Research Article

Viscosity Approximation Method for System of Variational Inclusions Problems and Fixed-Point Problems of a Countable Family of Nonexpansive Mappings

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We propose new iterative schemes for finding the common element of the set of common fixed points of countable family of nonexpansive mappings, the set of solutions of the variational inequality problem for relaxed cocoercive and Lipschitz continuous, the set of solutions of system of variational inclusions problem, and the set of solutions of equilibrium problems in a real Hilbert space by using the viscosity approximation method. We prove strong convergence theorem under some parameters. The results in this paper unify and generalize some well-known results in the literature.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping S of C into itself is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S; that is, $F(S) = \{x \in C : Sx = x\}$. If $C \subset H$ is nonempty, closed and convex and let $S : C \rightarrow C$ be a nonexpansive mapping, then $F(S)$ is closed and convex and $F(S) \neq \emptyset$, when C is bounded; see, for example, [1, 2]. The metric projection, P_C, onto a given nonempty, closed and convex subset C, satisfies the nonexpansive with $F(P_C) = C$. A mapping $B : C \rightarrow C$ is called monotone if $\langle Bx - By, x - y \rangle \geq 0$ for all $x, y \in C$. A mapping $B : C \rightarrow C$ is called β-inverse-strongly monotone if there exists a constant $\beta > 0$ such that

$\langle Bx - By, x - y \rangle \geq \beta \|x - y\|^2$, for all $x, y \in C$. A mapping $B : C \to C$ is called relaxed (ϕ, ω)-cocoercive if there exists $\phi, \omega > 0$ such that

$$\langle Bx - By, x - y \rangle \geq (-\phi)\|Bx - By\|^2 + \omega\|x - y\|^2, \quad \forall x, y \in C. \tag{1.1}$$

A mapping $B : C \to C$ is said to be ξ-Lipschitz continuous if there exists $\xi \geq 0$ such that

$$\|Bx - By\| \leq \xi\|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

Let $B : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ a multivalued mapping. The variational inclusion problem is to find $\tilde{x} \in H$ such that

$$\theta \in B(\tilde{x}) + M(\tilde{x}), \tag{1.3}$$

where θ is the zero vector in H. The set of solutions of problem (1.3) is denoted by $I(B, M)$. If $M = \partial\psi_C$, where C is a nonempty closed convex subset of H and $\partial\psi_C : H \to [0, +\infty]$ is the indicator function of C; that is,

$$\psi_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \tag{1.4}$$

then, the variational inclusion problem (1.3) is equivalent to the variational inequality problems denoted by $\mathrm{VI}(C, B)$ which is to find $\tilde{x} \in C$ such that

$$\langle B\tilde{x}, y - \tilde{x} \rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

In 2003, Takahashi and Toyoda [3] to find $x^* \in F(S) \cap \mathrm{VI}(C, B)$ introduced the following iterative scheme:

$$\begin{aligned} &x_0 \in C \text{ chosen arbitrary,} \\ &x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \xi_n Bx_n), \quad \forall n \geq 0, \end{aligned} \tag{1.6}$$

where B is a β-inverse-strongly monotone mapping, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\xi_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(S) \cap \mathrm{VI}(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.6) converges weakly to some $x^* \in F(S) \cap \mathrm{VI}(C, B)$.

In 2008, Zhang et al. [4] to find $x^* \in F(S) \cap I(M, B)$. They introduced the following new iterative scheme:

$$\begin{aligned} &x_0 \in C \text{ chosen arbitrary,} \\ &y_n = J_{M,\lambda}(x_n - \lambda Bx_n), \\ &x_{n+1} = \alpha_n x + (1 - \alpha_n)Sy_n, \quad \forall n \geq 0, \end{aligned} \tag{1.7}$$

where $J_{M,\lambda} = (I + \lambda M)^{-1}$ is the resolvent operator associated with M and a positive number λ, $\{\alpha_n\}$ is a sequence in the interval $[0, 1]$.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $\tilde{x} \in C$ such that

$$F(\tilde{x}, y) \geq 0, \quad \forall y \in C. \tag{1.8}$$

The set of solutions of (1.8) is denoted by $\mathrm{EP}(F)$. Many problems in applied sciences, such as monotone inclusion problems, variational inequality problems, saddle point problems, Nash equilibria in noncooperative games, as well as certain fixed-point problems reduce to finding some element to $\mathrm{EP}(F)$ in Hilbert and Banach spaces (see [5–14]).

Given any $r > 0$. The operator $T_r : H \to C$ defined by

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\}, \tag{1.9}$$

is called the resolvent of F (see [5, 6]).

It is shown in [6] that, under suitable hypotheses on F (to be stated precisely in Section 2), $T_r : H \to C$ is single valued and firmly nonexpansive and satisfies

$$F(T_r) = \mathrm{EP}(F), \quad \forall r > 0. \tag{1.10}$$

Using this result, for finding an element of $F(S) \cap \mathrm{VI}(C, B) \cap \mathrm{EP}(F)$, Su et al. [15] introduced the following iterative scheme by the viscosity approximation method in Hilbert spaces:

$$\begin{aligned} &x_0 \in C \text{ chosen arbitrary,} \\ &x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_C (I - \xi_n B) T_{r_n} x_n, \quad \forall n \geq 0, \end{aligned} \tag{1.11}$$

where $f : C \to C$ is a contraction (i.e., $\|f(x) - f(y)\| \leq \psi \|x - y\|$, for all $x, y \in C$ and $0 \leq \psi < 1$) and $\{\alpha_n\} \subset (0, 1)$, $\xi_n \subset (0, 2\beta)$, and $r_n \subset (0, \infty)$ satisfy some appropriate conditions. Furthermore, they prove $\{x_n\}$ converges strongly to the same point $x^* \in F(S) \cap \mathrm{VI}(C, B) \cap \mathrm{EP}(F)$, where $x^* = P_{F(S) \cap \mathrm{VI}(C,B) \cap \mathrm{EP}(F)} f(x^*)$.

In this paper, motivated and inspired by the above facts, we introduce a new iterative scheme for finding a common element of the set of solutions of the variational inequalities for μ-Lipschitz continuous and relaxed (ϕ, ω)-cocoercive mapping, the set of solutions to the variational inclusion for family of α-inverse strongly monotone mappings, the set of fixed points of a countable family of nonexpansive mappings, and the set of solutions of an equilibrium problem in a real Hilbert space by using the viscosity approximation method. Strong convergence results are derived under suitable conditions in a real Hilbert space.

2. Preliminaries

In this section, we will recall some basic notations and collect some conclusions that will be used in the next section.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. We denote strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and

weak convergence by $x_n \rightharpoonup x$. Let C be nonempty closed convex subset of H. Recall that for all $x \in H$ there exists a unique nearest point in C to x denoted $P_C x$; that is, $\|x - P_C x\| \le \|x - y\|$, for all $y \in C$. The mapping P_C is nonexpansive; that is, $\|P_C x - P_C y\| \le \|x - y\|$, for all $x, y \in H$. The mapping P_C is firmly nonexpansive; that is, $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y \rangle$, for all $x, y \in H$. It is well known that

$$\tilde{x} \in \text{VI}(C, B) \Leftrightarrow \tilde{x} = P_C(\tilde{x} - \lambda B \tilde{x}), \quad \forall \lambda > 0. \tag{2.1}$$

A set-valued mapping $M : H \rightarrow 2^H$ is called monotone if, for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g \rangle \ge 0$. A monotone mapping $M : H \rightarrow 2^H$ is called maximal, if its graph of any Graph $(M) := \{(x, f) \in H \times H \mid f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if for all $(x, f) \in H \times H, \langle x - y, f - g \rangle \ge 0$, for all $(y, g) \in$ Graph (M) (the graph of mapping M) implies that $f \in Mx$.

Definition 2.1. Let $M : H \rightarrow 2^H$ be a multivalued maximal monotone mapping; then the set-valued mapping $J_{M,\lambda} : H \rightarrow H$ defined by

$$J_{M,\lambda}(\tilde{x}) = (I + \lambda M)^{-1}(\tilde{x}), \quad \forall \tilde{x} \in H, \tag{2.2}$$

is called the resolvent operator associated with M, where λ is any positive number and I is the identity mapping.

Lemma 2.2 (see [16]). *Let $M : H \rightarrow 2^H$ be a maximal monotone mapping and let $B : H \rightarrow H$ be a Lipschitz continuous mapping. Then the mapping $M + B : H \rightarrow 2^H$ is a maximal monotone mapping.*

Lemma 2.3 (see [16, 17]).

(1) *The resolvent operator $J_{M,\lambda}$ is single valued and nonexpansive for all $\lambda > 0$; that is,*

$$\left\| J_{M,\lambda}(x) - J_{M,\lambda}(y) \right\| \le \|x - y\|, \quad \forall x, y \in H, \forall \lambda > 0. \tag{2.3}$$

(2) *The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly monotone; that is,*

$$\left\| J_{M,\lambda}(x) - J_{M,\lambda}(y) \right\|^2 \le \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y) \rangle, \quad \forall x, y \in H. \tag{2.4}$$

Lemma 2.4 (see [17]).

(1) *Let $\tilde{x} \in H$ is a solution of problem (1.3) if and only if $\tilde{x} = J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$; that is,*

$$I(B, M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0. \tag{2.5}$$

(2) *If $\lambda \in [0, 2\beta]$, then $I(B, M)$ is a closed convex subset in H.*

Lemma 2.5 (see [18]). *Each Hilbert space H satisfies Opial's condition; that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \tag{2.6}$$

holds for each $y \in H$ with $y \neq x$.

Lemma 2.6 (see [19]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n \to \infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n \to \infty} \|z_n - x_n\| = 0$.*

Lemma 2.7 (see [20]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n)a_n + \delta_n, \quad n \geq 0, \tag{2.7}$$

where $\{b_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} b_n = \infty$,

(2) $\limsup_{n \to \infty} \delta_n / b_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.8. *Let H be a real Hilbert space. Then hold the following identities:*

(i) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \ \forall t \in [0,1], \ \forall x, y \in H$,

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \ \forall x, y \in H$.

Lemma 2.9 (see [21]). *Let C be a nonempty closed subset of a Banach space, and let $\{S_n\}$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_nz\| : z \in C\} < \infty$. Then, for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by*

$$Sy = \lim_{n \to \infty} S_n y, \quad \forall y \in C. \tag{2.8}$$

Then $\lim_{n \to \infty} \sup\{\|Sz - S_nz\| : z \in C\} = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0, \ \forall \ x, y \in C$;

(A3) for each $x, y, z \in C, \ \lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.10 (see [5]). *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \tag{2.9}$$

Lemma 2.11 (see [6]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\}, \tag{2.10}$$

for all $x \in H$. Then, the following hold:

(i) *T_r is single valued;*

(ii) *T_r is firmly nonexpansive; that is, for any $x, y \in H \|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$;*

(iii) *$F(T_r) = \mathrm{EP}(F)$;*

(iv) *$\mathrm{EP}(F)$ is closed and convex.*

Lemma 2.12 (see [22]). *Let H be a Hilbert space and M a maximal monotone on H. Then, the following holds:*

$$\|J_{M,r}x - J_{M,s}x\|^2 \leq \frac{r - s}{r}\langle J_{M,r}x - J_{M,s}x, J_{M,r}x - x \rangle, \quad \forall s, r > 0, \ x \in H, \tag{2.11}$$

where $J_{M,r} = (I + rM)^{-1}$ and $J_{M,s} = (I + sM)^{-1}$.

3. Main Results

In this section, we will use the viscosity approximation method to prove a strong convergence theorem for finding a common element of the set of fixed points of a countable family of nonexpansive mappings, the set of solutions of the variational inequality problem for relaxed cocoercive and Lipschitz continuous mappings, the set of solutions of system of variational inclusions, and the set of solutions of equilibrium problem in a real Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $B : C \to H$ be relaxed (ϕ, ω)-cocoercive and μ-Lipschitz continuous with $\omega > \phi\mu^2$, for some $\phi, \omega, \mu > 0$. Let $\mathcal{G} = \{G_k : k = 1, 2, 3, \ldots, N\}$ be a finite family of β-inverse strongly monotone mappings from C into H, and let F be a bifunction from $C \times C \to \mathbb{R}$ satisfing (A1)–(A4). Let $f : C \to C$ be a contraction with coefficient ψ $(0 \leq \psi < 1)$, and let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that*

$$\Omega : \bigcap_{n=1}^{\infty} F(S_n) \cap \left(\bigcap_{k=1}^{N} I(G_k, M_k) \right) \cap \mathrm{VI}(C, B) \cap \mathrm{EP}(F) \neq \emptyset. \tag{3.1}$$

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{aligned}
&x_1 = x \in C \text{ chosen arbitrarily,} \\
&y_n = J_{M_N, \lambda_{N,n}}(I - \lambda_{N,n}G_N) \ldots J_{M_2, \lambda_{2,n}}(I - \lambda_{2,n}G_2)J_{M_1, \lambda_{1,n}}(I - \lambda_{1,n}G_1)T_{r_n}x_n, \\
&x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n P_C(y_n - \xi_n B y_n), \quad \forall n \geq 1,
\end{aligned} \tag{3.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $\{\xi_n\}, \{r_n\} \subset (0,\infty)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(C4) $\{\xi_n\} \subset [a,b]$ *for some* a, b *with* $0 \le a \le b \le 2(\omega - \phi\mu^2)/\mu^2$ *and* $\lim_{n\to\infty} |\xi_{n+1} - \xi_n| = 0$,

(C5) $\{\lambda_{k,n}\}_{k=1}^{N} \subset [c,d] \subset (0,2\beta)$ *and* $\lim_{n\to\infty} |\lambda_{k,n+1} - \lambda_{k,n}| = 0$, *for each* $k \in \{1,2,\ldots,N\}$,

(C6) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty$ for any bounded subset K of C. Let S be a mapping of C into itself defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$ and suppose that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $x^* \in \Omega$, where $x^* = P_\Omega f(x^*)$.

Proof. First, we prove that the mapping $P_\Omega f : H \to C$ has a unique fixed point.

In fact, since $f : C \to C$ is a contraction with $\psi \in [0,1)$ and $P_\Omega f : H \to \Omega$ is also a contraction, we obtain

$$\|P_\Omega f(x) - P_\Omega f(y)\| \le \|f(x) - f(y)\| \le \psi\|x - y\|, \quad \forall x, y \in C. \tag{3.3}$$

Therefore, there exists a unique element $x^* \in C$ such that $x^* = P_\Omega f(x^*)$, where

$$\Omega : \bigcap_{n=1}^{\infty} F(S_n) \cap \left(\bigcap_{k=1}^{N} I(G_k, M_k) \right) \cap \mathrm{VI}(C, B) \cap \mathrm{EP}(F). \tag{3.4}$$

Now, we prove that $(I - \xi_n B)$ is nonexpansive.

Indeed, for any $x, y \in C$, since $B : C \to H$ is a μ-Lipschitz continuous and relaxed (ϕ, ω)-cocoercive mappings with $\omega > \phi\mu^2$ and $\xi_n \le 2(\omega - \phi\mu^2)/\mu^2$, we obtain

$$
\begin{aligned}
\|(I - \xi_n B)x - (I - \xi_n B)y\|^2 &= \|(x - y) - \xi_n(Bx - By)\|^2 \\
&= \|x - y\|^2 - 2\xi_n\langle x - y, Bx - By\rangle + \xi_n^2\|Bx - By\|^2 \\
&\le \|x - y\|^2 - 2\xi_n\left\{-\phi\|Bx - By\|^2 + \omega\|x - y\|^2\right\} + \xi_n^2\|Bx - By\|^2 \\
&\le \|x - y\|^2 + 2\xi_n\phi\mu^2\|x - y\|^2 - 2\xi_n\omega\|x - y\|^2 + \xi_n^2\mu^2\|x - y\|^2 \\
&= \left(1 + 2\xi_n\phi\mu^2 - 2\xi_n\omega + \xi_n^2\mu^2\right)\|x - y\|^2 \\
&= \left(1 - \xi_n\mu^2\left[\frac{2(\omega - \phi\mu^2)}{\mu^2} - \xi_n\right]\right)\|x - y\|^2 \\
&\le \left(1 - \xi_n\mu^2\left[\frac{2(\omega - \phi\mu^2)}{\mu^2} - b\right]\right)\|x - y\|^2.
\end{aligned}
\tag{3.5}
$$

Setting

$$\zeta = \frac{\mu^2}{2}\left[\frac{2(\omega - \phi\mu^2)}{\mu^2} - b\right] > 0, \tag{3.6}$$

thus,

$$\left\| (I - \xi_n B) x - (I - \xi_n B) y \right\|^2 \le (1 - 2\xi_n \zeta) \left\| x - y \right\|^2 \le (1 - \xi_n \zeta)^2 \left\| x - y \right\|^2, \tag{3.7}$$

which implies that

$$\left\| (I - \xi_n B) x - (I - \xi_n B) y \right\| \le (1 - \xi_n \zeta) \left\| x - y \right\| \le \left\| x - y \right\|. \tag{3.8}$$

Hence $(I - \xi_n B)$ is nonexpansive.

We divide the proof of Theorem 3.1 into five steps.

Step 1. We show that the sequence $\{x_n\}$ is bounded.

Now, let $\tilde{x} \in \Omega$ and if $\{T_{r_n}\}$ is a sequence of mappings defined as in Lemma 2.11, then $\tilde{x} = P_C (\tilde{x} - \lambda_n B\tilde{x}) = T_{r_n}\tilde{x}$, and let $u_n = T_{r_n} x_n$. So, we have

$$\|u_n - \tilde{x}\| = \|T_{r_n} x_n - T_{r_n}\tilde{x}\| \le \|x_n - \tilde{x}\|. \tag{3.9}$$

For $k \in \{1, 2, \ldots, N\}$ and for any positive integer number n, we define the operator $\Upsilon_n^k : C \to H$ as follows:

$$\begin{aligned} \Upsilon_n^0 x &= x, \\ \Upsilon_n^k x &= J_{M_k, \lambda_{k,n}} (I - \lambda_{k,n} G_k) \ldots J_{M_2, \lambda_{2,n}} (I - \lambda_{2,n} G_2) J_{M_1, \lambda_{1,n}} (I - \lambda_{1,n} G_1) x, \end{aligned} \tag{3.10}$$

for all n, we get $y_n = \Upsilon_n^N u_n$. On the other hand, since $G_k : C \to H$ is β-inverse strongly monotone and $\lambda_{k,n} \subset [c,d] \subset (0, 2\beta)$, then $J_{M_k, \lambda_{k,n}} (I - \lambda_{k,n} G_k)$ is nonexpansive. Thus Υ_n^k is nonexpansive. From Lemma 2.4(1), we have $\tilde{x} = \Upsilon_n^N \tilde{x}$. It follows that

$$\left\| y_n - \tilde{x} \right\| = \left\| \Upsilon_n^N u_n - \Upsilon_n^N \tilde{x} \right\| \le \|u_n - \tilde{x}\| \le \|x_n - \tilde{x}\|. \tag{3.11}$$

Setting $v_n = P_C(y_n - \xi_n B y_n)$ and $I - \xi_n B$ is a nonexpansive mapping, we obtain

$$\begin{aligned} \|v_n - \tilde{x}\| &= \left\| P_C (y_n - \xi_n B y_{\mathrm{n}}) - P_C(\tilde{x} - \xi_n B\tilde{x}) \right\| \\ &\le \left\| (y_n - \xi_n B y_n) - (\tilde{x} - \xi_n B\tilde{x}) \right\| \\ &= \left\| (I - \xi_n B) y_n - (I - \xi_n B)\tilde{x} \right\| \\ &\le \left\| y_n - \tilde{x} \right\| \le \|x_n - \tilde{x}\|. \end{aligned} \tag{3.12}$$

From (3.2) and (3.12), we deduce that

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n v_n - \tilde{x}\| \\
&\leq \alpha_n \|f(x_n) - \tilde{x}\| + \beta_n \|x_n - \tilde{x}\| + \gamma_n \|v_n - \tilde{x}\| \\
&\leq \alpha_n \|f(x_n) - f(\tilde{x})\| + \alpha_n \|f(\tilde{x}) - \tilde{x}\| + \beta_n \|x_n - \tilde{x}\| + \gamma_n \|x_n - \tilde{x}\| \\
&\leq \alpha_n \psi \|x_n - \tilde{x}\| + \alpha_n \|f(\tilde{x}) - \tilde{x}\| + (1 - \alpha_n)\|x_n - \tilde{x}\| \\
&\leq (1 - \alpha_n(1 - \psi))\|x_n - \tilde{x}\| + \alpha_n \|f(\tilde{x}) - \tilde{x}\| \\
&= (1 - \alpha_n(1 - \psi))\|x_n - \tilde{x}\| + \alpha_n(1 - \psi)\frac{\|f(\tilde{x}) - \tilde{x}\|}{(1 - \psi)} \\
&\leq \max\left\{ \|x_n - \tilde{x}\|, \frac{\|f(\tilde{x}) - \tilde{x}\|}{1 - \psi} \right\}.
\end{aligned}
\tag{3.13}
$$

It follows from induction that

$$
\|x_n - \tilde{x}\| \leq \max\left\{ \|x_1 - \tilde{x}\|, \frac{\|f(\tilde{x}) - \tilde{x}\|}{1 - \psi} \right\}, \quad \forall n \geq 1.
\tag{3.14}
$$

Therefore, $\{x_n\}$ is bounded and hence so are $\{v_n\}$, $\{y_n\}$, $\{u_n\}$, $\{By_n\}$, and $\{S_n v_n\}$.

Step 2. We claim that $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.
 By the definition of T_r, $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we get

$$
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in H,
\tag{3.15}
$$

$$
F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in H.
\tag{3.16}
$$

Taking $y = u_{n+1}$ in (3.15) and $y = u_n$ in (3.16), we have

$$
F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0,
\tag{3.17}
$$

and hence

$$
F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0.
\tag{3.18}
$$

So, from (A2) we have

$$
\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0,
\tag{3.19}
$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.20}$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\
&\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\},
\end{aligned} \tag{3.21}$$

and hence

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n|,
\end{aligned} \tag{3.22}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$.

Notice from Lemma 2.12 that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \left\| \Upsilon_{n+1}^N u_{n+1} - \Upsilon_n^N u_n \right\| \\
&\leq \left\| \left(u_{n+1} - \lambda_{k,n+1} G_k \Upsilon_{n+1}^k u_{n+1}\right) - \left(u_n - \lambda_{k,n} G_k \Upsilon_n^k u_n\right) \right\| \\
&\quad + \left\| J_{M_k}, \lambda_{k,n+1}\left(u_n - \lambda_{k,n} G_k \Upsilon_n^k u_n\right) - J_{M_k}, \lambda_{k,n}\left(u_n - \lambda_{k,n} G_k \Upsilon_n^k u_n\right) \right\| \\
&\leq \|u_{n+1} - u_n\| + |\lambda_{k,n+1} - \lambda_{k,n}| \left\| G_k \Upsilon_n^k u_n \right\| \\
&\quad + \frac{|\lambda_{k,n+1} - \lambda_{k,n}|}{\lambda_{k,n+1}} \left\| J_{M_k}, \lambda_{k,n+1}\left(u_n - \lambda_{k,n} G_k \Upsilon_n^k u_n\right) - \left(u_n - \lambda_{k,n} G_k \Upsilon_n^k u_n\right) \right\| \\
&\leq \|u_{n+1} - u_n\| + 2 M_2 |\lambda_{k,n+1} - \lambda_{k,n}| \\
&\leq \|x_{n+1} - x_n\| + \frac{M_1}{c}|r_{n+1} - r_n| + 2 M_2 |\lambda_{k,n+1} - \lambda_{k,n}|,
\end{aligned} \tag{3.23}$$

where M_2 is an appropriate constant such that

$$M_2 = \max\left\{ \sup_{n \geq 1}\left\{ \left\| G_k \Upsilon_n^k u_n \right\| \right\}, \ \sup_{n \geq 1}\left\{ \frac{\left\| J_{M_k}, \lambda_{k,n+1}\left(u_n - \lambda_{k,n} G_k \Upsilon_n^k u_n\right) - \left(u_n - \lambda_{k,n} G_k \Upsilon_n^k u_n\right) \right\|}{J_{M_k}, \lambda_{k,n+1}} \right\} \right\}. \tag{3.24}$$

Since $I - \xi_n B$ is nonexpansive mappings, we have the following estimates:

$$
\begin{aligned}
\|v_{n+1} - v_n\| &\leq \|P_C(y_{n+1} - \xi_{n+1} B y_{n+1}) - P_C(y_n - \xi_n B y_n)\| \\
&\leq \|(y_{n+1} - \xi_{n+1} B y_{n+1}) - (y_n - \xi_n B y_n)\| \\
&= \|(y_{n+1} - \xi_{n+1} B y_{n+1}) - (y_n - \xi_{n+1} B y_n) + (\xi_n - \xi_{n+1}) B y_n\| \\
&\leq \|(y_{n+1} - \xi_{n+1} B y_{n+1}) - (y_n - \xi_{n+1} B y_n)\| + |\xi_n - \xi_{n+1}| \|B y_n\| \\
&= \|(I - \xi_{n+1} B) y_{n+1} - (I - \xi_{n+1} B) y_n\| + |\xi_n - \xi_{n+1}| \|B y_n\| \\
&\leq \|y_{n+1} - y_n\| + |\xi_n - \xi_{n+1}| \|B y_n\|.
\end{aligned}
\tag{3.25}
$$

Substituting (3.23) into (3.25), we obtain

$$
\begin{aligned}
\|v_{n+1} - v_n\| &\leq \|x_{n+1} - x_n\| + \frac{M_1}{c} |r_{n+1} - r_n| + 2M_2 |\lambda_{k,n+1} - \lambda_{k,n}| \\
&\quad + |\xi_n - \xi_{n+1}| \|B y_n\|.
\end{aligned}
\tag{3.26}
$$

Indeed, define $x_{n+1} = (1 - \beta_n) z_n + \beta_n x_n$ for all $n \in \mathbb{N}$. It follows that

$$
z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n f(x_n) + \gamma_n S_n v_n}{1 - \beta_n}.
\tag{3.27}
$$

Thus, we have

$$
\begin{aligned}
\|z_{n+1} - z_n\| &= \left\| \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} S_{n+1} v_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n S_n v_n}{1 - \beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (f(x_{n+1}) - f(x_n)) + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} (S_{n+1} v_{n+1} - S_n v_n) \right. \\
&\quad \left. + \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) f(x_n) + \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) S_n v_n \right\| \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|f(x_{n+1}) - f(x_n)\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|S_{n+1} v_{n+1} - S_n v_n\| \\
&\quad + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|f(x_n) - S_n v_n\| \\
&\leq \frac{\psi \alpha_{n+1}}{1 - \beta_{n+1}} \|x_{n+1} - x_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|S_{n+1} v_{n+1} - S_n v_n\| \\
&\quad + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|f(x_n) - S_n v_n\|.
\end{aligned}
\tag{3.28}
$$

Now, compute

$$
\begin{aligned}
\|S_{n+1} v_{n+1} - S_n v_n\| &\leq \|S_{n+1} v_{n+1} - S_{n+1} v_n\| + \|S_{n+1} v_n - S_n v_n\| \\
&\leq \|v_{n+1} - v_n\| + \|S_{n+1} v_n - S_n v_n\| \\
&\leq \|x_{n+1} - x_n\| + \frac{M_1}{c} |r_{n+1} - r_n| + |\xi_n - \xi_{n+1}| \|B y_n\| \\
&\quad + 2M_2 |\lambda_{k,n+1} - \lambda_{k,n}| + \|S_{n+1} v_n - S_n v_n\|.
\end{aligned}
\tag{3.29}
$$

Combining (3.28) and (3.29), we have

$$
\begin{aligned}
\|z_{n+1} - z_n\| &\leq \frac{\psi \alpha_{n+1}}{1 - \beta_{n+1}} \|x_{n+1} - x_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left\{ \|x_{n+1} - x_n\| + \frac{M_1}{c} |r_{n+1} - r_n| + |\xi_n - \xi_{n+1}| \|By_n\| \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + 2M_2 |\lambda_{k,n+1} - \lambda_{k,n}| + \|S_{n+1}v_n - S_n v_n\| \right\} \\
&\quad + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|f(x_n) - S_n v_n\| \\
&\leq \|x_{n+1} - x_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left\{ \frac{M_1}{c} |r_{n+1} - r_n| + |\xi_n - \xi_{n+1}| \|By_n\| \right. \\
&\qquad\qquad\qquad\qquad\quad \left. + 2M_2 |\lambda_{k,n+1} - \lambda_{k,n}| \right\} + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|S_{n+1}v_n - S_n v_n\| \\
&\quad + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|f(x_n) - S_n v_n\|.
\end{aligned}
\tag{3.30}
$$

It follows that

$$
\begin{aligned}
\|z_{n+1} - z_n\| &- \|x_{n+1} - x_n\| \\
&\leq \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left\{ \frac{M_1}{c} |r_{n+1} - r_n| + |\xi_n - \xi_{n+1}| \|By_n\| + 2M_2 |\lambda_{k,n+1} - \lambda_{k,n}| \right\} \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|S_{n+1}v_n - S_n v_n\| + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|f(x_n) - S_n v_n\| \\
&\leq \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left\{ \frac{M_1}{c} |r_{n+1} - r_n| + |\xi_n - \xi_{n+1}| \|By_n\| + 2M_2 |\lambda_{k,n+1} - \lambda_{k,n}| \right\} \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \sup\{ \|S_{n+1}z - S_n z\| : z \in \{v_n\} \} + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|f(x_n) - S_n v_n\|.
\end{aligned}
\tag{3.31}
$$

This together with conditions (C1)–(C6) and $\lim_{n \to \infty} \sup\{ \|S_{n+1}z - S_n z\| : z \in \{v_n\} \} = 0$ implies that

$$
\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.
\tag{3.32}
$$

Hence, by Lemma 2.6, we obtain $\|z_n - x_n\| \to 0$ as $n \to \infty$. It then follows that

$$
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n) \|z_n - x_n\| = 0.
\tag{3.33}
$$

By (3.26), we also have

$$
\lim_{n \to \infty} \|v_{n+1} - v_n\| = 0.
\tag{3.34}
$$

Step 3. We claim that $\lim_{n\to\infty}\|Sv_n - v_n\| = 0$.

Since $\{G_k : k = 1, 2, 3, \ldots, N\}$ is β-inverse strongly monotone mappings, by the choice of $\{\lambda_{k,n}\}$ for given $\tilde{x} \in \Omega$ and $k \in \{0, 1, 2, \ldots, N-1\}$, we also have

$$
\begin{aligned}
&\left\|\Upsilon_n^{k+1} u_n - \tilde{x}\right\|^2 \\
&= \left\|J_{M_{k+1},\lambda_{k+1,n}}(I - \lambda_{k+1,n}G_{k+1})\Upsilon_n^k u_n - J_{M_{k+1},\lambda_{k+1,n}}(I - \lambda_{k+1,n}G_{k+1})\tilde{x}\right\|^2 \\
&\leq \left\|(I - \lambda_{k+1,n}G_{k+1})\Upsilon_n^k u_n - (I - \lambda_{k+1,n}G_{k+1})\tilde{x}\right\|^2 \\
&= \left\|\left(\Upsilon_n^k u_n - \lambda_{k+1,n}G_{k+1}\Upsilon_n^k u_n\right) - (\tilde{x} - \lambda_{k+1,n}G_{k+1}\tilde{x})\right\|^2 \\
&= \left\|\left(\Upsilon_n^k u_n - \tilde{x}\right) - \lambda_{k+1,n}\left(G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\right)\right\|^2 \\
&= \left\|\Upsilon_n^k u_n - \tilde{x}\right\|^2 - 2\lambda_{k+1,n}\left\langle\Upsilon_n^k u_n - \tilde{x}, G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\right\rangle + \lambda_{k+1,n}^2\left\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\right\|^2 \\
&\leq \left\|\Upsilon_n^k u_n - \tilde{x}\right\|^2 - 2\lambda_{k+1,n}\beta\left\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\right\| + \lambda_{k+1,n}^2\left\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\right\|^2 \\
&\leq \|u_n - \tilde{x}\|^2 - 2\lambda_{k+1,n}\beta\left\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\right\| + \lambda_{k+1,n}^2\left\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\right\|^2 \\
&\leq \|x_n - \tilde{x}\|^2 + \lambda_{k+1,n}(\lambda_{k+1,n} - 2\beta)\left\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\right\|^2.
\end{aligned}
\tag{3.35}
$$

Form (3.13), we have

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq \alpha_n\|f(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 + \gamma_n\|v_n - \tilde{x}\|^2 \\
&\leq \alpha_n\|f(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 + \gamma_n\|y_n - \tilde{x}\|^2 \\
&= \alpha_n\|f(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 + \gamma_n\left\|\Upsilon_n^N u_n - \tilde{x}\right\|^2 \\
&\leq \alpha_n\|f(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 + \gamma_n\left\|\Upsilon_n^{k+1} u_n - \tilde{x}\right\|^2 \\
&\leq \alpha_n\|f(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 \\
&\quad + \gamma_n\left\{\|x_n - \tilde{x}\|^2 + \lambda_{k+1,n}(\lambda_{k+1,n} - 2\beta)\left\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\right\|^2\right\} \\
&\leq \alpha_n\|f(x_n) - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 + \gamma_n\lambda_{k+1,n}(\lambda_{k+1,n} - 2\beta)\left\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\right\|^2.
\end{aligned}
\tag{3.36}
$$

It follows that

$$
\begin{aligned}
&\gamma_n\lambda_{k+1,n}(2\beta - \lambda_{k+1,n})\left\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\right\|^2 \\
&\leq \gamma_n c(2\beta - d)\left\|G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x}\right\|^2 \\
&\leq \|x_n - x_{n+1}\|(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|) + \alpha_n\|f(x_n) - \tilde{x}\|^2.
\end{aligned}
\tag{3.37}
$$

By condition (C2), (3.33), and $\liminf_{n\to\infty} \gamma_n > 0$, we obtain

$$\lim_{n\to\infty} \left\| G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x} \right\| = 0. \tag{3.38}$$

From Lemma 2.3(2) and as $I - \lambda_{k+1,n}G_{k+1}$ is nonexpansive, we have

$$
\begin{aligned}
&\left\| \Upsilon_n^{k+1} u_n - \tilde{x} \right\|^2 \\
&= \left\| J_{M_{k+1},\lambda_{k+1,n}}(I - \lambda_{k+1,n}G_{k+1})\Upsilon_n^k u_n - J_{M_{k+1},\lambda_{k+1,n}}(I - \lambda_{k+1,n}G_{k+1})\tilde{x} \right\|^2 \\
&\leq \left\langle (I - \lambda_{k+1,n}G_{k+1})\Upsilon_n^k u_n - (I - \lambda_{k+1,n}G_{k+1})\tilde{x}, \Upsilon_n^{k+1} u_n - \tilde{x} \right\rangle \\
&= \frac{1}{2}\left\{ \left\| (I - \lambda_{k+1,n}G_{k+1})\Upsilon_n^k u_n - (I - \lambda_{k+1,n}G_{k+1})\tilde{x} \right\|^2 + \left\| \Upsilon_n^{k+1} u_n - \tilde{x} \right\|^2 \right. \\
&\quad \left. - \left\| (I - \lambda_{k+1,n}G_{k+1})\Upsilon_n^k u_n - (I - \lambda_{k+1,n}G_{k+1})\tilde{x} - \left(\Upsilon_n^{k+1} u_n - \tilde{x} \right) \right\|^2 \right\} \\
&\leq \frac{1}{2}\left\{ \left\| \Upsilon_n^k u_n - \tilde{x} \right\|^2 + \left\| \Upsilon_n^{k+1} u_n - \tilde{x} \right\|^2 - \left\| \left(\Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right) - \lambda_{k+1,n}\left(G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x} \right) \right\|^2 \right\} \\
&\leq \frac{1}{2}\left\{ \left\| \Upsilon_n^k u_n - \tilde{x} \right\|^2 + \left\| \Upsilon_n^{k+1} u_n - \tilde{x} \right\|^2 - \left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\|^2 \right. \\
&\quad \left. - \lambda_{k+1,n}^2 \left\| G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x} \right\|^2 + 2\lambda_{k+1,n}\left\langle \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n, G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x} \right\rangle \right\},
\end{aligned}
\tag{3.39}
$$

which yields that

$$
\begin{aligned}
&\left\| \Upsilon_n^{k+1} u_n - \tilde{x} \right\|^2 \\
&\leq \left\| \Upsilon_n^k u_n - \tilde{x} \right\|^2 - \left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\|^2 + 2\lambda_{k+1,n}\left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\| \left\| G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x} \right\| \\
&\leq \left\| u_n - \tilde{x} \right\|^2 - \left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\|^2 + 2\lambda_{k+1,n}\left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\| \left\| G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x} \right\| \\
&\leq \left\| x_n - \tilde{x} \right\|^2 - \left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\|^2 + 2\lambda_{k+1,n}\left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\| \left\| G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x} \right\|.
\end{aligned}
\tag{3.40}
$$

Substituting (3.40) into (3.36), we obtain

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq \alpha_n \left\| f(x_n) - \tilde{x} \right\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \left\| \Upsilon_n^{k+1} u_n - \tilde{x} \right\|^2 \\
&\leq \alpha_n \left\| f(x_n) - \tilde{x} \right\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \left\{ \|x_n - \tilde{x}\|^2 - \left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\|^2 \right. \\
&\quad \left. + 2\lambda_{k+1,n}\left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\| \left\| G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x} \right\| \right\} \\
&\leq \alpha_n \left\| f(x_n) - \tilde{x} \right\|^2 + \|x_n - \tilde{x}\|^2 - \gamma_n \left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\|^2 \\
&\quad + 2\lambda_{k+1,n}\gamma_n \left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\| \left\| G_{k+1}\Upsilon_n^k u_n - G_{k+1}\tilde{x} \right\|.
\end{aligned}
\tag{3.41}
$$

It follows that

$$
\begin{aligned}
\gamma_n \left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\|^2 &\leq \|x_n - x_{n+1}\| (\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|) + \alpha_n \|f(x_n) - \tilde{x}\|^2 \\
&\quad + 2\lambda_{k+1,n} \gamma_n \left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\| \left\| G_{k+1} \Upsilon_n^k u_n - G_{k+1} \tilde{x} \right\|.
\end{aligned}
\tag{3.42}
$$

By condition (C2), (3.33), (3.38), and $\liminf_{n \to \infty} \gamma_n > 0$, we obtain

$$
\lim_{n \to \infty} \left\| \Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n \right\| = 0.
\tag{3.43}
$$

For $\tilde{x} \in \Omega$, we obtain

$$
\begin{aligned}
\|v_n - \tilde{x}\|^2 &= \left\| P_C(y_n - \xi_n B y_n) - P_C(\tilde{x} - \xi_n B \tilde{x}) \right\|^2 \\
&\leq \left\| (y_n - \xi_n B y_n) - (\tilde{x} - \xi_n B \tilde{x}) \right\|^2 \\
&= \left\| (y_n - \tilde{x}) - \xi_n (B y_n - B \tilde{x}) \right\|^2 \\
&\leq \|y_n - \tilde{x}\|^2 - 2\xi_n \langle y_n - \tilde{x}, B y_n - B \tilde{x} \rangle + \xi_n^2 \|B y_n - B \tilde{x}\|^2 \\
&\leq \|y_n - \tilde{x}\|^2 - 2\xi_n \left\{ -\phi \|B y_n - B \tilde{x}\|^2 + \omega \|y_n - \tilde{x}\|^2 \right\} + \xi_n^2 \|B y_n - B \tilde{x}\|^2 \\
&\leq \|y_n - \tilde{x}\|^2 + 2\xi_n \phi \|B y_n - B \tilde{x}\|^2 - 2\xi_n \omega \|y_n - \tilde{x}\|^2 + \xi_n^2 \|B y_n - B \tilde{x}\|^2 \\
&\leq \|y_n - \tilde{x}\|^2 + 2\xi_n \phi \|B y_n - B \tilde{x}\|^2 - \frac{2\xi_n \omega}{\mu^2} \|B y_n - B \tilde{x}\|^2 + \xi_n^2 \|B y_n - B \tilde{x}\|^2 \\
&\leq \|x_n - \tilde{x}\|^2 + \left(2\xi_n \phi + \xi_n^2 - \frac{2\xi_n \omega}{\mu^2} \right) \|B y_n - B \tilde{x}\|^2.
\end{aligned}
\tag{3.44}
$$

On the other hand, we have

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \left\| \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n v_n - \tilde{x} \right\|^2 \\
&\leq \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|S_n v_n - \tilde{x}\|^2 \\
&\leq \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|v_n - \tilde{x}\|^2 \\
&\leq \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 \\
&\quad + \gamma_n \left\{ \|x_n - \tilde{x}\|^2 + \left(2\xi_n \phi + \xi_n^2 - \frac{2\xi_n \omega}{\mu^2} \right) \|B y_n - B \tilde{x}\|^2 \right\} \\
&= \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|x_n - \tilde{x}\|^2 \\
&\quad + \gamma_n \left(2\xi_n \phi + \xi_n^2 - \frac{2\xi_n \omega}{\mu^2} \right) \|B y_n - B \tilde{x}\|^2 \\
&\leq \alpha_n \|f(x_n) - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 + \gamma_n \left(2\xi_n \phi + \xi_n^2 - \frac{2\xi_n \omega}{\mu^2} \right) \|B y_n - B \tilde{x}\|^2.
\end{aligned}
\tag{3.45}
$$

It follows that

$$
\begin{aligned}
\left(\frac{2a\omega}{\mu^2} - b^2 - 2b\phi\right) & \gamma_n \|By_n - B\tilde{x}\|^2 \\
& \leq \left(\frac{2\xi_n\omega}{\mu^2} - \xi_n^2 - 2\xi_n\phi\right)\gamma_n \|By_n - B\tilde{x}\|^2 \\
& \leq \|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 + \alpha_n \|f(x_n) - \tilde{x}\|^2 \\
& \leq \|x_n - x_{n+1}\|(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|) + \alpha_n \|f(x_n) - \tilde{x}\|^2.
\end{aligned}
\tag{3.46}
$$

It now follows from the last inequality, conditions (C2), (3.33), and $\liminf_{n\to\infty}\gamma_n > 0$ that

$$
\lim_{n\to\infty} \|By_n - B\tilde{x}\| = 0.
\tag{3.47}
$$

Since P_C is firmly nonexpansive, we have

$$
\begin{aligned}
\|v_n - \tilde{x}\|^2 &= \|P_C(y_n - \xi_n By_n) - P_C(\tilde{x} - \xi_n B\tilde{x})\|^2 \\
&= \|P_C(I - \xi_n B)y_n - P_C(I - \xi_n B)\tilde{x}\|^2 \\
&\leq \langle (I - \xi_n B)y_n - (I - \xi_n B)\tilde{x}, v_n - \tilde{x} \rangle \\
&= \frac{1}{2}\Big\{ \|(I - \alpha_n B)y_n - (I - \xi_n B)\tilde{x}\|^2 + \|v_n - \tilde{x}\|^2 \\
&\qquad - \|(I - \xi_n B)y_n - (I - \xi_n B)\tilde{x} - (v_n - \tilde{x})\|^2 \Big\} \\
&\leq \frac{1}{2}\Big\{ \|y_n - \tilde{x}\|^2 + \|v_n - \tilde{x}\|^2 - \|(y_n - v_n) - \xi_n(By_n - B\tilde{x})\|^2 \Big\} \\
&\leq \frac{1}{2}\Big\{ \|y_n - \tilde{x}\|^2 + \|v_n - \tilde{x}\|^2 - \|y_n - v_n\|^2 \\
&\qquad -\xi_n^2 \|By_n - B\tilde{x}\|^2 + 2\xi_n\langle y_n - v_n, By_n - B\tilde{x}\rangle \Big\},
\end{aligned}
\tag{3.48}
$$

which yields that

$$
\begin{aligned}
\|v_n - \tilde{x}\|^2 &\leq \|y_n - \tilde{x}\|^2 - \|y_n - v_n\|^2 + 2\xi_n\|y_n - v_n\|\|By_n - B\tilde{x}\| \\
&\leq \|x_n - \tilde{x}\|^2 - \|y_n - v_n\|^2 + 2\xi_n\|y_n - v_n\|\|By_n - B\tilde{x}\|.
\end{aligned}
\tag{3.49}
$$

Substituting (3.49) into (3.45), we obtain

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq \alpha_n\|f(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 + \gamma_n\|v_n - \tilde{x}\|^2 \\
&\leq \alpha_n\|f(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 \\
&\quad + \gamma_n\Big\{ \|x_n - \tilde{x}\|^2 - \|y_n - v_n\|^2 + 2\xi_n\|y_n - v_n\|\|By_n - B\tilde{x}\| \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|x_n - \tilde{x}\|^2 - \gamma_n \|y_n - v_n\|^2 \\
&\quad + 2\gamma_n \xi_n \|y_n - v_n\| \|By_n - B\tilde{x}\| \\
&\leq \alpha_n \|f(x_n) - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - \gamma_n \|y_n - v_n\|^2 + 2\gamma_n \xi_n \|y_n - v_n\| \|By_n - B\tilde{x}\|.
\end{aligned}
\tag{3.50}
$$

It follows that

$$
\begin{aligned}
\gamma_n \|y_n - v_n\|^2 &\leq \|x_n - x_{n+1}\| (\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|) + \alpha_n \|f(x_n) - \tilde{x}\|^2 \\
&\quad + 2\gamma_n \xi_n \|y_n - v_n\| \|By_n - B\tilde{x}\|.
\end{aligned}
\tag{3.51}
$$

By condition (C2), (3.33), (3.47), and $\liminf_{n\to\infty} \gamma_n > 0$, we obtain

$$
\lim_{n\to\infty} \|y_n - v_n\| = 0.
\tag{3.52}
$$

On the other hand, in the light of Lemma 2.11(ii) T_{r_n} is firmly nonexpansive; so we have

$$
\begin{aligned}
\|u_n - \tilde{x}\|^2 &= \|T_{r_n} x_n - T_{r_n} \tilde{x}\|^2 \\
&\leq \langle T_{r_n} x_n - T_{r_n} \tilde{x}, x_n - \tilde{x} \rangle = \langle u_n - \tilde{x}, x_n - \tilde{x} \rangle \\
&= \frac{1}{2} \left(\|u_n - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - \|x_n - u_n\|^2 \right),
\end{aligned}
\tag{3.53}
$$

which implies that

$$
\|u_n - \tilde{x}\|^2 \leq \|x_n - \tilde{x}\|^2 - \|x_n - u_n\|^2.
\tag{3.54}
$$

Form (3.45), we have

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|v_n - \tilde{x}\|^2 \\
&\leq \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|y_n - \tilde{x}\|^2 \\
&\leq \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|u_n - \tilde{x}\|^2 \\
&\leq \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \left\{ \|x_n - \tilde{x}\|^2 - \|x_n - u_n\|^2 \right\} \\
&= \alpha_n \|f(x_n) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|x_n - \tilde{x}\|^2 - \gamma_n \|x_n - u_n\|^2 \\
&\leq \alpha_n \|f(x_n) - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - \gamma_n \|x_n - u_n\|^2.
\end{aligned}
\tag{3.55}
$$

It follows that

$$
\gamma_n \|x_n - u_n\|^2 \leq \|x_n - x_{n+1}\| (\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|) + \alpha_n \|f(x_n) - \tilde{x}\|^2.
\tag{3.56}
$$

By condition (C2), (3.33), and $\liminf_{n\to\infty} \gamma_n > 0$, we obtain

$$
\lim_{n\to\infty} \|x_n - u_n\| = 0.
\tag{3.57}
$$

Observe that

$$x_{n+1} - x_n = \alpha_n (f(x_n) - x_n) + \gamma_n (S_n v_n - x_n). \tag{3.58}$$

By condition (C2) and (3.33), we have

$$\lim_{n \to \infty} \gamma_n \| S_n v_n - x_n \| = \lim_{n \to \infty} (\| x_{n+1} - x_n \| - \alpha_n \| f(x_n) - x_n \|) = 0. \tag{3.59}$$

Since

$$\| S v_n - u_n \| \leq \| S v_n - x_n \| + \| x_n - u_n \|, \tag{3.60}$$

from (3.57) and (3.59), we have

$$\lim_{n \to \infty} \| S_n v_n - u_n \| = 0. \tag{3.61}$$

Form (3.55), we have

$$
\begin{aligned}
\| x_{n+1} - \tilde{x} \|^2 \\
&\leq \alpha_n \| f(x_n) - \tilde{x} \|^2 + \| x_n - \tilde{x} \|^2 - \gamma_n \| x_n - u_n \|^2 \\
&\leq \alpha_n \| f(x_n) - \tilde{x} \|^2 + \| x_n - \tilde{x} \|^2 - \gamma_n \| (x_n - y_n) + (y_n - u_n) \|^2 \\
&\leq \alpha_n \| f(x_n) - \tilde{x} \|^2 + \| x_n - \tilde{x} \|^2 - \gamma_n \left\{ \| x_n - y_n \|^2 + 2 \| x_n - y_n \| \| y_n - u_n \| + \| y_n - u_n \|^2 \right\} \\
&= \alpha_n \| f(x_n) - \tilde{x} \|^2 + \| x_n - \tilde{x} \|^2 - \gamma_n \| x_n - y_n \|^2 - 2\gamma_n \| x_n - y_n \| \| y_n - u_n \| - \gamma_n \| y_n - u_n \|^2 \\
&\leq \alpha_n \| f(x_n) - \tilde{x} \|^2 + \| x_n - \tilde{x} \|^2 - \gamma_n \| x_n - y_n \|^2.
\end{aligned}
\tag{3.62}
$$

It follows that

$$\gamma_n \| x_n - y_n \|^2 \leq \| x_n - x_{n+1} \| (\| x_n - \tilde{x} \| + \| x_{n+1} - \tilde{x} \|) + \alpha_n \| f(x_n) - \tilde{x} \|^2. \tag{3.63}$$

By condition (C2), (3.33), and $\liminf_{n \to \infty} \gamma_n > 0$, we obtain

$$\lim_{n \to \infty} \| x_n - y_n \| = 0. \tag{3.64}$$

Since

$$\| u_n - y_n \| \leq \| u_n - x_n \| + \| x_n - y_n \|, \tag{3.65}$$

from (3.57) and (3.64), we have

$$\lim_{n \to \infty} \| u_n - y_n \| = 0. \tag{3.66}$$

Furthermore, by the triangular inequality we also have

$$\|S_n v_n - v_n\| \leq \|S_n v_n - u_n\| + \|u_n - y_n\| + \|y_n - v_n\|. \tag{3.67}$$

From (3.52), (3.61), and (3.66), we have

$$\lim_{n \to \infty} \|S_n v_n - v_n\| = 0. \tag{3.68}$$

Applying Lemma 2.9 and (3.68), we have

$$\begin{aligned}
\|Sv_n - v_n\| &\leq \|Sv_n - S_n v_n\| + \|S_n v_n - v_n\| \\
&\leq \sup\{\|Sz - S_n z\| : z \in \{v_n\}\} + \|S_n v_n - v_n\| \longrightarrow 0.
\end{aligned} \tag{3.69}$$

Step 4. We claim that $\limsup_{n \to \infty} \langle f(x^*) - x^*, x_n - x^* \rangle \leq 0$.
Indeed, we choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n \to \infty} \langle f(x^*) - x^*, Sv_n - x^* \rangle = \lim_{i \to \infty} \langle f(x^*) - x^*, Sv_{n_i} - x^* \rangle. \tag{3.70}$$

Without loss of generality, let $\{v_{n_i}\} \rightharpoonup z \in C$. From $\|Sv_n - v_n\| \to 0$, we obtain $Sv_{n_i} \rightharpoonup z$. Then, (3.70) reduces to

$$\limsup_{n \to \infty} \langle f(x^*) - x^*, Sv_n - x^* \rangle = \langle f(x^*) - x^*, z - x^* \rangle. \tag{3.71}$$

In order to show $\langle f(x^*) - x^*, z - x^* \rangle \leq 0$, it suffices to show that

$$z \in \Omega : \bigcap_{n=1}^{\infty} F(S_n) \cap \left(\bigcap_{k=1}^{N} I(G_k, M_k) \right) \cap \mathrm{VI}(C, B) \cap \mathrm{EP}(F). \tag{3.72}$$

Firstly, we will show $z \in F(S) = \bigcap_{n=1}^{\infty} F(S_n)$.
Assume $z \notin F(S)$. By Opial's theorem (Lemma 2.5) and $\|Sv_n - v_n\| \to 0$, we have

$$\begin{aligned}
\liminf_{i \to \infty} \|v_{n_i} - z\| &< \liminf_{i \to \infty} \|v_{n_i} - Sz\| \\
&= \liminf_{i \to \infty} \|v_{n_i} - Sv_{n_i} + Sv_{n_i} - Sz\| \\
&= \liminf_{i \to \infty} \|Sv_{n_i} - Sz\| \\
&\leq \liminf_{i \to \infty} \|v_{n_i} - z\|.
\end{aligned} \tag{3.73}$$

This is a contradiction. Thus, we obtain $z \in F(S)$.

Next, we will show that $z \in \mathrm{VI}(C, B)$.

Let

$$Tw_1 = \begin{cases} Bw_1 + N_C w_1, & w_1 \in C; \\ \emptyset, & w_1 \notin C. \end{cases} \tag{3.74}$$

Since B is relaxed (ϕ, ω)-cocoercive, μ-Lipschitz continuous with $\omega > \phi\mu^2$, we obtain

$$\langle Bx - By, x - y \rangle \geq (-\phi)\|Bx - By\|^2 + \omega\|x - y\|^2 \geq \left(\omega - \phi\mu^2\right)\|x - y\|^2 \geq 0, \tag{3.75}$$

which yields that B is monotone. Then T is maximal monotone (see [23]). Let $(w_1, w_2) \in G(T)$. Since $w_2 - Bw_1 \in N_C(w_1)$ and $v_n \in C$, we have $\langle w_1 - v_n, w_2 - Bw_1 \rangle \geq 0$. On the other hand, from $v_n = P_C(y_n - \xi_n By_n)$, we have

$$\langle w_1 - v_n, v_n - (y_n - \xi_n By_n) \rangle \geq 0; \tag{3.76}$$

that is,

$$\left\langle w_1 - v_n, \frac{v_n - y_n}{\xi_n} + By_n \right\rangle \geq 0. \tag{3.77}$$

Therefore, we obtain

$$\begin{aligned} \langle w_1 - v_{n_i}, w_2 \rangle &\geq \langle w_1 - v_{n_i}, Bw_1 \rangle \\ &\geq \langle w_1 - v_{n_i}, Bw_1 \rangle - \left\langle w_1 - v_{n_i}, \frac{v_{n_i} - y_{n_i}}{\xi_{n_i}} + By_{n_i} \right\rangle \\ &= \left\langle w_1 - v_{n_i}, Bw_1 - By_{n_i} - \frac{v_{n_i} - y_{n_i}}{\xi_{n_i}} \right\rangle \\ &= \langle w_1 - v_{n_i}, Bw_1 - Bv_{n_i} \rangle + \langle w_1 - v_{n_i}, Bv_{n_i} - By_{n_i} \rangle \\ &\quad - \left\langle w_1 - v_{n_i}, \frac{v_{n_i} - y_{n_i}}{\xi_{n_i}} \right\rangle \\ &\geq \langle w_1 - v_{n_i}, Bv_{n_i} \rangle - \left\langle w_1 - v_{n_i}, \frac{v_{n_i} - y_{n_i}}{\xi_{n_i}} + By_{n_i} \right\rangle \\ &= \langle w_1 - v_{n_i}, Bv_{n_i} - By_{n_i} \rangle - \left\langle w_1 - v_{n_i}, \frac{v_{n_i} - y_{n_i}}{\xi_{n_i}} \right\rangle. \end{aligned} \tag{3.78}$$

Noting that $\|v_{n_i} - y_{n_i}\| \to 0$ and B is relaxed (ϕ, ω)-cocoercive and (3.78), we obtain

$$\langle w_1 - z, w_2 \rangle \geq 0. \tag{3.79}$$

Since T is maximal monotone, we have $z \in T^{-1}0$, and hence $z \in \mathrm{VI}(C, B)$.

Now, we will show that $z \in \bigcap_{k=1}^{N} I(G_k, M_k)$.

For this purpose, let $k \in \{1, 2, 3, \ldots, N\}$ and G_k is β-inverse strongly monotone, G_k is an $1/\beta$-Lipschitz continuous monotone mapping. From Lemma 2.2, we know that $M_k + G_k$ is maximal monotone. Let $(v, g) \in G(M_k + G_k)$; that is, $g - G_k v \in M_k(v)$. On the other hand, since $\Upsilon_{n_i}^k u_{n_i} = J_{M_k, \lambda_{k, n_i}}(\Upsilon_{n_i}^{k-1} u_{n_i} - \lambda_{k, n_i} G_k \Upsilon_{n_i}^{k-1} u_{n_i})$, we have

$$\Upsilon_{n_i}^k u_{n_i} - \lambda_{k, n_i} G_k \Upsilon_{n_i}^k u_{n_i} \in (I + \lambda_{k, n_i} M_k)\left(\Upsilon_{n_i}^k u_{n_i}\right); \tag{3.80}$$

that is,

$$\frac{1}{\lambda_{k, n_i}}\left(\Upsilon_{n_i}^{k-1} u_{n_i} - \Upsilon_{n_i}^k u_{n_i} - \lambda_{k, n_i} G_k \Upsilon_{n_i}^{k-1} u_{n_i}\right) \in M_k\left(\Upsilon_{n_i}^k u_{n_i}\right). \tag{3.81}$$

By virtue of the maximal monotonicity of $M_k + G_k$, we have

$$\left\langle v - \Upsilon_{n_i}^k u_{n_i}, g - G_k v - \frac{1}{\lambda_{k, n_i}}\left(\Upsilon_{n_i}^{k-1} u_{n_i} - \Upsilon_{n_i}^k u_{n_i} - \lambda_{k, n_i} G_k \Upsilon_{n_i}^{k-1} u_{n_i}\right)\right\rangle \geq 0, \tag{3.82}$$

and so

$$
\begin{aligned}
\left\langle v - \Upsilon_{n_i}^k u_{n_i}, g \right\rangle &\geq \left\langle v - \Upsilon_{n_i}^k u_{n_i}, G_k v + \frac{1}{\lambda_{k, n_i}}\left(\Upsilon_{n_i}^{k-1} u_{n_i} - \Upsilon_{n_i}^k u_{n_i} - \lambda_{k, n_i} G_k \Upsilon_{n_i}^{k-1} u_{n_i}\right)\right\rangle \\
&= \left\langle v - \Upsilon_{n_i}^k u_{n_i}, G_k v - G_k \Upsilon_{n_i}^k u_{n_i} + G_k \Upsilon_{n_i}^k u_{n_i} - G_k \Upsilon_{n_i}^{k-1} u_{n_i} \right. \\
&\qquad \left. + \frac{1}{\lambda_{k, n_i}}\left(\Upsilon_{n_i}^{k-1} u_{n_i} - \Upsilon_{n_i}^k u_{n_i}\right)\right\rangle \\
&\geq 0 + \left\langle v - \Upsilon_{n_i}^k u_{n_i}, G_k \Upsilon_{n_i}^k u_{n_i} - G_k \Upsilon_{n_i}^{k-1} u_{n_i} \right\rangle \\
&\quad + \left\langle v - \Upsilon_{n_i}^k u_{n_i}, \frac{1}{\lambda_{k, n_i}}\left(\Upsilon_{n_i}^{k-1} u_{n_i} - \Upsilon_{n_i}^k u_{n_i}\right)\right\rangle.
\end{aligned} \tag{3.83}
$$

From $\|\Upsilon_n^k u_n - \Upsilon_n^{k+1} u_n\| \to 0$, we also obtain that $\Upsilon_{n_i}^k u_{n_i} \rightharpoonup z$ and $\{G_k : k = 1, 2, 3, \ldots, N\}$ are Lipschitz continuous; we have

$$\lim_{n \to \infty}\left\langle v - \Upsilon_{n_i}^k u_{n_i}, g \right\rangle = \left\langle v - z, g \right\rangle \geq 0. \tag{3.84}$$

Since $M_k + G_k$ is maximal monotone, we have $\theta \in (M_k + G_k)(z)$; that is, $z \in \bigcap_{k=1}^N I(G_k, M_k)$.

Finally, we will show that $z \in EP(F)$.

Since $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.85}$$

If follows from (A2) that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq -F(u_n, y) \geq F(y, u_n), \tag{3.86}$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, u_{n_i}). \tag{3.87}$$

Since $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$ and $u_{n_i} \rightharpoonup z$, it follows by (A4) that $F(y, z) \leq 0$ for all $y \in H$. For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1-t)z$. Since $y \in H$ and $z \in H$, we have $y_t \in H$, and hence $F(y_t, z) \leq 0$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, z) \leq tF(y_t, y), \tag{3.88}$$

and hence $F(y_t, y) \geq 0$. From (A3), we have $F(z, y) \geq 0$ for all $y \in H$ and hence $z \in \mathrm{EP}(F)$. Therefore, it follows that $z \in \Omega$.

Since $x^* = P_\Omega f(x^*)$, we have

$$
\begin{aligned}
\limsup_{n \to \infty} \langle f(x^*) - x^*, x_n - x^* \rangle &= \limsup_{n \to \infty} \langle f(x^*) - x^*, Sv_n - x^* \rangle \\
&= \lim_{i \to \infty} \langle f(x^*) - x^*, Sv_{n_i} - x^* \rangle \\
&= \langle f(x^*) - x^*, z - x^* \rangle \leq 0.
\end{aligned}
\tag{3.89}
$$

On the other hand, we have

$$
\begin{aligned}
\limsup_{n \to \infty} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle &\leq \limsup_{n \to \infty} \langle f(x^*) - x^*, x_{n+1} - x_n \rangle \\
&\quad + \limsup_{n \to \infty} \langle f(x^*) - x^*, x_n - x^* \rangle.
\end{aligned}
\tag{3.90}
$$

Since $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and (3.91), we have

$$\limsup_{n \to \infty} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \leq 0. \tag{3.91}$$

Step 5. We claim that $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

Indeed, from (3.2) and (3.12), we obtain

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n v_n - x^*, x_{n+1} - x^* \rangle \\
&= \alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle + \beta_n \langle x_n - x^*, x_{n+1} - x^* \rangle + \gamma_n \langle S_n v_n - x^*, x_{n+1} - x^* \rangle \\
&\leq \frac{1}{2}\beta_n \left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \right) + \frac{1}{2}\gamma_n \left(\|v_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \right) \\
&\quad + \alpha_n \langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\leq \frac{1}{2}(1 - \alpha_n) \left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \right) + \frac{1}{2}\alpha_n \left(\|f(x_n) - f(x^*)\|^2 + \|x_{n+1} - x^*\|^2 \right) \\
&\quad + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\leq \frac{1}{2}\left[1 - \alpha_n \left(1 - \psi^2 \right) \right] \|x_n - x^*\|^2 + \frac{1}{2}(1 - \alpha_n) \|x_{n+1} - x^*\|^2 + \frac{1}{2}\alpha_n \|x_{n+1} - x^*\|^2 \\
&\quad + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle,
\end{aligned}
\tag{3.92}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \left[1 - \alpha_n \left(1 - \psi^2 \right) \right] \|x_n - x^*\|^2 + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&= (1 - b_n)\|x_n - x^*\|^2 + \delta_n,
\end{aligned}
\tag{3.93}
$$

where $b_n = \alpha_n(1 - \psi^2)$ and $\delta_n = 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle$. It is easy to see that $b_n \to 0$, $\sum_{n=1}^{\infty} b_n = \infty$, and $\limsup_{n\to\infty} \delta_n / b_n \leq 0$. Applying Lemma 2.7 to (3.93), we conclude that

$$
x_n \to x^* = P_\Omega f(x^*).
\tag{3.94}
$$

Consequently, also $\{y_n\}$ converges strongly to x^*. The proof is now complete. □

As in [21, Theorem 4.1], we can generate a sequence $\{S_n\}$ of nonexpansive mappings satisfying condition $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty$ for any bounded subset K of C by using convex combination of general sequence $\{T_k\}$ of nonexpansive mappings with a common fixed point.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $B : C \to H$ be relaxed (ϕ, ω)-cocoercive and μ-Lipschitz continuous with $\omega > \phi\mu^2$, for some $\phi, \omega, \mu > 0$. Let $\mathcal{G} = \{G_k : k = 1, 2, 3, \ldots, N\}$ be a finite family of β-inverse strongly monotone mappings from C into H, and let F be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4). Let $f : C \to C$ be a contraction with coefficient $\psi (0 \leq \psi < 1)$, and let $\{\delta_n^k\}$ be a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ such that*

$$
\Omega : F\left(\bigcap_{k=1}^{\infty} F(T_k) \right) \cap \left(\bigcap_{k=1}^{N} I(G_k, M_k) \right) \cap \mathrm{VI}(C, B) \cap \mathrm{EP}(F) \neq \emptyset.
\tag{3.95}
$$

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$
\begin{aligned}
x_1 &= x \in C \text{ chosen arbitrarily,} \\
y_n &= J_{M_N,\lambda_{N,n}}(I - \lambda_{N,n}G_N) \cdots J_{M_2,\lambda_{2,n}}(I - \lambda_{2,n}G_2)J_{M_1,\lambda_{1,n}}(I - \lambda_{1,n}G_1)T_{r_n}x_n, \\
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{k=1}^{n} \delta_n^k T_k P_C(y_n - \xi_n By_n), \quad \forall n \geq 1,
\end{aligned}
\tag{3.96}
$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $\{\xi_n\}, \{r_n\} \subset (0,\infty)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C4) $\{\xi_n\} \subset [a,b]$ *for some* a,b *with* $0 \leq a \leq b \leq 2(\omega - \phi\mu^2)/\mu^2$ *and* $\lim_{n\to\infty}|\xi_{n+1} - \xi_n| = 0$,

(C5) $\{\lambda_{k,n}\}_{k=1}^{N} \subset [c,d] \subset (0,2\beta)$ *and* $\lim_{n\to\infty}|\lambda_{k,n+1} - \lambda_{k,n}| = 0$, *for each* $k \in \{1,2,\ldots,N\}$,

(C6) $\liminf_{n\to\infty}r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$,

(C7) $\sum_{k=1}^{n}\delta_n^k$, *for all* $n \in \mathbb{N}$, $\lim_{n\to\infty}\delta_n^k > 0$, *for all* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{k}\sum_{k=1}^{n}|\delta_{n+1}^k - \delta_n^k| < \infty$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $x^* \in \Omega$, where $x^* = P_\Omega f(x^*)$.

In Theorem 3.1, taking $N = 1$ and $S_n = S$, then we have the following corollary.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $B : C \to H$ be relaxed (ϕ,ω)-cocoercive and μ-Lipschitz continuous with $\omega > \phi\mu^2$, for some $\phi,\omega,\mu > 0$. Let G be an β-inverse strongly monotone mappings from C into H, and let F be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4). Let $f : C \to C$ be a contraction with coefficient ψ $(0 \leq \psi < 1)$, and let S be a nonexpansive mappings of C into itself such that*

$$
\Omega : F(S) \cap I(G,M) \cap VI(C,B) \cap EP(F) \neq \emptyset.
\tag{3.97}
$$

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$
\begin{aligned}
x_1 &= x \in C \text{ chosen arbitrarily,} \\
F(u_n,y) &+ \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
y_n &= J_{M,\lambda_n}(I - \lambda_n G)u_n, \\
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n S P_C(y_n - \xi_n By_n), \quad \forall n \geq 1,
\end{aligned}
\tag{3.98}
$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $\{\xi_n\}, \{r_n\} \subset (0,\infty)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C4) $\{\xi_n\} \subset [a,b]$ *for some* a,b *with* $0 \leq a \leq b \leq 2(\omega - \phi\mu^2)/\mu^2$ *and* $\lim_{n\to\infty}|\xi_{n+1} - \xi_n| = 0$,

(C5) $\{\lambda_n\} \subset [c,d] \subset (0,2\beta)$ *and* $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$,

(C6) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then, the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* $x^* \in \Omega$, *where* $x^* = P_\Omega f(x^*)$.

Acknowledgments

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