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Research Article

On the q-Genocchi Numbers and Polynomials with Weight α and Weak Weight β

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We construct a new type of q-Genocchi numbers and polynomials with weight α and weak weight $\beta : G_{n,q}^{(\alpha,\beta)}, G_{n,q}^{(\alpha,\beta)}(x)$, respectively. Some interesting results and relationships are obtained.

1. Introduction

The Genocchi numbers and polynomials possess many interesting properties and are arising in many areas of mathematics and physics. Recently, many mathematicians have studied in the area of the q-Genocchi numbers and polynomials (see [1–13]). In this paper, we construct a new type of q-Genocchi numbers $G_{n,q}^{(\alpha,\beta)}$ and polynomials $G_{n,q}^{(\alpha,\beta)}(x)$ with weight α and weak weight β.

Throughout this paper, we use the following notations. By $\mathbb{Z}_p$, we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assumes that $|q-1|_p < p^{-(1/p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. Throughout this paper, we use the notation

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}, \tag{1.1}$$

cf. [1–13].

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. For

$$f \in UD(\mathbb{Z}_p) = \{f \mid f : \mathbb{Z}_p \longrightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}, \tag{1.2}$$

the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \tag{1.3}$$

cf. [3–6].

If we take $f_1(x) = f(x+1)$ in (1.1), then we easily see that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{1.4}$$

From (1.4), we obtain

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \tag{1.5}$$

where $f_n(x) = f(x+n)$ (cf. [3–6]).

As-well-known definition, the Genocchi polynomials are defined by

$$\begin{aligned} F(t) &= \frac{2t}{e^t+1} = e^{Gt} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \\ F(t,x) &= \frac{2t}{e^t+1} e^{xt} = e^{G(x)t} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \end{aligned} \tag{1.6}$$

with the usual convention of replacing $G^n(x)$ by $G_n(x)$. In the special case, $x = 0$, $G_n(0) = G_n$ are called the n-th Genocchi numbers (cf. [1–11]).

These numbers and polynomials are interpolated by the Genocchi zeta function and Hurwitz-type Genocchi zeta function, respectively.

$$\begin{aligned} \zeta_G(s) &= 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \\ \zeta_G(s,x) &= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}. \end{aligned} \tag{1.7}$$

Our aim in this paper is to define q-Genocchi numbers $G_{n,q}^{(\alpha,\beta)}$ and polynomials $G_{n,q}^{(\alpha,\beta)}(x)$ with weight α and weak weight β. We investigate some properties which are related to q-Genocchi numbers $G_{n,q}^{(\alpha,\beta)}$ and polynomials $G_{n,q}^{(\alpha,\beta)}(x)$ with weight α and weak weight β. We also derive the existence of a specific interpolation function which interpolates q-Genocchi numbers $G_{n,q}^{(\alpha,\beta)}$ and polynomials $G_{n,q}^{(\alpha,\beta)}(x)$ with weight α and weak weight β at negative integers.

2. q-Genocchi Numbers and Polynomials with Weight α and Weak Weight β

Our primary goal of this section is to define q-Genocchi numbers $G_{n,q}^{(\alpha,\beta)}$ and polynomials $G_{n,q}^{(\alpha,\beta)}(x)$ with weight α and weak weight β. We also find generating functions of q-Genocchi numbers $G_{n,q}^{(\alpha,\beta)}$ and polynomials $G_{n,q}^{(\alpha,\beta)}(x)$ with weight α and weak weight β.

For $\alpha \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, q-Genocchi numbers $G_{n,q}^{(\alpha,\beta)}$ are defined by

$$G_{n,q}^{(\alpha,\beta)} = n\int_{\mathbb{Z}_p} [x]_{q^\alpha}^{n-1} d\mu_{-q^\beta}(x). \tag{2.1}$$

By using p-adic q-integral on $\mathbb{Z}_p$, we obtain

$$\begin{aligned} n\int_{\mathbb{Z}_p} [x]_{q^\alpha}^{n-1} d\mu_{-q^\beta}(x) &= n \lim_{N\to\infty} \frac{1}{[p^N]_{-q^\beta}} \sum_{x=0}^{p^N-1} [x]_{q^\alpha}^{n-1} \left(-q^\beta\right)^x \\ &= n[2]_{q^\beta} \left(\frac{1}{1-q^\alpha}\right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \frac{1}{1+q^{\alpha l+\beta}} \\ &= n[2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m q^{\beta m} [m]_{q^\alpha}^{n-1}. \end{aligned} \tag{2.2}$$

By (2.1), we have

$$G_{n,q}^{(\alpha,\beta)} = n[2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m q^{\beta m} [m]_{q^\alpha}^{n-1}. \tag{2.3}$$

From the above, we can easily obtain that

$$\begin{aligned} F_q^{(\alpha,\beta)}(t) &= \sum_{n=0}^{\infty} G_{n,q}^{(\alpha,\beta)} \frac{t^n}{n!} \\ &= t[2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m q^{\beta m} e^{[m]_{q^\alpha} t}. \end{aligned} \tag{2.4}$$

Thus, q-Genocchi numbers $G_{n,q}^{(\alpha,\beta)}$ with weight α and weak weight β are defined by means of the generating function

$$F_q^{(\alpha,\beta)}(t) = t[2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m q^{\beta m} e^{[m]_{q^\alpha} t}. \tag{2.5}$$

Using similar method as above, we introduce q-Genocchi polynomials $G_{n,q}^{(\alpha,\beta)}(x)$ with weight α and weak weight β.

$G_{n,q}^{(\alpha,\beta)}(x)$ are defined by

$$G_{n,q}^{(\alpha,\beta)}(x) = n\int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^{n-1} d\mu_{-q^\beta}(y). \tag{2.6}$$

By using p-adic q-integral, we have

$$G_{n,q}^{(\alpha,\beta)}(x) = n[2]_{q^\beta}\left(\frac{1}{1-q^\alpha}\right)^{n-1}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l q^{\alpha x l}\frac{1}{1+q^{\alpha l+\beta}}. \tag{2.7}$$

By using (2.6) and (2.7), we obtain

$$F_q^{(\alpha,\beta)}(t,x) = \sum_{n=0}^{\infty} G_{n,q}^{(\alpha,\beta)}(x)\frac{t^n}{n!} = t[2]_{q^\beta}\sum_{m=0}^{\infty}(-1)^m q^{\beta m} e^{[m+x]_{q^\alpha}t}. \tag{2.8}$$

Remark 2.1. In (2.8), we simply see that

$$\begin{aligned}\lim_{q\to 1} F_q^{(\alpha,\beta)}(t,x) &= 2t\sum_{m=0}^{\infty}(-1)^m e^{(m+x)t} \\ &= \frac{2t}{1+e^t}e^{xt} \\ &= F(t,x).\end{aligned} \tag{2.9}$$

Since $[x+y]_{q^\alpha} = [x]_{q^\alpha} + q^{\alpha x}[y]_{q^\alpha}$, we easily obtain that

$$\begin{aligned}G_{n+1,q}^{(\alpha,\beta)}(x) &= (n+1)\int_{\mathbb{Z}_p}[x+y]_{q^\alpha}^n d\mu_{-q^\beta}(y) \\ &= q^{-\alpha x}\sum_{k=0}^{n+1}\binom{n+1}{k}[x]_{q^\alpha}^{n+1-k} q^{\alpha x k} G_{k,q}^{(\alpha,\beta)} \\ &= q^{-\alpha x}\left([x]_{q^\alpha} + q^{\alpha x} G_q^{(\alpha,\beta)}\right)^{n+1} \\ &= (n+1)[2]_{q^\beta}\sum_{m=0}^{\infty}(-1)^m q^{\beta m}[x+m]_{q^\alpha}^n.\end{aligned} \tag{2.10}$$

Observe that, if $q \to 1$, then $G_{n,q}^{(\alpha,\beta)} \to G_n$ and $G_{n,q}^{(\alpha,\beta)}(x) \to G_n(x)$.

By (2.7), we have the following complement relation.

Theorem 2.2. *Property of complement*

$$G_{n,q^{-1}}^{(\alpha,\beta)}(1-x) = (-1)^{n-1} q^{\alpha(n-1)} G_{n,q}^{(\alpha,\beta)}(x). \tag{2.11}$$

By (2.7), we have the following distribution relation.

Theorem 2.3. *For any positive integer m (=odd), one has*

$$G_{n,q}^{(\alpha,\beta)}(x) = \frac{[2]_{q^\beta}}{[2]_{q^{\beta m}}}[m]_{q^\alpha}^{n-1}\sum_{i=0}^{m-1}(-1)^i q^{\beta i} G_{n,q^m}^{(\alpha,\beta)}\left(\frac{i+x}{m}\right), \quad n \in \mathbb{Z}^+. \tag{2.12}$$

By (1.5), (2.1), *and* (2.6), *one easily sees that*

$$m[2]_{q^\beta}\sum_{l=0}^{n-1}(-1)^{n-1-l}q^{\beta l}[l]_{q^\alpha}^{m-1} = q^{\beta n}G_{m,q}^{(\alpha,\beta)}(n) + (-1)^{n-1}G_{m,q}^{(\alpha,\beta)}. \tag{2.13}$$

Hence, we have the following theorem.

Theorem 2.4. *Let $m \in \mathbb{Z}^+$.*
If $n \equiv 0 \pmod 2$, then

$$q^{\beta n}G_{m,q}^{(\alpha,\beta)}(n) - G_{m,q}^{(\alpha,\beta)} = m[2]_{q^\beta}\sum_{l=0}^{n-1}(-1)^{l+1}q^{\beta l}[l]_{q^\alpha}^{m-1}. \tag{2.14}$$

If $n \equiv 1 \pmod 2$, then

$$q^{\beta n}G_{m,q}^{(\alpha,\beta)}(n) + G_{m,q}^{(\alpha,\beta)} = m[2]_{q^\beta}\sum_{l=0}^{n-1}(-1)^{l}q^{\beta l}[l]_{q^\alpha}^{m-1}. \tag{2.15}$$

From (1.4), *one notes that*

$$\begin{aligned}
[2]_{q^\beta}t &= q^\beta\int_{\mathbb{Z}_p} te^{[x+1]_{q^\alpha}t}d\mu_{-q^\beta}(x) + \int_{\mathbb{Z}_p} te^{[x]_{q^\alpha}t}d\mu_{-q^\beta}(x) \\
&= \sum_{n=0}^{\infty}\left(q^\beta\int_{\mathbb{Z}_p} n[x+1]_{q^\alpha}^{n-1}d\mu_{-q^\beta}(x) + \int_{\mathbb{Z}_p} n[x]_{q^\alpha}^{n-1}d\mu_{-q^\beta}(x)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(q^\beta G_{n,q}^{(\alpha,\beta)}(1) + G_{n,q}^{(\alpha,\beta)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.16}$$

Therefore, we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{Z}^+$, one has*

$$q^\beta G_{n,q}^{(\alpha,\beta)}(1) + G_{n,q}^{(\alpha,\beta)} = \begin{cases} [2]_{q^\beta}, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases} \tag{2.17}$$

By Theorem 2.4 and (2.10), we have the following corollary.

Corollary 2.6. *For $n \in \mathbb{Z}^+$, one has*

$$q^{\beta-\alpha}\left(q^{\alpha}G_q^{(\alpha,\beta)}+1\right)^n + G_{n,q}^{(\alpha,\beta)} = \begin{cases} [2]_{q^{\beta}}, & \text{if } n=1, \\ 0, & \text{if } n \neq 1. \end{cases} \tag{2.18}$$

with the usual convention of replacing $(G_q^{(\alpha,\beta)})^n$ by $G_{n,q}^{(\alpha,\beta)}$.

3. The Analogue of the Genocchi Zeta Function

By using q-Genocchi numbers and polynomials with weight α and weak weight β, q-Genocchi zeta function and Hurwitz q-Genocchi zeta functions are defined. These functions interpolate the q-Genocchi numbers and q-Genocchi polynomials with weight α and weak weight β, respectively. In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. From (2.4), we note that

$$\begin{aligned} \left.\frac{d^{k+1}}{dt^{k+1}} F_q^{(\alpha,\beta)}(t)\right|_{t=0} &= (k+1)[2]_{q^{\beta}} \sum_{m=0}^{\infty} (-1)^m q^{\beta m} [m]_{q^{\alpha}}^k \\ &= G_{k+1,q}^{(\alpha,\beta)}, \quad (k \in \mathbb{N}). \end{aligned} \tag{3.1}$$

By using the above equation, we are now ready to define q-Genocchi zeta functions.

Definition 3.1. Let $s \in \mathbb{C}$. We define

$$\zeta_q^{(\alpha,\beta)}(s) = [2]_{q^{\beta}} \sum_{n=1}^{\infty} \frac{(-1)^n q^{\beta n}}{[n]_{q^{\alpha}}^s}. \tag{3.2}$$

Note that $\zeta_q^{(\alpha,\beta)}(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $q \to 1$, then $\zeta_q^{(\alpha,\beta)}(s) = \zeta(s)$ which is the Genocchi zeta functions. Relation between $\zeta_q^{(\alpha,\beta)}(s)$ and $G_{k,q}^{(\alpha,\beta)}$ is given by the following theorem.

Theorem 3.2. *For $k \in \mathbb{N}$, we have*

$$\zeta_q^{(\alpha,\beta)}(-k) = \frac{G_{k+1,q}^{(\alpha,\beta)}}{k+1}. \tag{3.3}$$

Observe that $\zeta_q^{(\alpha,\beta)}(s)$ *function interpolates* $G_{k,q}^{(\alpha,\beta)}$ *numbers at nonnegative integers. By using* (2.3), *one notes that*

$$\begin{aligned}\left.\frac{d^{k+1}}{dt^{k+1}}F_q^{(\alpha,\beta)}(t,x)\right|_{t=0} &= (k+1)[2]_{q^\beta}\sum_{m=0}^{\infty}(-1)^m q^{\beta m}[x+m]_{q^\alpha}^k \\ &= G_{k+1,q}^{(\alpha,\beta)}(x), \quad (k\in\mathbb{N}),\end{aligned} \tag{3.4}$$

$$\left(\frac{d}{dt}\right)^{k+1}\left.\left(\sum_{n=0}^{\infty}G_{n,q}^{(\alpha,\beta)}(x)\frac{t^n}{n!}\right)\right|_{t=0} = G_{k+1,q}^{(\alpha,\beta)}(x), \quad \text{for } k\in\mathbb{N}. \tag{3.5}$$

By (3.2) and (3.5), we are now ready to define the Hurwitz q-Genocchi zeta functions.

Definition 3.3. Let $s\in\mathbb{C}$. We define

$$\zeta_q^{(\alpha,\beta)}(s,x) = [2]_{q^\beta}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\beta n}}{[n+x]_{q^\alpha}^s}. \tag{3.6}$$

Note that $\zeta_q^{(\alpha,\beta)}(s,x)$ is a meromorphic function on $\mathbb{C}$.

Remark 3.4. It holds that

$$\lim_{q\to 1}\zeta_q^{(\alpha,\beta)}(s,x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+x)^s}. \tag{3.7}$$

Relation between $\zeta_q^{(\alpha)}(s,x)$ and $G_{k,q}^{(\alpha)}(x)$ is given by the following theorem.

Theorem 3.5. *For* $k\in\mathbb{N}$, *one has*

$$\zeta_q^{(\alpha,\beta)}(-k,x) = \frac{G_{k+1,q}^{(\alpha,\beta)}(x)}{k+1}. \tag{3.8}$$

Observe that $\zeta_q^{(\alpha,\beta)}(-k,x)$ *function interpolates* $G_{k,q}^{(\alpha,\beta)}(x)$ *numbers at nonnegative integers.*

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