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Research Article

Asymptotic Stability of a Class of Impulsive Delay Differential Equations

G. L. Zhang, M. H. Song, and M. Z. Liu

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

Correspondence should be addressed to M. H. Song, songmh@lsec.cc.ac.cn

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This paper is concerned with a class of linear impulsive delay differential equations. Asymptotic stability of analytic solutions of this kind of equations is studied by the property of delay differential equations without impulsive perturbations. New numerical methods for this kind of equations are constructed. The convergence and asymptotic stability of the methods for this kind of equations are studied.

1. Introduction

Impulsive differential equations arise widely in the study of medicine, biology, economics, and engineering, and so forth In recent years, theory of impulsive differential delay equations (IDDEs) has been an object of active research (see [1–18] and the reference therein). The results about the existence and uniqueness of IDDEs have been studied in [2, 3, 10]. And the stability of IDDEs have attracted increasing interest in both theoretical research and practical applications (see [1, 3, 4, 6–18] and the reference therein). In this paper, we use the property of delay differential equations without impulsive perturbations to study asymptotic stability of analytic solutions of a class of linear impulsive delay differential equations.

There are a few papers on numerical methods of impulsive differential equations. In [5], Covachev et al. obtained a convergent difference approximation for a nonlinear impulsive ordinary system in a Banach space. In [9, 19], the authors studied the stability of Runge-Kutta methods for linear impulsive ordinary differential equations. In [20], Ding et al. studied the convergence property of an Euler method for IDDEs. In this paper, we construct new numerical methods for a class of linear impulsive delay differential equations. The convergence and asymptotic stability of the methods for this kind of equations are studied.

2. Stability of Analytic Solutions of a Class of Linear IDDEs

In this paper, we consider the following equation:

$$\begin{aligned} x'(t) &= px(t) + qx(t-\tau), \quad t \geq 0,\ t \neq k\tau,\ k = 0,1,2,\ldots, \\ x(t) &= rx(t^-), \quad t = k\tau,\ k = 0,1,2,\ldots, \\ x(t) &= \Phi(t), \quad t \in [-\tau, 0), \end{aligned} \tag{2.1}$$

where $r \neq 0$, $\tau > 0$, p and q are real constants, Φ is a continuous function on $[-\tau, 0)$, and $x'(t)$ denotes the right-hand derivative of $x(t)$.

Definition 2.1. The zero solution of (2.1) is said to be asymptotically stable, if

$$\lim_{t\to\infty} x(t) = 0, \tag{2.2}$$

where $x(t)$ is the solution of (2.1) for any initial function $\Phi \in C([-\tau, 0), R)$.

We define $\Phi(0) = \lim_{t\to 0^-}\Phi(t)$ and the delay differential equations without impulsive perturbations as follows:

$$\begin{aligned} y'(t) &= \left(\frac{\ln r}{\tau} + p\right) y(t) + qy(t-\tau), \quad t \geq 0, \\ y(t) &= \Psi(t) = r^{t/\tau+1}\Phi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{2.3}$$

In the rest of the paper, $\{t\}$ is defined as $\{t\} = t - [t]$ where $[\cdot]$ denotes the greatest integer function.

Theorem 2.2. *Assume that $x(t)$ is the solution of* (2.1) *and $y(t) = r^{\{t/\tau\}}x(t)$, $t \in [-\tau, \infty)$, then $y(t)$ is the solution of* (2.3)*. On the other hand, assume that $y(t)$ is the solution of* (2.3) *and $x(t) = r^{-\{t/\tau\}}x(t)$, $t \in [-\tau, \infty)$, then $x(t)$ is the solution of* (2.1).

Proof. (i) We prove that $y(t)$ is continuous if $x(t)$ is the solution of (2.1). Obviously,

$$y(t) = r^{\{t/\tau\}}x(t) = r^{t/\tau-k}x(t) \tag{2.4}$$

is continuous on $[k\tau, (k+1)\tau)$, $k = -1, 0, 1, \ldots$. We know that $y(t) = \Psi(t) = r^{t/\tau+1}\Phi(t)$, $t \in [-\tau, 0]$ is continuous. Therefore,

$$\begin{aligned} y(0^-) &= \lim_{t\to 0^-} r^{t/\tau+1}x(t) = rx(0^-) = r\Phi(0^-), \\ y(0) &= x(0) = rx(0^-) = r\Phi(0^-). \end{aligned} \tag{2.5}$$

Hence, $y(0) = y(0^-)$. So $y(t)$ is continuous at $t = 0$. It follows from

$$\begin{aligned} y((k+1)\tau) &= r^{\{((k+1)\tau)/\tau\}} x((k+1)\tau) = r^{\{k+1\}} x((k+1)\tau) = x((k+1)\tau) = rx\left((k+1)\tau^-\right), \\ y\left((k+1)\tau^-\right) &= \lim_{t \to (k+1)\tau^-} r^{\{t/\tau\}} x(t) = rx\left((k+1)\tau^-\right), \end{aligned} \tag{2.6}$$

that $y(t)$ is continuous at $t = (k+1)\tau,\ k = 0, 1, 2, \ldots$. Hence $y(t)$ is continuous on $t \in [-\tau, \infty)$.

(ii) We prove that $y(t)$ is the solution of (2.3) if $x(t)$ is the solution of (2.1). For $t \in [k\tau, (k+1)\tau),\ k = 0, 1, \ldots$, we have

$$\begin{aligned} y'(t) &= r^{t/\tau - k} x(t) \frac{\ln r}{\tau} + r^{t/\tau - k} x'(t) \\ &= r^{t/\tau - k} x(t) \frac{\ln r}{\tau} + r^{t/\tau - k} (px(t) + qx(t-\tau)) \\ &= \left(p + \frac{\ln r}{\tau}\right)\left(r^{t/\tau - k} x(t)\right) + q\left(r^{((t-\tau)/\tau) - (k-1)} x(t-\tau)\right) \\ &= \left(p + \frac{\ln r}{\tau}\right) y(t) + qy(t-\tau). \end{aligned} \tag{2.7}$$

Because $y'(t)$ denotes the right-hand derivative of $y(t)$, $y'(t) = (p + \ln r/\tau) y(t) + qy(t-\tau)$, hence $y(t)$ is the solution of (2.3).

(iii) We prove that $x(t)$ is the solution of (2.1) if $y(t)$ is the solution of (2.3). For $t \in [k\tau, (k+1)\tau),\ k = 0, 1, \ldots,$

$$\begin{aligned} x'(t) &= -r^{-(t/\tau - k)} y(t) \frac{\ln r}{\tau} + r^{-(t/\tau - k)} y'(t) \\ &= -r^{-(t/\tau - k)} y(t) \frac{\ln r}{\tau} + r^{-(t/\tau - k)} \left(\left(p + \frac{\ln r}{\tau}\right) y(t) + qy(t-\tau)\right) \\ &= p\left(r^{-(t/\tau - k)} y(t)\right) + q\left(r^{-(t/\tau - k)} y(t-\tau)\right) \\ &= p\left(r^{-(t/\tau - k)} y(t)\right) + q\left(r^{-((t-\tau)/\tau - (k-1))} y(t-\tau)\right) = px(t) + qx(t-\tau). \end{aligned} \tag{2.8}$$

Obviously,

$$\begin{aligned} x((k+1)\tau) &= r^{-\{k+1\}} y((k+1)\tau) = y((k+1)\tau), \\ x\left((k+1)\tau^-\right) &= \lim_{t \to (k+1)\tau^-} r^{-\{t/\tau\}} y(t) = r^{-1} y((k+1)\tau). \end{aligned} \tag{2.9}$$

So $x((k+1)\tau) = rx((k+1)\tau^-),\ k = 0, 1, \ldots$. Consequently, $x(t)$ is the solution of (2.1). □

By Theorem 2.2, we obtain the following corollary.

Corollary 2.3. *Assume that* $\lim_{t\to\infty} x(t) = 0$, *where* $x(t)$ *is the solution of* (2.1). *Then* $\lim_{t\to\infty} y(t) = 0$, *where* $r^{\{t/\tau\}}x(t)$, $t \in [-\tau,\infty)$, *and* $y(t)$ *also is the solution of* (2.3). *On the other hand, assume that* $\lim_{t\to\infty} y(t) = 0$, *where* $y(t)$ *is the solution of* (2.3). *Then* $\lim_{t\to\infty} x(t) = 0$, *where* $x(t) = r^{-\{t/\tau\}}x(t)$, $t \in [-\tau,\infty)$, *and* $x(t)$ *also is the solution of* (2.1).

Lemma 2.4 (see [21]). *Suppose* $\alpha_0 = \max\{\mathrm{Re}\,\lambda : \lambda - (\ln r/\tau + p) - qe^{-\lambda\tau} = 0\}$ *and* $y(t)$ *is the solution of* (2.3). *Then, for any* $\alpha > \alpha_0$, *there is a constant* $K = K(\alpha)$ *such that*

$$\left|y(t)\right| \le Ke^{\alpha t}\|\Psi\|_\tau, \quad t \ge 0, \qquad \|\Psi\|_\tau = \sup_{-\tau\le\theta\le 0}\left|r^{\theta/\tau+1}\Phi(\theta)\right|. \tag{2.10}$$

Theorem 2.5. *Assume that* $|r|\exp((p+|q|)\tau) < 1$. *Then the zero solution of* (2.1) *is asymptotically stable.*

Proof. First we prove that $\alpha_0 < 0$, where

$$\alpha_0 = \max\left\{\Re\lambda : \lambda - \left(\frac{\ln r}{\tau} + p\right) - qe^{-\lambda\tau} = 0\right\}. \tag{2.11}$$

Suppose that $\alpha_0 \ge 0$. Then there exist a λ_0 such that $\Re\lambda_0 \ge 0$ and $\lambda_0 - (\ln r/\tau + p) - qe^{-\lambda_0\tau} = 0$. So

$$\Re\lambda_0 \le \left(\frac{\ln|r|}{\tau} + p\right) + \left|q\right|e^{-\mathrm{Re}\,\lambda_0\tau} \le \frac{\ln|r|}{\tau} + p + \left|q\right|. \tag{2.12}$$

And $|r|\exp((p+|q|)\tau) < 1$ implies that

$$\frac{\ln|r|}{\tau} + p + \left|q\right| < 0. \tag{2.13}$$

Hence (2.12) and (2.13) imply that $\Re\lambda_0 < 0$, which is a contradiction.

Consequently, $\alpha_0 < 0$. By Lemma 2.4, we know that $\lim_{t\to\infty} y(t) = 0$, where $y(t)$ the solution of (2.3). By Corollary 2.3, we know that $\lim_{t\to\infty} x(t) = 0$, where $x(t)$ is the solution of (2.1). □

Corollary 2.6. *Assume that there is a constant* $\lambda > 0$, *such that* $|r| < e^{\lambda\tau}$ *and* $p + |q|e^{\lambda\tau} \le -\lambda$. *Then the zero solution of* (2.1) *is asymptotically stable.*

Proof. Obviously, $p + |q| \le p + |q|e^{\lambda\tau} \le -\lambda < 0$. And because $|r| < e^{\lambda\tau}$, so

$$|r|e^{(p+|q|)\tau} \le |r|e^{-\lambda\tau} < 1. \tag{2.14}$$

By Theorem 2.5, we have that the zero solution of (2.1) is asymptotically stable. □

Similarly, we can get the following corollary.

Corollary 2.7. *Assume that there is a constant* $\lambda > 0$, *such that* $|r| < e^{-\lambda\tau}$ *and* $p + |q||r|^{-1} \le \lambda$. *Then the zero solution of* (2.1) *is asymptotically stable.*

3. Numerical Solutions of IDDEs

We consider the Runge-Kutta methods for (2.3) as follows:

$$y_{n+1} = y_n + h\sum_{i=1}^{v} b_i\left(\left(\frac{\ln r}{\tau} + p\right)Y_n^i + qY_{n-m}^i\right),$$
$$Y_n^i = y_n + h\sum_{j=1}^{v} a_{ij}\left(\left(\frac{\ln r}{\tau} + p\right)Y_n^j + qY_{n-m}^j\right), \tag{3.1}$$

where $h = 1/m$, is a given stepsize with integer $m > 1$. v is referred to as the number of stages. The weights b_i, the abscissas $c_i = \sum_{i=1}^{v} a_{ij}$, and the matrix $A = [a_{i,j}]_{i,j=1}^{v}$ will be denoted by (A, b, c). $y_n = r^{nh/\tau+1}\Phi(nh)$, $n = -m, -m+1, \ldots, 0$, and y_n is an approximation of $y(nh)$, $n = 0, 1, \ldots$.

Define

$$x_n = \Re\left(r^{-\{n/m\}} y_n\right). \tag{3.2}$$

Then x_n is the numerical solution of (2.1).

Theorem 3.1. *Assume that the Runge-Kutta method (A, b, c) is of order p_1. Then the order of* (3.1) *and* (3.2) *for* (2.1) *is also p_1, when $\Psi \in C^{p_1}([-\tau, 0], R)$.*

Proof. In [22], it has been proved that the order of (3.1) is p_1, if the Runge-Kutta method $(A,\ b,\ c)$ is of order p_1. And for any fixed $T > 0$, and any $t_n = nh \le T$, we have $|x(t_n) - x_n| \le \sqrt{(x(t_n) - x_n)^2 + (\mathrm{Im}(r^{-\{n/m\}} y_n))^2} = |x(t_n) - r^{-\{n/m\}} y_n| = |r^{-\{t_n/\tau\}} y(t_n) - r^{-\{n/m\}} y_n| = |r^{-\{n/m\}}||y(t_n) - y_n| \le C|y(t_n) - y_n| = O(h^{p_1})$, where $C = \max\{1, |r|^{-1}\}$. Hence the order of (3.1) and (3.2) is p_1. □

By (3.2), we know that $x_n \to 0$ and $\mathrm{Im}(r^{-\{n/m\}} y_n) \to 0$, $n \to \infty$ if and only if $y_n \to 0$, $n \to \infty$. The definition of P-stability region and P-stable has been given in many papers, for example, [23]. The P-stability region of (3.1) is the set S_p of pairs of numbers (β, γ), $\beta = h(\ln r/\tau + p)$, $\gamma = hq$, such that the numerical solution $\{y_n\}_{n\ge 0}$ satisfies

$$\lim_{n\to\infty} y_n = 0 \tag{3.3}$$

for all constant delays τ and all initial function Ψ.

The Runge-Kutta method (3.1) is P-stable, if

$$S_p \supseteq \left\{(\beta, \gamma) \in C^2 : \mathrm{Re}\,\beta + |\gamma| < 0\right\}. \tag{3.4}$$

Therefore we obtain the following theorem.

Theorem 3.2. *If $|r|\exp((p+|q|)\tau) < 1$, then the solutions of* (2.3) *and* (2.1) *are asymptotically stable. In addition, if the Runge-Kutta method is P-stable, then the process* (3.1) *preserves asymptotically stable of* (2.3)*; consequently, the process* (3.1)-(3.2) *preserves asymptotically stable of* (2.1).

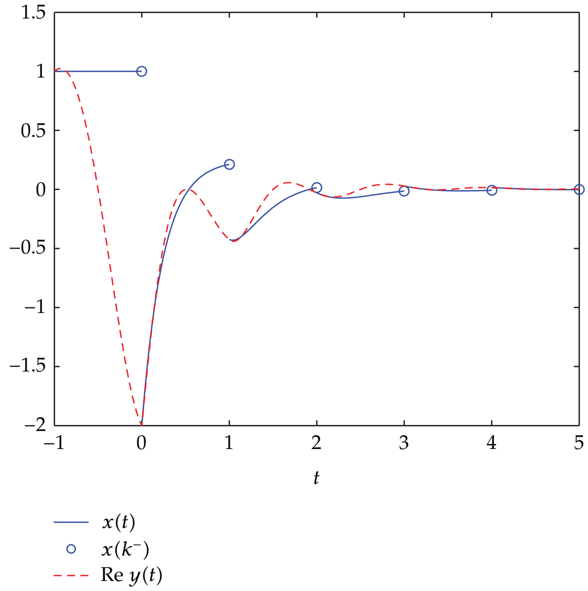

Figure 1: The analytic solution of (4.1) and the real part analytic solution of (4.3).

4. Numerical Experiments

Consider the following IDDE:

$$\begin{aligned} x'(t) &= -4x(t) + x(t-1), \quad t \geq 0,\ t \neq k,\ k = 0,1,2,\dots, \\ x(t) &= -2x(t^-), \quad t = k,\ k = 0,1,2,\dots, \\ x(t) &= \Phi(t), \quad t \in [-1,0). \end{aligned} \tag{4.1}$$

The analytic solution of (4.1) (see Figure 1) as $\Phi(t) \equiv 1,\ t \in [-1,0)$ is

$$x(t) = \begin{cases} 1, & t \in \ [-1,0), \\ -\dfrac{9}{4}e^{-4t} + \dfrac{1}{4}, & t \in \ [0,1), \\ \left(\dfrac{9}{2}e^{-4} - \dfrac{9}{4}t + \dfrac{27}{16}\right)e^{-4(t-1)} + \dfrac{1}{16}, & t \in \ [1,2), \\ -2\left(\left(\dfrac{9}{2}e^{-4} - \dfrac{9}{2} + \dfrac{27}{16}\right)e^{-4} + \dfrac{1}{16}\right), & t = 2. \end{cases} \tag{4.2}$$

Note that $\epsilon_{0.5}$, ϵ_1, $\epsilon_{1.5}$, and ϵ_2 are the errors of the analytic solution and the numerical solution (3.1) (3.2) for (4.1) at $t = 0.5,\ 1,\ 1.5,$ and 2, respectively.

In Tables 1 and 2, we have listed the errors at $t = 0.5, 1.1.5$ and 2 of the explicit Euler method and the Trapezoid method.

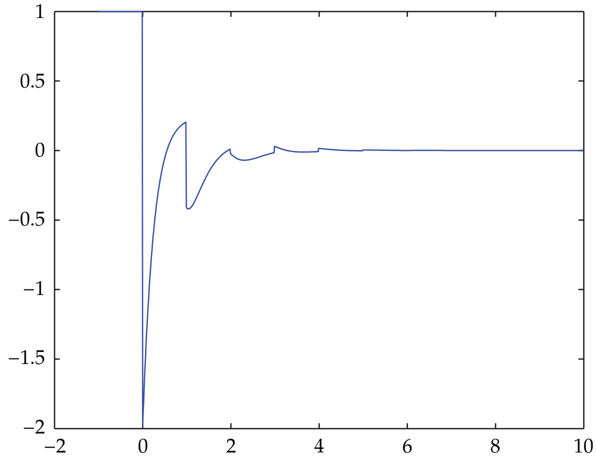

Figure 2: The numerical solution of (4.1), when (3.1) is the Trapezoid method as $h = 1/100$.

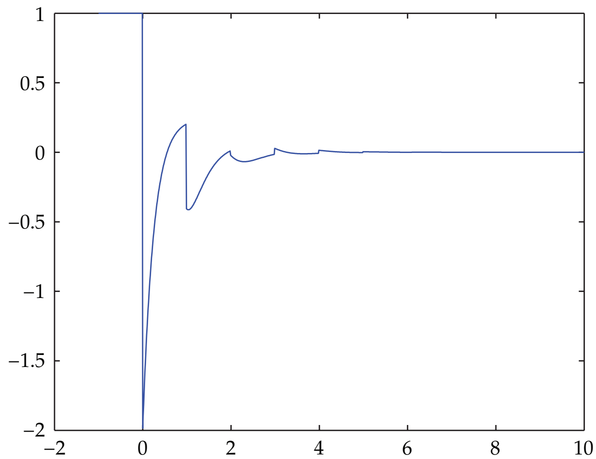

Figure 3: The numerical solution of (4.1), when (3.1) is the implicit Euler method as $h = 1/100$.

Obviously, $|r|e^{(p+|q|)\tau} = 2e^{-4+1} = 2e^{-3} < 1$, so the zero solution of (4.1) is asymptotically stable by Theorem 2.5. Assume that $x(t)$ is the solution of (4.1) and $y(t) = (-2)^{\{t\}}x(t)$, for $t \in [-1, \infty)$. Then $y(t)$ is the solution of the following equation:

$$\begin{aligned} y'(t) &= (\ln(-2) - 4)y(t) + y(t-1), \quad t \geq 0, \ t \neq k, \ k = 0, 1, 2, \ldots, \\ y(t) &= (-2)^{t+1}\Phi(t), \quad t \in [-1, 0). \end{aligned} \tag{4.3}$$

Because $\Re(\ln(-2) - 4) + 1 < 0$, we know that $y(t) \to 0$ as $t \to \infty$ (see Figure 1).

The numerical process (3.1) and (3.2) for (4.1) is asymptotically stable as (3.2) is the Trapezoid method (see Figure 2) and the implicit Euler method (see Figure 3).

Table 1

Stepsize	$\epsilon_{0.5}$	ϵ_1	$\epsilon_{1.5}$	ϵ_2
$h = 1/50$	0.007237306	0.013957231	0.009751641	0.000998634
$h = 1/100$	0.003480570	0.006575988	0.004535880	0.000421508
$h = 1/200$	0.001706846	0.003192398	0.002188540	0.000193107
$h = 1/400$	0.000845200	0.0015728959	0.001075069	0.000092351
Ratio	2.045995126	2.070658147	2.086056611	2.214324691

Table 2

Stepsize	$\epsilon_{0.5}$	ϵ_1	$\epsilon_{1.5}$	ϵ_2
$h = 1/50$	$3.50207e-4$	$2.52492e-4$	$2.21015e-4$	$2.45482e-4$
$h = 1/100$	$8.74678e-5$	$6.30749e-5$	$5.52490e-5$	$6.13071e-5$
$h = 1/200$	$2.18617e-5$	$1.57657e-5$	$1.38119e-5$	$1.53228e-5$
$h = 1/400$	$5.46510e-6$	$3.94124e-6$	$3.45297e-6$	$3.83045e-6$
Ratio	4.00167	4.00133	4.00015	4.00181

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