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Research Article

On Some Pursuit and Evasion Differential Game Problems for an Infinite Number of First-Order Differential Equations

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We study pursuit and evasion differential game problems described by infinite number of first-order differential equations with function coefficients in Hilbert space l_2. Problems involving integral, geometric, and mix constraints to the control functions of the players are considered. In each case, we give sufficient conditions for completion of pursuit and for which evasion is possible. Consequently, strategy of the pursuer and control function of the evader are constructed in an explicit form for every problem considered.

1. Introduction

The books by Friedman [1], Isaacs [2], Krasovskiĭ and Subbotin [3], Lewin [4], Petrosyan [5], and Pontryagin [6] among others are fundamental to the study of differential games.

Many works are devoted to differential game problems described by both ordinary differential equations in R^n and partial differential equations. In particular, pursuit and evasion differential game problems involving distributed parameter systems are of increasing interest (see, e.g., [7–14]).

Satimov and Tukhtasinov [10, 11] studied pursuit and evasion problems described by the parabolic equation

$$z_t - Az = -u + v, \qquad z|_{t=0} = z_0(x), \qquad z|_{S_T} = 0, \tag{1.1}$$

where $z = z(t,x)$ is unknown function; $x = (x_1, x_2, \dots, x_n) \in \Omega \subset R^n,\ n \geq 1$ is parameter in a bounded domain Ω; $t \in [0,T],\ T > 0$; $u = u(t,x),\ v = v(t,x)$ are control functions of the players; $S_T = \{(x,t) | x \in \partial\Omega, t \in (0,T)\}$, A is a differential operator defined in the space $L_2(\Omega)$. In this problem, the authors used the fact in [15] that under certain conditions, the problem (1.1) has a unique generalized (in the sense of distribution) solution of the form

$$z(t,x) = \sum_{k=1}^{\infty} z_k(t)\psi_k(x), \tag{1.2}$$

where the functions $z_k(t)$, $0 \le t \le T$, $k = 1,2,\ldots$, constitute the solution of the Cauchy problem for the following infinite system of differential equations and initial conditions:

$$\dot{z}_k + \lambda_k z_k = -u_k(t) + v_k(t), \quad z_k(0) = z_{0k}, \quad k = 1,2,\ldots, \tag{1.3}$$

and λ_k, $k = 1,2,\ldots$, satisfying the condition that $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$, are eigenvalues of the operator A, the functions $\psi_k(x)$, $k = 1,2,\ldots$, constitute an orthonormal and complete system of eigenfunctions of the operator A, $u_k(t), v_k(t)$, and z_{0k} are the Fourier coefficients in the expansion of $u(t,x), v(t,x)$, and $z_0(x)$, respectively, in the system $\{\psi_k(x)\}$.

The work above shows the significant relationship between differential game problems described by (1.1) in one side and those described by (1.3) in the other side. Therefore, it is logical to study the latter in an independent frame work (see, e.g., [16, 17]).

In the present paper, we solve pursuit and evasion problems described by system (1.3), with function coefficients λ_k, $k = 1,2,\ldots$ instead of constants coefficients considered in the previous study. Different forms of constraints on the controls of the players are considered.

2. Statement of the Problem

Let

$$l_2 = \left\{ \alpha = (\alpha_1, \alpha_2, \ldots) : \sum_{k=1}^{\infty} \alpha_k^2 < \infty \right\}, \tag{2.1}$$

with inner product and norm

$$\begin{gathered} \langle \alpha, \beta \rangle = \sum_{k=1}^{\infty} \alpha_k \beta_k, \alpha, \beta \in l_2, \quad \|\alpha\| = \left(\sum_{k=1}^{\infty} \alpha_k^2 \right)^{1/2}, \\ \|w(\cdot)\|_{L_2(0,T,l_2)} = \left(\sum_{k=1}^{\infty} \int_0^T w_k^2(s)\,ds \right)^{1/2}. \end{gathered} \tag{2.2}$$

Let

$$L_2(0,T,l_2) = \left\{ w(t) = (w_1(t), w_2(t), \ldots) : \|w(\cdot)\|_{L_2(0,T,l_2)} < \infty, w_k(\cdot) \in L_2(0,T) \right\}, \tag{2.3}$$

where $T, T > 0$, is a given number.

We examine a pursuit and evasion differential game problems described by the following infinite system of differential equations

$$\dot{z}_k(t) + \lambda_k(t) z_k(t) = -u_k(t) + v_k(t), \quad z_k(0) = z_{k0}, \quad k = 1, 2, \ldots, \tag{2.4}$$

where $z_k, u_k, v_k \in R^1$, $k = 1, 2, \ldots$, $z_0 = (z_{10}, z_{20}, \ldots) \in l_2, u_k, v_k, \ldots$, are control parameters of pursuer and evader respectively, $\lambda_k(t)$, $k = 1, 2, \ldots$, are bounded, non-negative continuous functions on the interval $[0, T]$ such that $\lambda_k(0) = 0$, $k = 1, 2, \ldots$.

Definition 2.1. A function $w(\cdot), w : [0, T] \to l_2$, with measurable coordinates $w_k(t)$, $0 \le t \le T$, $k = 1, 2, \ldots$, subject to

$$\sum_{k=1}^{\infty} \int_0^T w_k^2(s) ds \le \rho^2 \left(\sum_{k=1}^{\infty} w_k^2(t) \le \rho^2, t \in [0, T] \right), \tag{2.5}$$

where ρ is a positive number, is referred to as an admissible control subject to integral constraint (resp., geometric constraint).

We denote the set of all admissible controls with respect to integral constraint by $S_1(\rho)$ and with respect to geometric constraint by $S_2(\rho)$.

The control $u(\cdot) = (u_1(\cdot), u_2(\cdot), \ldots)$ of the pursuer and $v(\cdot) = (v_1(\cdot), v_2(\cdot), \ldots)$ of the evader are said to be admissible if they satisfy one of the following conditions

$$\left(\sum_{k=1}^{\infty} \int_0^T u_k^2(s) ds \right)^{1/2} \le \rho, \qquad \left(\sum_{k=1}^{\infty} \int_0^T v_k^2(s) ds \right)^{1/2} \le \sigma, \tag{2.6}$$

$$\left(\sum_{k=1}^{\infty} u_k^2(t) \right)^{1/2} \le \rho, \ t \in [0, T], \qquad \left(\sum_{k=1}^{\infty} v_k^2(t) \right)^{1/2} \le \sigma, \ t \in [0, T], \tag{2.7}$$

$$\left(\sum_{k=1}^{\infty} \int_0^T u_k^2(s) ds \right)^{1/2} \le \rho, \qquad \left(\sum_{k=1}^{\infty} v_k^2(t) \right)^{1/2} \le \sigma, \ t \in [0, T], \tag{2.8}$$

$$\left(\sum_{k=1}^{\infty} u_k^2(t) \right)^{1/2} \le \rho, \ t \in [0, T], \qquad \left(\sum_{k=1}^{\infty} \int_0^T v_k^2(s) ds \right)^{1/2} \le \sigma, \tag{2.9}$$

where ρ and σ are positive constants. We will call the system (2.4) in which $u(\cdot)$ and $v(\cdot)$ satisfy inequalities (2.6) (resp., (2.7), (2.8), and (2.9)), game G_1 (resp., G_2, G_3, G_4).

Definition 2.2. A function $z(t) = (z_1(t), z_2(t), \ldots)$, $0 \le t \le T$, is called the solution of the system (2.4) if each coordinate $z_k(t)$

(i) is absolutely continuous and almost everywhere on $[0, T]$ satisfies (2.4),

(ii) $z(\cdot) \in C(0, T; l_2)$.

Definition 2.3. A function

$$U(t,v), \qquad U:[0,T]\times l_2 \longrightarrow l_2 \tag{2.10}$$

is referred to as the strategy of the pursuer with respect to integral constraint if:

(1) for any admissible control of the evader $v = v(t),\ t \in [0,T]$, the system (2.4) has a unique solution at $u = u(t, v_1(t), v_2(t), \ldots)$,

(2) $U(\cdot, v(\cdot)) \in S_1(\rho)$.

In a similar way, we define strategy of the pursuer with respect to geometric constraint.

Definition 2.4. One will say that pursuit can be completed in the game G_1 (resp., G_2, G_3) from an initial position z_0, if there exists a strategy of the pursuer to ensure that $z(t) = 0$ for some $t \in [0,T]$ and for any admissible control of the evader $v(\cdot)$, where $z(t)$ is the solution to (2.4).

Definition 2.5. One will say that pursuit can be completed in the game G_4 from an initial position z_0, if for arbitrary $\varepsilon > 0$, there exists a strategy of the pursuer to ensure that $\|z(t)\| \le \varepsilon$ for some $t \in [0,T]$ and for any admissible control of the evader $v(\cdot)$, where $z(t)$ is the solution to (2.4).

Definition 2.6. One will say that evasion is possible in the game G_1 (resp., G_2, G_3, G_4) from the initial position $z_0 \neq 0$, if there exists a function $v(t) \in S_1(\sigma)$ $(v(t) \in S_2(\sigma), v(t) \in S_2(\sigma), v(t) \in S_1(\sigma))$ such that, for arbitrary function $u_0(t) \in S_1(\rho)$ $(u_0(t) \in S_2(\rho), u_0(t) \in S_1(\rho), u_0(t) \in S_2(\rho))$, the solution $z(t)$ of (2.4) does not vanish, that is, $z(t) \neq 0$ for any $t \in [0,T]$.

The problem is to find

(1) conditions on the initial state z_0 for which pursuit can be completed for a finite time;

(2) conditions for which evasion is possible from any initial position $z_0 \neq 0$ in the differential game G_i, for $i = 1,2,3,4$.

In problems 1 and 2, different forms of constraints on the controls of the players are to be considered.

3. Differential Game Problem

The kth equation in (2.4) has a unique solution of the form

$$z_k(t) = e^{-\alpha_k(t)}\left(z_{k0} - \int_0^t u_k(s)e^{\alpha_k(s)}ds + \int_0^t v_k(s)e^{\alpha_k(s)}ds\right), \tag{3.1}$$

where $\alpha_k(t) = \int_0^t \lambda_k(s)ds$.

It has been proven in [18] that the solution $z(t) = (z_1(t), z_2(t), \ldots)$ of (2.4), where $z_k,\ k = 1,2,\ldots$ defined by (3.1), belongs to the space $C(0,T;l_2)$.

Let

$$\begin{aligned}
\Upsilon &= \left\{ z_0 = (z_{10}, z_{20}, \ldots) \mid \exists k = j : z_{j0}^2 \leq \frac{\rho^2}{4}, A_j(t) \geq 1 \right\}, \\
\Upsilon_1(T) &= \left\{ z_0 = (z_{10}, z_{20}, \ldots) \mid \sum_{k=1}^{\infty} \frac{z_{k0}^2}{A_k(T)} \leq (\rho - \sigma)^2 \right\}, \\
\Upsilon_2(T) &= \left\{ z_0 = (z_{10}, z_{20}, \ldots) \mid \sum_{k=1}^{\infty} \frac{z_{k0}^2}{B_k^2(T)} \leq (\rho - \sigma)^2 \right\}, \\
\Upsilon_3(T) &= \left\{ z_0 = (z_{10}, z_{20}, \ldots) \mid \|z_0\| + \frac{\sigma^2}{\varepsilon} \sup_k A_k(T) \leq \rho T, \varepsilon > 0 \right\},
\end{aligned} \tag{3.2}$$

where $A_k(T) = \int_0^T e^{2\alpha_k(s)} ds$ and $B_k(T) = \int_0^T e^{\alpha_k(s)} ds$.

3.1. Pursuit Differential Game

Theorem 3.1. *If $\rho \geq \sigma$ then from the initial position $z_0 \in \Upsilon_1(T)$, pursuit can be completed in the game G_1.*

Proof. Let define the pursuer's strategy as

$$u_k(t) = \begin{cases} z_{k0} \mathrm{A}_k^{-1}(T) e^{\alpha_k(t)} + v_k(t), & 0 \leq t \leq T, \\ 0, & t > T. \end{cases} \tag{3.3}$$

The admissibility of this strategy follows from the relations

$$\begin{aligned}
\left(\sum_{k=1}^{\infty} \int_0^T u_k^2(s) ds \right)^{1/2} &= \left(\sum_{k=1}^{\infty} \int_0^T \left| z_{k0} A_k^{-1}(T) e^{\alpha_k(s)} + v_k(s) \right|^2 ds \right)^{1/2} \\
&\leq \left(\sum_{k=1}^{\infty} \int_0^T \left(\left| z_{k0} A_k^{-1}(T) e^{\alpha_k(s)} \right| + |v_k(s)| \right)^2 ds \right)^{1/2} \\
&\leq \left(\sum_{k=1}^{\infty} \int_0^T \left| z_{k0} A_k^{-1}(T) e^{\alpha_k(s)} \right|^2 ds \right)^{1/2} + \left(\sum_{k=1}^{\infty} \int_0^T |v_k(s)|^2 ds \right)^{1/2} \\
&\leq \left(\sum_{k=1}^{\infty} z_{k0}^2 A_k^{-1}(T) \right)^{1/2} + \sigma \\
&= \rho - \sigma + \sigma = \rho,
\end{aligned} \tag{3.4}$$

here we used the Minkowski inequality and the fact that $z_0 \in \Upsilon_1(T)$.

Suppose that the pursuer uses the strategy (3.3), one can easily see that for any admissible control of the evader $z_k(T) = 0,\ k = 1, 2, \ldots$, that is,

$$\begin{aligned} z_k(T) &= e^{-\alpha_k(T)}\left(z_{k0} - \int_0^T z_{k0} A_k^{-1}(T) e^{2\alpha_k(s)} ds \right) \\ &= e^{-\alpha_k(T)}(z_{k0} - z_{k0}) = 0. \end{aligned} \tag{3.5}$$

Therefore, pursuit can be completed in the game G_1. This ends the proof of the theorem. □

Theorem 3.2. *If $\rho \geq \sigma$ then from the initial position $z_0 \in Y_2(T)$, pursuit can be completed in the game G_2.*

Proof. We define the pursuer's strategy as

$$u_k(t) = \begin{cases} z_{k0} B_k^{-1}(T) + v_k(t), & 0 \leq t \leq T, \\ 0, & t > T. \end{cases} \tag{3.6}$$

The inclusion $u(\cdot) \in S_2(\rho)$ follows from the relations

$$\begin{aligned} \left(\sum_{k=1}^{\infty} u_k^2(t) \right)^{1/2} &= \left(\sum_{k=1}^{\infty} \left| z_{k0} B_k^{-1}(T) + v_k(t) \right|^2 \right)^{1/2} \\ &\leq \left(\sum_{k=1}^{\infty} \left(\left| z_{k0} B_k^{-1}(T) \right| + |v_k(t)| \right)^2 \right)^{1/2} \\ &\leq \left(\sum_{k=1}^{\infty} \left| z_{k0} B_k^{-1}(T) \right|^2 \right)^{1/2} + \left(\sum_{k=1}^{\infty} |v_k(t)|^2 \right)^{1/2} \\ &= \rho - \sigma + \sigma = \rho, \end{aligned} \tag{3.7}$$

here we used the Minkowski inequality and the fact that $z_0 \in Y_2(T)$.

Suppose that the pursuer uses the strategy (3.6). One can easily see that $z_k(T) = 0,\ k = 1, 2, \ldots$, that is,

$$\begin{aligned} z_k(T) &= e^{-\alpha_k(T)}\left(z_{k0} - \int_0^T z_{k0} B_k^{-1}(T) e^{\alpha_k(s)} ds \right) \\ &= e^{-\alpha_k(T)}(z_{k0} - z_{k0}) = 0. \end{aligned} \tag{3.8}$$

Therefore, pursuit can be completed in the game G_2. This completes the proof of the theorem. □

Theorem 3.3. *If $\rho \geq \sigma$ and $z_0 \in Y_1(T)$ at some $T \in (0, T]$, then pursuit can be completed in the game G_3.*

Proof. Suppose, as contained in the hypothesis of the theorem, that $z_0 \in Y_1(T)$, $T \in (0,1]$ and let $v_0(t)$ be an arbitrary admissible control of the evader.

Let the pursuer use the strategy $u(t) = (u_1(t), u_2(t), \ldots)$ defined by

$$u_k(t) = \begin{cases} z_{k0} A_k^{-1}(T) e^{\alpha_k(t)} + v_{0k}(t), & 0 \le t \le T, \\ 0, & t > T. \end{cases} \tag{3.9}$$

Then, using (3.1), we have

$$\begin{aligned} z(T) &= e^{-\alpha_k(T)} \left(z_{k0} - \int_0^T z_{k0} A_k^{-1}(T) e^{2\alpha_k(t)} ds \right) \\ &= e^{-\alpha_k(T)} (z_{k0} - z_{k0}) = 0. \end{aligned} \tag{3.10}$$

We now show the admissibility of the strategy used by the pursuer. From the inclusion $v_0(t) \in S_2(\sigma)$ we can deduce that

$$\left(\sum_{k=1}^{\infty} \int_0^T v_{0k}^2(s) ds \right)^{1/2} \le \sigma \sqrt{T}, \tag{3.11}$$

$$\begin{aligned} \left(\sum_{k=1}^{\infty} \int_0^T u_k^2(s) ds \right)^{1/2} &= \left(\sum_{k=1}^{\infty} \int_0^T \left| z_{k0} A_k^{-1}(T) e^{\alpha_k(s)} + v_{0k}(s) \right|^2 ds \right)^{1/2} \\ &\le \left(\sum_{k=1}^{\infty} \int_0^T \left(\left| z_{k0} A_k^{-1}(T) e^{\alpha_k(s)} \right| + |v_k(s)| \right)^2 ds \right)^{1/2} \\ &\le \left(\sum_{k=1}^{\infty} \int_0^T \left| z_{k0} A_k^{-1}(T) e^{\alpha_k(s)} \right|^2 ds \right)^{1/2} + \left(\sum_{k=1}^{\infty} \int_0^T |v_{0k}(s)|^2 ds \right)^{1/2} \\ &\le \left(\sum_{k=1}^{\infty} z_{k0}^2 A_k^{-1}(T) \right)^{1/2} + \sigma \sqrt{T} \\ &= \rho - \sigma + \sigma \sqrt{T} \le \rho, \end{aligned} \tag{3.12}$$

recall that $T \in (0,1]$ and (3.11). This completes the proof. □

Theorem 3.4. *For arbitrary $\rho > 0$, $\sigma > 0$ and initial position $z_0 \in Y_3(T)$, pursuit can be completed in the game G_4.*

Proof. Let v_0 be an arbitrary admissible control function of the evader. When the pursuer uses the admissible control function

$$u_k(t) = z_{k0} T_1^{-1} e^{-\alpha_k(t)}, \qquad k = 1, 2, \ldots, \ 0 \le t \le T_1, \tag{3.13}$$

for time $T_1 = \|z_0\|\rho^{-1}$, the solution (3.1) of (2.4) becomes

$$z_k(T_1) = e^{-\alpha_k(T_1)} \int_0^{T_1} v_{0k}(s) e^{\alpha_k(s)} ds. \tag{3.14}$$

Then for arbitrary positive number ε, it is obvious that either

(1) $\|z(T_1)\| \le \varepsilon$, or

(2) $\|z(T_1)\| > \varepsilon$.

If (1) is true then the proof is complete. Obviously $T_1 \le T$.

Suppose that (1) is not true then (2) must hold. We now assume that $z_0 = z(T_1)$ and repeat previous argument by setting

$$u_k(t) = z_{k0}(T_1) T_2^{-1} e^{-\alpha_k(t)}, \quad k = 1, 2, \ldots, \ 0 \le t \le T_2, \tag{3.15}$$

with time $T_2 = \|z(T_1)\|\rho^{-1}$ (we will later prove that the sum of T_i is less than or equal to T). For this step the solution (3.1) becomes

$$z_k(T_1 + T_2) = e^{-\alpha_k(T_2)} \int_0^{T_2} v_{0k}(T_1 + s) e^{\alpha_k(s)} ds. \tag{3.16}$$

Yet again, we have either of the following cases holding:

(1) $\|z(T_1 + T_2)\| \le \varepsilon$, or

(2) $\|z(T_1 + T_2)\| > \varepsilon$.

If (1) holds then the game is completed in the time $T_1 + T_2$, else we assume $z_0 = z_0(T_1 + T_2)$ and repeat the process again and so on.

We now proof a claim that the game will be completed before nth finite step, where

$$n = \left\lceil \frac{\sigma^2 \sup_k A_k(T)}{\varepsilon^2} \right\rceil. \tag{3.17}$$

Note that the existence of the supreme of the sequence $A_1(T), A_2(T), \ldots$, follows from the fact that $\lambda_1(t), \lambda_2(t), \ldots$ is a bounded sequence of continuous functions and $t \in [0, T]$.

Suppose that it is possible that the game can continue for nth step. In this case, we must have

$$\sum_{i=1}^{n} \sigma_i^2 \le \sigma^2. \tag{3.18}$$

But in the first instance, we have

$$|z_k(T_1)|^2 \le e^{-2\alpha_k(T_1)} \int_0^{T_1} v_{0k}^2(s) ds \int_0^{T_1} e^{2\alpha_k(s)} ds \le \sup_k A_k(T) \int_0^{T_1} v_{0k}^2(s) ds, \tag{3.19}$$

here we used (3.14) and Cauchy-Schwarz inequality.

Therefore,

$$\|z(T_1)\|^2 \le \sup_k A_k(T) \sum_{k=1}^{\infty} \int_0^{T_1} v_{0k}^2(s)\,ds = \sup_k A_k(T)\sigma_1^2, \tag{3.20}$$

and by using the assumption that $\|z(T_1)\| > \varepsilon$, we have

$$\sigma_1^2 > \frac{\varepsilon^2}{\sup_k A_k(T)}. \tag{3.21}$$

Since the right hand side of this inequality is independent of n, we can conclude that

$$\sigma_n^2 > \frac{\varepsilon^2}{\sup_k A_k(T)}. \tag{3.22}$$

Using this inequality and definition of n, we have

$$\sum_{i=1}^{n} \sigma_i^2 > \frac{n\varepsilon^2}{\sup_k A_k(T)} > \sigma^2, \tag{3.23}$$

contradicting (3.18). Hence, pursuit must be completed for the initial position $z_0 \in Y_3(T)$ before the nth step. Furthermore, the pursuit time is given by $T(z_0) = T_1 + T_2 + \cdots + T_{n-1}$, and the inclusion $T(z_0) \in [0, T]$ is satisfied. Indeed (see (3.20), definition of n and that $z_0 \in Y_3(T)$),

$$\begin{aligned} T(z_0) &= \frac{\|z_0\|}{\rho} + \frac{\|z(T_1)\|}{\rho} + \cdots + \frac{\|z(T_{n-2})\|}{\rho} \\ &\le \frac{1}{\rho}\left(\|z_0\| + \sqrt{\sup_k A_k(T)} \sum_{i=1}^{n-2} \sigma_i \right) \\ &\le \frac{1}{\rho}\left(\|z_0\| + \sigma\sqrt{(n-2)\sup_k A_k(T)} \right) \\ &\le \frac{1}{\rho}\left(\|z_0\| + \frac{\sigma^2}{\varepsilon} \sup_k A_k(T) \right) \le T. \end{aligned} \tag{3.24}$$

This proves the theorem. □

3.2. Evasion Differential Game

Theorem 3.5. *If $\sigma - \rho \ge 0$ then evasion is possible in the game G_1 from the initial position $z_0 \ne 0$.*

Proof. Suppose that

$$\sigma - \rho \ge 0, \tag{3.25}$$

and let $u_0(t)$ be an arbitrary control of the pursuer subjected to integral constraint. We construct the control function of the evader as follows:

$$v_k(t) = \begin{cases} A_j^{-1/2}(T)\rho e^{\alpha_j(t)}, & k = j, \\ 0, & k \neq j. \end{cases} \tag{3.26}$$

This control function belongs to $S_1(\sigma)$. Indeed,

$$\sum_{k=1}^{\infty} \int_0^T v_k^2(s)ds = A_j^{-1}(T)\rho^2 A_j(T) \le \sigma^2 \tag{3.27}$$

we have used (3.26) and (3.25).

Our goal now is to show that $z_j(t) \neq 0$ for any $t \in [0, T]$ as defined by (3.1). Substituting (3.26) into (3.1) and using the Cauchy-Schwartz inequality, we have

$$\begin{aligned} z_j(t) &\ge e^{-\alpha_j(t)}\left(z_{j0} + \rho\sqrt{A_j(t)} - \rho\sqrt{A_j(t)}\right) \\ &= z_{j0}e^{-\alpha_j(t)} > 0, \end{aligned} \tag{3.28}$$

for any $t \in [0, T]$. It follows that $z(t) \neq 0$ on the interval $[0, T]$. Hence, evasion is possible in the game G_1 from the given initial position $z_0 \neq 0$. The proof of the theorem is complete. □

Theorem 3.6. *Suppose that $\sigma \ge \rho$ or there exists a number $k = j$ such that $z_{0j} > 0$ and $\sigma - \rho\sqrt{A_j(T)} \ge 0$. Then from the initial position $z_0 \neq 0$, evasion is possible in the game G_2.*

Proof. Suppose that $\sigma \ge \rho$ and that $z_0 \neq 0$. The later condition means that $z_{k0} \neq 0$ for some $k = j$. We construct the control function of the evader as follows:

$$v_k(t) = \begin{cases} \rho, & k = j, \\ 0, & k \neq j. \end{cases} \tag{3.29}$$

It is obvious that this control belongs to the set $S_2(\sigma)$.

To be definite, let $z_{j0} > 0$. Using (3.29) and (3.1), we have

$$\begin{aligned} z_j(t) &> e^{-\alpha_j(t)}\left(z_{j0} + \rho\int_0^t e^{\alpha_j(s)}ds - \rho\int_0^t e^{\alpha_j(s)}ds\right) \\ &= e^{-\alpha_j(t)}z_{j0} > 0. \end{aligned} \tag{3.30}$$

This means that evasion is possible from the initial position $z_0 \neq 0$ in the game G_2.

We now prove the theorem with the alternative condition. Suppose that there exists a number $k = j$ such that $z_{0j} > 0$ and $\sigma - \rho\sqrt{A_j(T)} \geq 0$. Let the control of the evader be as follows:

$$v_k(t) = \begin{cases} \dfrac{\sqrt{TA_j(T)}}{t+e+T}\rho e^{-\alpha_j(t)}, & k = j, \\ 0, & k \neq j. \end{cases} \tag{3.31}$$

We show that this control satisfies the geometric constraint:

$$\sum_{k=1}^{\infty} v_k^2(t) = \frac{TA_j(T)}{(t+e+T)^2}\rho^2 e^{-2\alpha_j(t)} \leq \rho^2 A_j(T) \leq \sigma^2. \tag{3.32}$$

When the evader uses the control (3.31), the non-vanishing of $z_j(t)$ in the interval $[0,T]$ for any admissible control of the pursuer u_{i0}, can be seen from the following (see (3.1))

$$z_j(t) \geq e^{-\alpha_j(t)}\left(z_{j0} + \rho\sqrt{TA_j(T)}\ln(t+e+T) - \rho\sqrt{TA_j(T)}\right) > 0, \tag{3.33}$$

we use the fact that $\ln(t+e+T) > 1$ for any $t \in [0,T]$.

Therefore, $z(t) \neq 0,\ t \in [0,T]$. This completes the proof of the theorem. □

Theorem 3.7. *If $\sigma - \rho\sqrt{T} \geq 0$ then evasion is possible from the initial position $z_0 \neq 0$ in the game G_4.*

Proof. Suppose that $z_0 \neq 0$ and that $\sigma - \rho\sqrt{T} \geq 0$. We construct the control function of the evader as follows:

$$v_k(t) = \begin{cases} \rho\sqrt{\dfrac{T}{A_j(T)}}e^{\alpha_j(t)}, & k = j, \\ 0, & k \neq j. \end{cases} \tag{3.34}$$

We now show that this control satisfies the integral constraint

$$\sum_{k=1}^{\infty}\int_0^T v_k^2(s)ds = \rho^2\frac{T}{A_j(T)}\int_0^T e^{2\alpha_j(s)}ds \leq \rho^2 T \leq \sigma^2. \tag{3.35}$$

When the evader uses the control (3.34), our task is to show that $z_j(t)$ does not vanish in the interval $[0,T]$ for any admissible control of the pursuer u_{i0}.

For definiteness let $z_{j0} > 0$. Substituting (3.34) into (3.1), we have

$$z_j(t) \geq e^{-\alpha_j(t)}\left(z_{j0} + \rho\sqrt{\frac{T}{A_j(T)}}A_j(T) - \rho\sqrt{TA_j(T)}\right) = e^{-\alpha_j(t)}z_{j0} > 0. \tag{3.36}$$

Therefore, $z(t) \neq 0,\ t \in [0,T]$. This means that evasion is possible from initial position $z_0 \neq 0$ in game G_4. This ends the proof of the theorem. □

Theorem 3.8. *If $z_0 \in Y$ and $\sigma \geq 2\rho e^{\alpha_j(T)}$ for some $k = j$, then from the initial position $z_0 \neq 0$ evasion is possible in the game G_3.*

Proof. Suppose that $z_0 \in Y$ and that there exists $k = j$ such that

$$\sigma \geq 2\rho e^{\alpha_j(T)}. \tag{3.37}$$

We construct the control function of the evader as follows:

$$v_k(t) = \begin{cases} (2z_{j0} + \rho)e^{\alpha_j(t)}, & k = j, \\ 0, & k \neq j. \end{cases} \tag{3.38}$$

The inclusion $v(\cdot) \in S_2(\sigma)$ follows from the following

$$\begin{aligned} \sum_{k=1}^{\infty} v_k^2(t) &= \left[(2z_{j0} + \rho)e^{\alpha_j(t)}\right]^2 \\ &\leq 8z_{j0}^2 e^{2\alpha_j(t)} + 2\rho^2 e^{2\alpha_j(t)} \\ &\leq 2\rho^2 e^{2\alpha_j(t)} + 2\rho^2 e^{2\alpha_j(t)} \\ &\leq 4\rho^2 e^{2\alpha_j(T)} \leq \sigma^2. \end{aligned} \tag{3.39}$$

we used (3.34); (3.37) and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$.

Let the evader use the control (3.34) and for definiteness let $z_{j0} > 0$. Using (3.1) and the Cauchy-Schwartz inequality, we have

$$\begin{aligned} z_j(t) &\geq e^{-\alpha_j(t)}\left(z_{j0} - \rho\sqrt{A_j(t)} + 2z_{j0}A_j(t) + \rho\sqrt{A_j(t)}\right) \\ &= e^{-\alpha_j(t)}\left(z_{j0} + 2z_{j0}\mathrm{A}_j(t)\right) > 0. \end{aligned} \tag{3.40}$$

Therefore, we have $z_j(t) > 0$, $0 \leq t \leq T$, that is, evasion is possible in the game G_3. This ends the proof of the theorem. □

4. Conclusion

This paper is closely related to [10, 11]. However, the game model considered in this paper is a better generalization to the one in the last cited papers. The constant coefficients of the game model considered in the cited papers are specific to function coefficients considered in this papers. Sufficient conditions for which pursuit can be completed and for which evasion is possible with various form of constraints on the control of the players have been established.

For future works, optimal pursuit and multiplayers game problems described by the model considered in this paper can be investigated. As there are four different possible combinations of geometric and integral constraints on the control functions of the two players of the game, there would be four different problems to be studied.

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