66667
t			0.75				1.00	
x	0.25	0.50	0.75	1.00	0.25	0.50	0.75	1.00
VIM	0.03615	0.14461	0.32536	0.57842	0.03247	0.12989	0.29225	0.51955
VIMLW	0.03577	0.14443	0.32491	0.57751	0.03128	0.12932	0.29083	0.51421
Exact	0.03571	0.14286	0.32143	0.57143	0.03125	0.12500	0.28125	0.50000

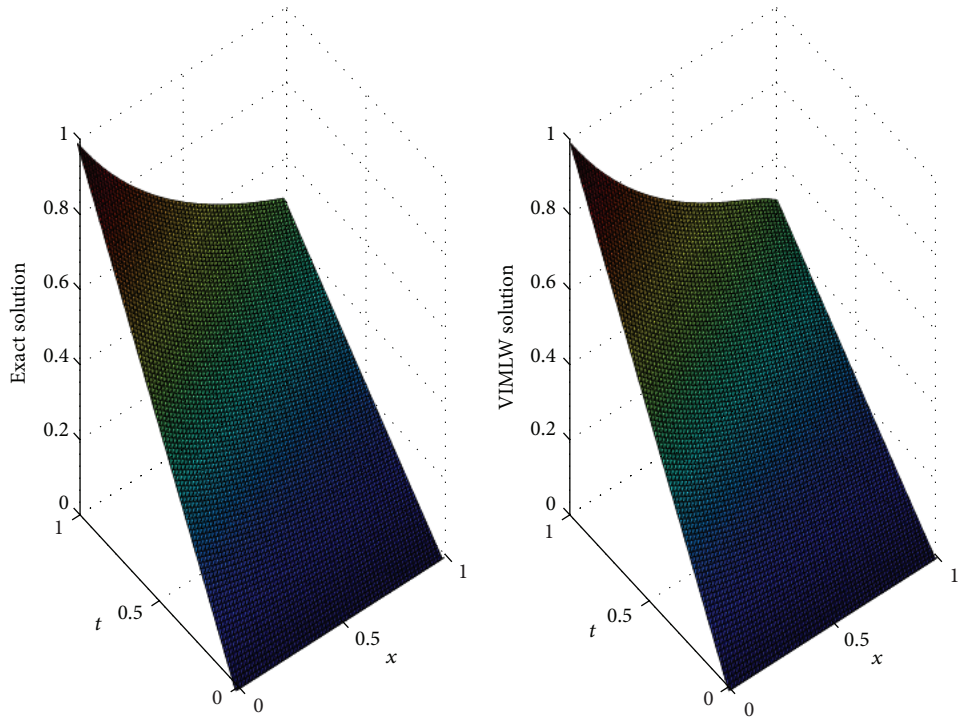

Figure 1: Exact solution and VIMLW approximate solution of Example 4.

$$
\begin{aligned}
&= \Psi^T(x) C_k \Psi(t) + \Psi^T(x) \left[L_k + N_k - G\right] \\
&\quad \times HP\Psi(t) \left[1 \;\; 1 \;\; \cdots \;\; 1\right] S^T \Psi(t) \\
&= \Psi^T(x) C_k \Psi(t) + \Psi^T(x) \left[L_k + N_k - G\right] \\
&\quad \times HP\Psi(t) \Psi^T(t) S \left[1 \;\; 1 \;\; \cdots \;\; 1\right]^T \\
&= \Psi^T(x) C_k \Psi(t) + \Psi^T(x) \left[L_k + N_k - G\right] \\
&\quad \times HPH\Psi(t)\, s,
\end{aligned} \tag{50}
$$

where $s = [1\ 1 \cdots 1] S [1\ 1 \cdots 1]^T$.

Finally, we get the iteration formula as follows:

$$
C_{k+1} = C_k + \left[L_k + N_k - G\right] HPH \left[1 \;\; 1 \;\; \cdots \;\; 1\right] S \left[1 \;\; 1 \;\; \cdots \;\; 1\right]^T. \tag{51}
$$

5. Numerical Examples

To demonstrate the effectiveness and good accuracy of the VIMLW, four different examples will be examined.

Example 4. Consider the regularized long-wave (RLW) equation [39]:

$$
u_t - u_{xxt} + \left(\frac{u^2}{2}\right)_x = 0, \quad -\infty < x < \infty, \; t > 0 \tag{52}
$$

with the initial condition $u(x, 0) = x$ and the exact solution is $u(x, t) = (x/(1 + t))$.

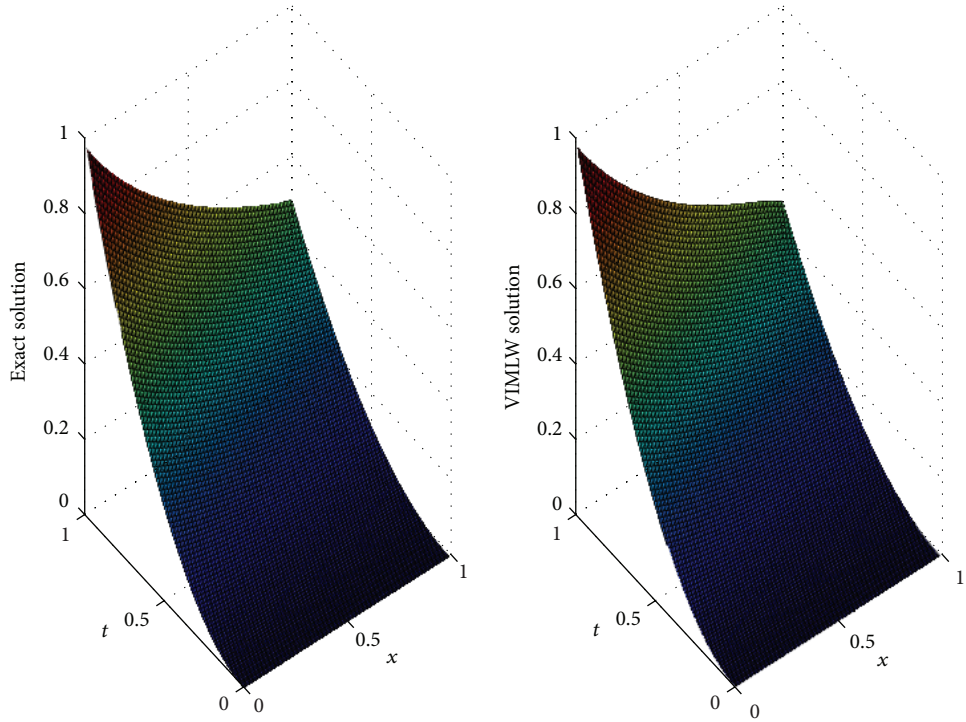

Figure 2: Exact solution and VIMLW approximate solution of Example 5.

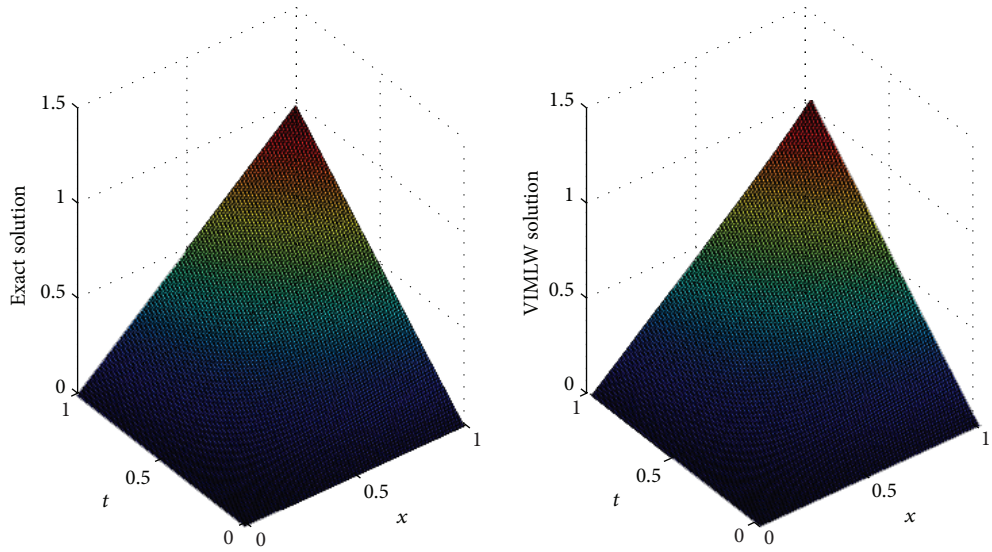

Figure 3: Exact solution and VIMLW approximate solution of Example 6.

By assuming $u_k(x,t) = \Psi^T(x)C_k\Psi(t)$ and from (52), we have

$$L\left[u_k\right] = \frac{\partial u_k}{\partial t} - \frac{\partial^3 u_k}{\partial x^2 \partial t} = \Psi^T(x)\, L_k \Psi(t),$$
$$N\left[u_k\right] = u_k \frac{\partial u_k}{\partial x} = \Psi^T(x)\, N_k \Psi(t), \tag{53}$$

where $L_k = C_k D - (D^T)^2 C_k D$, $N_k = (D^T C_k) \otimes C_k$.

We utilize the methods presented in this paper to solve (52) with $M = 16$ and $k = 1$. Table 1 shows the approximate solutions for (52) obtained for different points using the variational iteration and VIMLW method. Figure 1 presents the Exact solution and VIMLW approximate solution of Example 4. Note that only the fifth-order term of their solutions is used in evaluating the approximate solutions for Example 4. We can see that the approximate solution obtained with VIMLW gives almost the same results as that

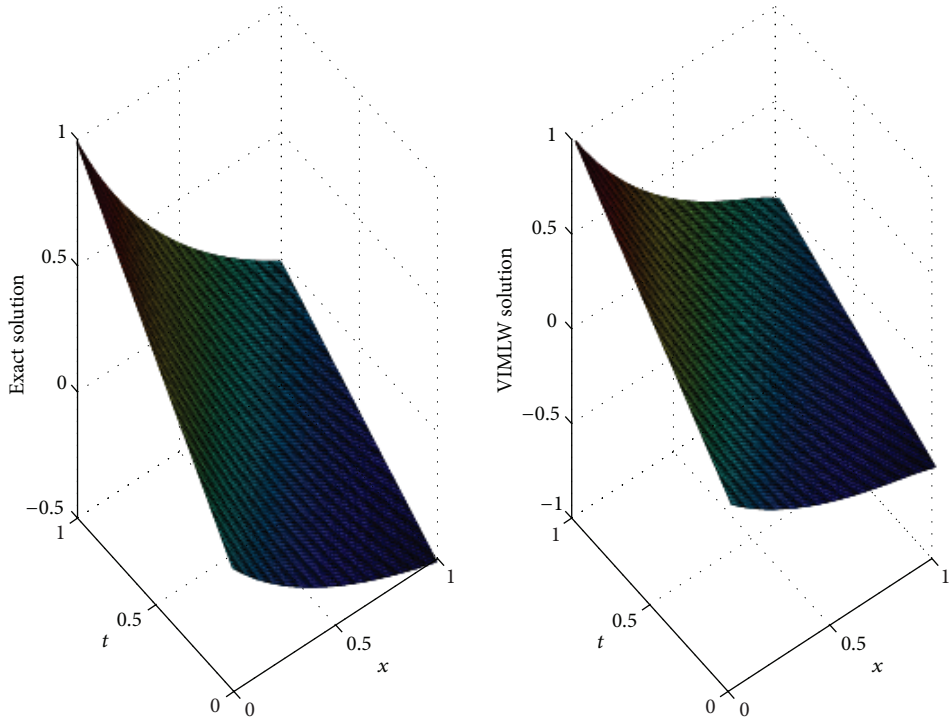

Figure 4: Exact solution and VIMLW approximate solution of Example 7.

with VIM. It indicates that the approximate solution is quite close to the exact one.

Example 5. Consider the following equation [39]:

$$u_t + \frac{1}{2}uu_{xx} = u_{xxx}, \quad -\infty < x < \infty, \ t > 0 \tag{54}$$

with the initial conditions $u(x,0) = x^2$, and the exact solution is $u(x,t) = (x^2/(1+t))$.

By assuming $u_k(x,t) = \Psi^T(x)C_k\Psi(t)$ and from (54), we have

$$\begin{aligned} L_k[u_k] &= \frac{\partial u_k}{\partial t} - \frac{\partial^3 u_k}{\partial x^3} = \Psi^T(x) L_k \Psi(t), \\ N_k[u_k] &= \frac{u_k}{2}\frac{\partial^2 u_k}{\partial x^2} = \Psi^T(x) N_k \Psi(t), \end{aligned} \tag{55}$$

where $L_k = (D^T)^3 C_k D$, $N_k = ((1/2)(D^T)^2 C_k) \otimes C_k$.

We employ the methods presented in this paper to solve (54) with $M = 16$ and $k = 1$. The numerical results are presented in Table 2 and shown in Figure 2. It is to be noted that only the fifth-order terms are used in evaluating the approximate solutions. The results obtained using the VIMLW are in good agreement with the results of VIM.

Example 6. We consider the following equation [40]:

$$\begin{aligned} u_t(x,t) + u(x,t)\,u_x(x,t) &= g(x,t), \\ t > 0, x \in R, \ 0 < \alpha \le 1 & \end{aligned} \tag{56}$$

with the initial conditions $u(x,0) = 0$ and the exact solution is $u(x,t) = xt$, where $g(x,t) = x + xt^2$.

By assuming $u_k(x,t) = \Psi^T(x)C_k\Psi(t)$, $g(x,t) = \Psi^T(x)G\Psi(t)$, we have

$$\begin{aligned} L_k[u_k] &= u_t(x,t) - g(x,t) = \Psi^T(x) L_k \Psi(t), \\ N_k[u_k] &= u_k \frac{\partial u_k}{\partial x} = \Psi^T(x) N_k \Psi(t), \end{aligned} \tag{57}$$

where $L_k = C_k D - G$, $N_k = C_k \otimes (D^T C_k)$.

Table 3 shows the approximate solutions for (56) with $M = 16$ and $k = 1$ using the VIM and the VIMLW methods and the results are plotted in Figure 3. It is to be noted that only the fourth-order terms of VIM and VIMLW are used in evaluating the approximate solutions in Table 3. We observe that the approximate solution of (56) with VIMLW gives analogous results to that obtained by VIM, which shows that the approximate solution remains closed form to the exact one.

Example 7. Consider the following Burgers-Poisson (BP) equation of the form [41]:

$$u_t - u_{xxt} + u_x + uu_x = (3u_x u_{xx} + uu_{xxx}), \quad -\infty < x < \infty, \ t > 0 \tag{58}$$

with the initial conditions $u(x,0) = x$, and the exact solution is $u(x,t) = (1+x)/(1+t) - 1$.

Table 3: Numerical values when $t = 0.25, 0.50, 0.75$, and 1.0 for (56).

t	0.25				0.50			
x	0.25	0.50	0.75	1.00	0.25	0.50	0.75	1.00
VIM	0.06250	0.12500	0.18750	0.25000	0.12508	0.25015	0.37523	0.50030
VIMLW	0.06250	0.12500	0.18750	0.25000	0.12508	0.25015	0.37523	0.50030
Exact	0.06250	0.12500	0.18750	0.25000	0.12500	0.25000	0.37500	0.50000
t	0.75				1.00			
x	0.25	0.50	0.75	1.00	0.25	0.50	0.75	1.00
VIM	0.18882	0.37764	0.56646	0.75527	0.25992	0.51983	0.77975	1.03970
VIMLW	0.18882	0.37764	0.56646	0.75527	0.25992	0.51983	0.77975	1.03970
Exact	0.18750	0.37500	0.56250	0.75000	0.25000	0.50000	0.75000	1.00000

Table 4: Numerical values when $t = 0.25, 0.50, 0.75$, and 1.0 for (58).

t	0.25				0.50			
x	0.25	0.50	0.75	1.00	0.25	0.50	0.75	1.00
VIM	0.00009	0.20011	0.40013	0.60015	−0.16488	0.00215	0.16917	0.33620
VIMLW	0.00009	0.20011	0.40013	0.59864	−0.16489	0.00215	0.16920	0.33283
Exact	0.00000	0.20000	0.40000	0.60000	−0.16667	0.00000	0.16667	0.33333
t	0.75				1.00			
x	0.25	0.50	0.75	1.00	0.25	0.50	0.75	1.00
VIM	−0.27697	−0.13237	0.01224	0.15694	−0.35057	−0.22068	−0.09079	0.03909
VIMLW	−0.27701	−0.13237	0.01228	0.15213	−0.35070	−0.22068	−0.09078	0.03422
Exact	−0.28571	−0.14286	0.00000	0.14286	−0.37500	−0.25000	−0.12500	0.00000

By assuming $u_k(x,t) = \Psi^T(x) C_k \Psi(t)$, we have

$$L[u_k] = \frac{\partial u_k}{\partial t} - \frac{\partial^3 u_k}{\partial x^2 \partial t} + \frac{\partial u_k}{\partial x} = \Psi^T(x) L_k \Psi(t), \quad (59)$$

where $L_k = C_k D - (D^T)^2 C_k D + D^T C_k$.

And

$$N[u_k] = u_k \frac{\partial u_k}{\partial x} - 3\frac{\partial u_k}{\partial x}\frac{\partial^2 u_k}{\partial x^2} - u_k \frac{\partial^3 u_k}{\partial x^3} = \Psi^T(x) N_k \Psi(t), \quad (60)$$

where $N_k = C_k \oplus (D^T C_k) - 3(D^T C_k) \oplus [(D^T)^2 C_k] - C_k \oplus [(D^T)^3 C_k]$.

Table 4 shows the approximate solutions to (58) with $M = 16$ and $k = 1$ with VIM and VIMLW, and Figure 4 presents the Exact solution and VIMLW approximate solution of Example 7. Only the fourth-order terms are used in evaluating the approximate solutions in Table 4. From Table 4 and Figure 4 the approximate solution of the given Example 7 by using VIMLW is in good agreement with the results of VIM and it clearly appears that the approximate solution remains closed form to exact solution.

6. Conclusion

A new modification of variational iteration method using Legendre wavelets is proposed and employed to solve a number of nonlinear partial differential equations. The proposed method can give approximations of higher accuracy and closed form solutions if existed. There are four important points to make here. First, unlike the VIM, the VIMLW can easily overcome the difficulty arising in the evaluation integration and the derivative of nonlinear terms and does not need symbolic computation. Second, by using the properties of BPFs, operational matrices of product of Legendre wavelets are derived and utilized to deal with nonlinear terms. Third, compared with Legendre wavelets method, the VIMLW only needs a few iterations instead of solving a system of nonlinear algebraic equations. Fourth and most important, VIMLW is computer oriented and can use existing fast algorithms to reduce the computation cost.

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