Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 572326, 18 pages
doi:10.1155/2012/572326

Research Article

Hybrid Iterative Scheme by a Relaxed Extragradient Method for Equilibrium Problems, a General System of Variational Inequalities and Fixed-Point Problems of a Countable Family of Nonexpansive Mappings

Qiao-Li Dong,[1] Yan-Ni Guo,[1] and Fang Su[2]

[1] *College of Science, Civil Aviation University of China, Tianjin 300300, China*
[2] *College of Science, National University of Defense Technology, Changsha 410073, China*

Correspondence should be addressed to Qiao-Li Dong, dongql@lsec.cc.ac.cn

Received 29 November 2011; Accepted 5 December 2011

Academic Editor: Yonghong Yao

Based on the relaxed extragradient method and viscosity method, we introduce a new iterative method for finding a common element of solution of equilibrium problems, the solution set of a general system of variational inequalities, and the set of fixed points of a countable family of nonexpansive mappings in a real Hilbert space. Furthermore, we prove the strong convergence theorem of the studied iterative method. The results of this paper extend and improve the results of Ceng et al., (2008) , W. Kumam and P. Kumam, (2009) , Yao et al., (2010) and many others.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(F)$. The equilibrium problems covers, as special cases, monotone inclusion problems, saddle point problems, minimization problems, optimization problems, variational inequality problems, Nash equilibria in noncooperative games, and various forms of feasibility problems (see [1–4] and the references therein).

A mapping $A : C \to H$ is called *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha \|Au - Av\|^2, \quad \forall u, v \in C. \tag{1.2}$$

It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

We denote by $F(T)$ the set of fixed points of T. Recently, Wang and Guo [5] introduced an iterative scheme for a countable family of nonexpansive mappings.

Let C be a nonempty closed convex subset of a real Hilbert space H. For a given nonlinear operator $A : C \to H$, consider the following variational inequality problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad x \in C. \tag{1.4}$$

The set of solutions of the variational inequality (1.4) is denoted by $\mathrm{VI}(C, A)$ (see [6–9] and the references therein).

Let $A, B : C \to H$ be two mappings. Consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\left\langle \lambda Ay^* + x^* - y^*, x - x^*\right\rangle \geq 0, \quad \forall x \in C, \\ &\left\langle \mu Bx^* + y^* - x^*, x - y^*\right\rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{1.5}$$

which is called a general system of variational inequalities, where $\lambda > 0$ and $\mu > 0$ are two constants. The set of solutions of (1.5) is denoted by GSVI(A, B, and C). In particular, if $A = B$, then problem (1.5) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\left\langle \lambda Ay^* + x^* - y^*, x - x^*\right\rangle \geq 0, \quad \forall x \in C, \\ &\left\langle \mu Ax^* + y^* - x^*, x - y^*\right\rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{1.6}$$

which is defined by Verma [7] (see also [10]) and is called the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then problem (1.6) reduces to the classical variational inequality problem (1.4). Recently, Yao et al. [11] presented system of variational inequalities in Banach space. For solving problem (1.5), recently, Ceng et al. [12] introduced and studied a relaxed extragradient method. Based on the relaxed extragradient method and the viscosity approximation method, W. Kumam and P. Kumam [13] constructed a new viscosity-relaxed extragradient approximation method. Very recently, based on the extragradient method, Yao et al. [14] proposed an iterative method for finding a common element of the set of a general system of variational inequalities and the set of fixed points of a strictly pseudocontractive mapping in a real Hilbert space.

Motivated and inspired by the above works, in this paper, we introduce an iterative method based on the extragradient method and viscosity method for finding a common element of solution of equilibrium problems, the solution set of a general system of variational inequalities, and the set of fixed points of a countable family of nonexpansive mappings in a real Hilbert space. Furthermore, we prove the strong convergence theorem of the proposed iterative method.

2. Preliminaries

Let C be a closed convex subset of H, and let $T : C \to C$ be nonexpansive such that $F(T) \neq \emptyset$. For all $\widehat{x} \in F(T)$ and all $x \in C$, we have

$$\begin{aligned}\|x-\widehat{x}\|^2 \geq \|Tx - T\widehat{x}\|^2 &= \|Tx-\widehat{x}\|^2 = \|(Tx-x)+(x-\widehat{x})\|^2 \\ &= \|Tx-x\|^2 + \|x-\widehat{x}\|^2 + 2\langle Tx-x, x-\widehat{x}\rangle,\end{aligned} \tag{2.1}$$

and hence

$$\|Tx-x\|^2 \leq 2\langle x-Tx, x-\widehat{x}\rangle, \quad \forall \widehat{x} \in F(T), \forall x \in C. \tag{2.2}$$

Remark 2.1. Let A be α-inverse-strongly monotone. For all $x, y \in C$, $\lambda > 0$, we have

$$\begin{aligned}\left\|(I-\lambda A)x-(I-\lambda A)y\right\|^2 &= \left\|x-y\right\|^2 - 2\lambda\langle Ax-Ay, x-y\rangle + \lambda^2\left\|Ax-Ay\right\|^2 \\ &\leq \left\|x-y\right\|^2 + \lambda(\lambda-2\alpha)\left\|Ax-Ay\right\|^2.\end{aligned} \tag{2.3}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C}\left\|x-y\right\|. \tag{2.4}$$

The following characterizes the projection P_C.

Lemma 2.2. *Given that $x \in H$ and $y \in C$, then $P_C x = y$ if and only if there holds the inequality*

$$\langle x-y, y-z\rangle \geq 0 \quad \forall z \in C. \tag{2.5}$$

Lemma 2.3 (see [15]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(1-T)x_n\}$ converges strongly to y, then$(1-T)x = y$.*

Lemma 2.4 (see [16]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n, \quad \forall n \geq 0, \tag{2.6}$$

where two sequences $\{\gamma_n\} \subset (0,1)$ *and* $\{\delta_n\}$ *satisfy*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.5 (see [12]). *For given* $x^*, y^* \in C$, (x^*, y^*) *is a solution of problem* (1.5) *if and only if* x^* *is a fixed point of the mapping* $G : C \to C$ *defined by*

$$G(x) = P_C\left[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)\right], \quad \forall x \in C, \tag{2.7}$$

where $y^* = P_C(x^* - \mu Bx^*)$.

Note that the mapping G is nonexpansive provided that $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$. Throughout this paper, the set of fixed points of the mapping G is denoted by Γ.

Lemma 2.6 (see [1]). *Let* C *be a nonempty closed convex subset of* H *and* $F : C \times C \to \mathbb{R}$ *satisfy following conditions:*

(A1) $F(x,x) = 0, \forall x \in C$;

(A2) F *is monotone, that is,* $F(x,y) + F(y,x) \le 0, \ \forall x, y \in C$;

(A3) $\limsup_{t\to 0^+} F(tz + (1-t)\ x, y) \le F(x,y), \forall x, y,\ z \in C$;

(A4) *for each* $x \in C$, $F(x,\cdot)$ *is convex and lower semicontinuous.*

For $x \in C$ *and* $r > 0$, *set* $T_r^F : H \to C$ *to be*

$$T_r^F(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C \right\}. \tag{2.8}$$

Then T_r^F *is well defined and the following holds:*

(1) T_r^F *is single valued;*

(2) T_r^F *is firmly nonexpansive [17], that is, for any* $x, y \in E$,

$$\left\| T_r^F x - T_r^F y \right\|^2 \le \left\langle T_r^F x - T_r^F y, x - y \right\rangle; \tag{2.9}$$

(3) $F(T_r^F) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

By the proof of Lemma 5 in [2] (also see [3]), we have following lemma.

Lemma 2.7. *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and* $F : C \times C \to \mathbb{R}$ *be a bifunction. Let* $x \in C$ *and* $r_1, r_2 \in (0, \infty)$. *Then*

$$\|T_{r_1}x - T_{r_2}x\| \le \left|1 - \frac{r_2}{r_1}\right| (\|T_{r_1}x\| + \|x\|). \tag{2.10}$$

3. Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mappings $A, B : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and $\{T_n\}_{n=1}^{\infty} : C \to C$ be a countable family of nonexpansive mappings such that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap \Gamma \neq \emptyset$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1/2)$. Set $\beta_0 = 1$. For given $x_1 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be generated by*

$$\begin{gathered}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\
z_n = P_C(u_n - \mu B u_n), \\
y_n = \alpha_n f(x_n) + (1 - \alpha_n) P_C(z_n - \lambda A z_n), \\
x_{n+1} = \beta_n x_n + \sigma_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) T_i y_n + (1 - \beta_n)(1 - \sigma_n) P_C(z_n - \lambda A z_n),
\end{gathered} \tag{3.1}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, and sequences $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$, $\{\sigma_n\} \subset [0, 1]$, and $\{r_n\} \subset (r, \infty)$, $r > 0$, are such that

(i) *$\{\beta_n\}$ is strictly decreasing,*

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(iii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iv) $\sigma_n > 1/2(1 - \rho)$, $\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty$,

(v) $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $x^* = P_{\Omega} \cdot f(x^*)$, and (x^*, y^*) is a solution of the general system of variational inequalities (1.5), where $y^* = P_C(x^* - \mu B x^*)$.

Proof. The proof is divided into several steps.

Step 1. The sequence $\{x_n\}$ defined by (3.1) is bounded.

For each $x^* \in \Omega$, from Lemma 2.6, we have $u_n = T_{r_n} x_n$ and hence

$$\|u_n - x^*\| = \|T_{r_n} x_n - T_{r_n} x^*\| \leq \|x_n - x^*\|. \tag{3.2}$$

Since f is a ρ-contraction mapping, using (2.3) and (3.2), we have

$$\begin{aligned}
\|z_n - y^*\|^2 &= \|P_C(u_n - \mu B u_n) - P_C(x^* - \mu B x^*)\|^2 \\
&\leq \|(u_n - \mu B u_n) - (x^* - \mu B x^*)\|^2 \\
&\leq \|u_n - x^*\|^2 + \mu(\mu - 2\beta)\|B u_n - B x^*\|^2 \\
&\leq \|x_n - x^*\|^2 + \mu(\mu - 2\beta)\|B u_n - B x^*\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned} \tag{3.3}$$

Set $v_n = P_C(z_n - \lambda A z_n)$. Since P_C is nonexpansive, from (2.3), we have

$$\begin{aligned}\|v_n - x^*\|^2 &= \left\|P_C(z_n - \lambda A z_n) - P_C(y^* - \lambda A y^*)\right\|^2 \\ &\le \left\|z_n - y^*\right\|^2 + \lambda(\lambda - 2\alpha)\left\|Az_n - Ay^*\right\|^2 \\ &\le \left\|z_n - y^*\right\|^2.\end{aligned} \tag{3.4}$$

Hence we get

$$\begin{aligned}\left\|y_n - x^*\right\| &= \left\|\alpha_n f(x_n) + (1-\alpha_n)v_n - x^*\right\| \\ &\le \alpha_n\left\|f(x_n) - x^*\right\| + (1-\alpha_n)\left\|z_n - y^*\right\| \\ &\le \alpha_n\left(\left\|f(x_n) - f(x^*)\right\| + \left\|f(x^*) - x^*\right\|\right) + (1-\alpha_n)\|x_n - x^*\| \\ &\le \left(1 - \alpha_n(1-\rho)\right)\|x_n - x^*\| + \alpha_n\left\|f(x^*) - x^*\right\|.\end{aligned} \tag{3.5}$$

From (3.1) and (3.3)–(3.5), we get

$$\begin{aligned}\|x_{n+1} - x^*\| &= \left\|\beta_n x_n + \sigma_n\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i y_n + (1-\beta_n)(1-\sigma_n)v_n - x^*\right\| \\ &\le \beta_n\|x_n - x^*\| + \sigma_n\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\left\|T_i y_n - x^*\right\| + (1-\beta_n)(1-\sigma_n)\|v_n - x^*\| \\ &\le \beta_n\|x_n - x^*\| + \sigma_n\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\left\|y_n - x^*\right\| + (1-\beta_n)(1-\sigma_n)\left\|z_n - y^*\right\| \\ &\le \left(1 - \sigma_n(1-\beta_n)\right)\|x_n - x^*\| + \sigma_n(1-\beta_n)\left\|y_n - y^*\right\| \\ &\le \left(1 - \sigma_n(1-\beta_n)\right)\|x_n - x^*\| \\ &\quad + \sigma_n(1-\beta_n)\left[\left(1 - \alpha_n(1-\rho)\right)\|x_n - x^*\| + \alpha_n\left\|f(x^*) - x^*\right\|\right] \\ &\le \left[1 - \sigma_n(1-\beta_n)\alpha_n(1-\rho)\right]\|x_n - x^*\| \\ &\quad + \sigma_n(1-\beta_n)\alpha_n(1-\rho)\frac{1}{1-\rho}\left\|f(x^*) - x^*\right\| \\ &\le M,\end{aligned} \tag{3.6}$$

where $M = \max\{\|x_1 - x^*\| \;\; , 1/(1-\rho)\|f(x^*) - x^*\|\}$. Hence, $\{x_n\}$ is bounded and therefore $\{u_n\}$, $\{z_n\}$, $\{v_n\}$, and $\{y_n\}$ are also bounded.

Step 2. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $u_n = T_{r_n}x_n$ and $u_{n-1} = T_{r_{n-1}}x_{n-1}$, using Lemma 2.7, we have

$$\begin{aligned}\|u_n - u_{n-1}\| &= \|T_{r_n}x_n - T_{r_{n-1}}x_{n-1}\| \\ &\le \|T_{r_n}x_n - T_{r_n}x_{n-1}\| + \|T_{r_n}x_{n-1} - T_{r_{n-1}}x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + \left|1 - \frac{r_n}{r_{n-1}}\right|(\|u_{n-1}\| + \|x_{n-1}\|) \\ &\le \|x_n - x_{n-1}\| + \frac{1}{r}|r_{n-1} - r_n|L,\end{aligned} \tag{3.7}$$

where $L = \sup\{\|u_n\| + \|x_n\| : n = 1, 2, \ldots\}$. From (3.1) and (3.7), it follows that

$$\begin{aligned} \|z_n - z_{n-1}\| &= \left\|P_C(I-\mu B)u_n - P_C(I-\mu B)u_{n-1}\right\| \\ &\le \|u_n - u_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + \frac{1}{r}|r_{n-1} - r_n|L. \end{aligned} \tag{3.8}$$

From (3.1) and (3.8), we have

$$\begin{aligned} \left\|y_n - y_{n-1}\right\| &\le \alpha_n \left\|f(x_n) - f(x_{n-1})\right\| + (1-\alpha_n)\|P_C(I-\lambda A)z_n - P_C(I-\lambda A)z_{n-1}\| \\ &\le \alpha_n \rho \|x_n - x_{n-1}\| + (1-\alpha_n)\|z_n - z_{n-1}\| \\ &\le \alpha_n \rho \|x_n - x_{n-1}\| + (1-\alpha_n)\left(\|x_n - x_{n-1}\| + \frac{1}{r}|r_{n-1} - r_n|L\right) \\ &\le \left(1 - \alpha_n(1-\rho)\right)\|x_n - x_{n-1}\| + (1-\alpha_n)\frac{1}{r}|r_{n-1} - r_n|L. \end{aligned} \tag{3.9}$$

By definition of scheme (3.1), we have

$$\begin{aligned} x_{n+1} - x_n = \beta_n(x_n - x_{n-1}) &+ (\beta_n - \beta_{n-1})x_{n-1} + \sigma_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i)(T_i y_n - T_i y_{n-1}) \\ &+ (\sigma_n - \sigma_{n-1}) \sum_{i=1}^{n-1} (\beta_{i-1} - \beta_i) T_i y_{n-1} + \sigma_n(\beta_{n-1} - \beta_n) T_n y_{n-1} \\ &+ (1-\beta_n)(1-\sigma_n)[P_C(z_n - \lambda A z_n) - P_C(z_{n-1} - \lambda A z_{n-1})] \\ &- (1-\beta_{n-1})(1-\sigma_{n-1})P_C(z_{n-1} - \lambda A z_{n-1}) + (1-\beta_n)(1-\sigma_n)P_C(z_{n-1} - \lambda A z_{n-1}). \end{aligned} \tag{3.10}$$

Thus, from (3.8)–(3.10),

$$\begin{aligned} \|x_{n+1} - x_n\| \le \beta_n \|x_n - x_{n-1}\| &+ (\beta_{n-1} - \beta_n)\|x_{n-1}\| + \sigma_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) \left\|T_i y_n - T_i y_{n-1}\right\| \\ &+ |\sigma_n - \sigma_{n-1}| \sum_{i=1}^{n-1} (\beta_{i-1} - \beta_i) \left\|T_i y_{n-1}\right\| + \sigma_n (\beta_{n-1} - \beta_n) \left\|T_n y_{n-1}\right\| \\ &+ (1-\beta_n)(1-\sigma_n)\|P_C(z_n - \lambda A z_n) - P_C(z_{n-1} - \lambda A z_{n-1})\| \\ &+ \left|(1-\beta_n)(1-\sigma_n) - (1-\beta_{n-1})(1-\sigma_{n-1})\right| \|P_C(z_{n-1} - \lambda A z_{n-1})\| \\ \le \beta_n \|x_n - x_{n-1}\| &+ (\beta_{n-1} - \beta_n)\|x_{n-1}\| + \sigma_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) \left\|y_n - y_{n-1}\right\| \\ &+ |\sigma_n - \sigma_{n-1}| \sum_{i=1}^{n-1} (\beta_{i-1} - \beta_i) \left\|T_i y_{n-1}\right\| + (\beta_{n-1} - \beta_n) \left\|T_n y_{n-1}\right\| \end{aligned}$$

$$
\begin{aligned}
&+ (1-\beta_n)(1-\sigma_n)\|z_n - z_{n-1}\| + (\beta_{n-1}-\beta_n)(1-\sigma_n)\|P_C(I-\lambda A)z_{n-1}\| \\
&+ (1-\beta_{n-1})|\sigma_n - \sigma_{n-1}|\|P_C(I-\lambda A)z_{n-1}\| \\
\le{}& \beta_n\|x_n - x_{n-1}\| + (\beta_{n-1}-\beta_n)\|x_{n-1}\| + \sigma_n(1-\beta_n) \\
&\times \left[(1-\alpha_n(1-\rho))\|x_n - x_{n-1}\| + (1-\alpha_n)\frac{1}{r}|r_{n-1}-r_n|L\right] \\
&+ |\sigma_n - \sigma_{n-1}|\sum_{i=1}^{n-1}(\beta_{i-1}-\beta_i)\left\|T_i y_{n-1}\right\| \\
&+ (\beta_{n-1}-\beta_n)\left\|T_n y_{n-1}\right\| + (1-\beta_n)(1-\sigma_n)\left(\|x_n - x_{n-1}\| + \frac{1}{r}|r_{n-1}-r_n|L\right) \\
&+ (\beta_{n-1}-\beta_n)\|P_C(I-\lambda A)z_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|P_C(I-\lambda A)z_{n-1}\| \\
\le{}& \left[1-\sigma_n(1-\beta_n)\alpha_n(1-\rho)\right]\|x_n - x_{n-1}\| + (\beta_{n-1}-\beta_n)M \\
&+ \frac{1}{r}|r_{n-1}-r_n|L + |\sigma_n - \sigma_{n-1}|\sum_{i=1}^{n-1}(\beta_{i-1}-\beta_i)M + (\beta_{n-1}-\beta_n)M \\
&+ \left((\beta_{n-1}-\beta_n) + |\sigma_n - \sigma_{n-1}|\right)M \\
\le{}& \left[1-\sigma_n(1-\beta_n)\alpha_n(1-\rho)\right]\|x_n - x_{n-1}\| + \frac{1}{r}|r_{n-1}-r_n|L \\
&\quad + 3(\beta_{n-1}-\beta_n)M + 2|\sigma_n - \sigma_{n-1}|M,
\end{aligned}
\tag{3.11}
$$

where $M = \max\{\sup_{n\ge1}\|x_n\|,\ \ \sup_{i\ge1,n\ge1}\|T_i y_n\|,\ \ \sup_{n\ge1}\|\ P_C(I-\lambda A)z_n\|\}$. Since $\{\beta_n\}$ is strictly decreasing, we have $\sum_{n=2}^{\infty}(\beta_{n-1}-\beta_n) = \beta_1 < \infty$. Further, by assumption conditions (iv)-(v), we have

$$
\sum_{n=2}^{\infty}\left\{\frac{1}{r}|r_{n+1}-r_n|L + 3(\beta_{n-1}-\beta_n)M + 2|\sigma_n - \sigma_{n-1}|M\right\} < \infty. \tag{3.12}
$$

Thus, using Lemma 2.4, we have $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Step 3. $\lim_{n\to\infty}\|x_n - u_n\| = 0$.

For any $x^* \in \Omega$, it follows from Lemma 2.6 that

$$
\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \le \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^*\rangle = \langle u_n - x^*, x_n - x^*\rangle \\
&= \frac{1}{2}\left(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|u_n - x_n\|^2\right).
\end{aligned}
\tag{3.13}
$$

From (2.3), (3.1)–(3.4), we can get

$$
\begin{aligned}
\left\|y_n - x^*\right\|^2 &\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + (1-\alpha_n)\left(\left\|z_n - y^*\right\|^2 + \lambda(\lambda - 2\alpha)\left\|Az_n - Ay^*\right\|^2\right) \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + (1-\alpha_n)\left\|z_n - y^*\right\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + (1-\alpha_n)\|u_n - x^*\|^2.
\end{aligned}
\tag{3.14}
$$

From (3.1), (3.3)-(3.4), and (3.13)-(3.14), it follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\beta_n x_n + \sigma_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) T_i y_n + (1-\beta_n)(1-\sigma_n) v_n - x^*\right\|^2 \\
&\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\left\|\sigma_n \frac{1}{1-\beta_n}\sum_{i=1}^{n} (\beta_{i-1} - \beta_i) T_i y_n + (1-\sigma_n) v_n - x^*\right\|^2 \\
&\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\sigma_n \left\|\frac{1}{1-\beta_n}\sum_{i=1}^{n} (\beta_{i-1} - \beta_i) T_i y_n - x^*\right\|^2 \\
&\quad + (1-\beta_n)(1-\sigma_n)\|v_n - x^*\|^2 \\
&\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\sigma_n \frac{1}{1-\beta_n}\sum_{i=1}^{n} (\beta_{i-1} - \beta_i) \left\|T_i y_n - x^*\right\|^2 \\
&\quad + (1-\beta_n)(1-\sigma_n)\|v_n - x^*\|^2 \\
&\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\sigma_n \left\|y_n - x^*\right\|^2 + (1-\beta_n)(1-\sigma_n)\left\|z_n - y^*\right\|^2 \\
&\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\sigma_n \left[(1-\alpha_n)\|u_n - x^*\|^2 + \alpha_n \left\|f(x_n) - x^*\right\|^2\right] \\
&\quad + (1-\beta_n)(1-\sigma_n)\|u_n - x^*\|^2 \\
&\le \|x_n - x^*\|^2 + (1-\beta_n)\sigma_n\alpha_n\left(\left\|f(x_n) - x^*\right\|^2 - \|x_n - x^*\|^2\right) \\
&\quad - (1-\beta_n)(1-\sigma_n\alpha_n)\|u_n - x_n\|^2
\end{aligned}
\tag{3.15}
$$

which implies that

$$
\begin{aligned}
(1-\beta_n)(1-\sigma_n\alpha_n)\|u_n - x_n\|^2 \le{}& (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\
&+ \alpha_n \left\|f(x_n) - x^*\right\|^2.
\end{aligned}
\tag{3.16}
$$

Since $\{x_n\}$ is both bounded, using Step 2 and conditions (ii)–(iv), we conclude the result.

Step 4. $\lim_{n\to\infty}\|Az_n - Ay^*\| = 0$ and $\lim_{n\to\infty}\|Bx_n - Bx^*\| = 0$.

Using (3.3), (3.14), and (3.15), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\sigma_n\left\|y_n-x^*\right\|^2+(1-\beta_n)(1-\sigma_n)\left\|z_n-y^*\right\|^2\\
&\leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\sigma_n\\
&\quad\times\left[\alpha_n\left\|f(x_n)-x^*\right\|^2+(1-\alpha_n)\left(\left\|z_n-y^*\right\|^2+\lambda(\lambda-2\alpha)\left\|Az_n-Ay^*\right\|^2\right)\right]\\
&\quad+(1-\beta_n)(1-\sigma_n)\left\|z_n-y^*\right\|^2\\
&\leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\sigma_n(1-\alpha_n)\lambda(\lambda-2\alpha)\left\|Az_n-Ay^*\right\|^2\\
&\quad+\alpha_n\left\|f(x_n)-x^*\right\|^2+(1-\beta_n)\left\|z_n-y^*\right\|^2\\
&\leq \beta_n\|x_n-x^*\|^2+\lambda(\lambda-2\alpha)\left\|Az_n-Ay^*\right\|^2\\
&\quad+\alpha_n\left\|f(x_n)-x^*\right\|^2+(1-\beta_n)\left[\|x_n-x^*\|^2+\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2\right]\\
&\leq \|x_n-x^*\|^2+\lambda(\lambda-2\alpha)\left\|Az_n-Ay^*\right\|^2\\
&\quad+\alpha_n\left\|f(x_n)-x^*\right\|^2+\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2.
\end{aligned}\tag{3.17}$$

Therefore,

$$\begin{aligned}
&\lambda(2\alpha-\lambda)\left\|Az_n-Ay^*\right\|^2+\mu(2\beta-\mu)\|Bu_n-Bx^*\|^2\\
&\quad\leq(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|+\alpha_n\left\|f(x_n)-y^*\right\|^2.
\end{aligned}\tag{3.18}$$

From Step 2, using condition (iii), we get $\lim_{n\to\infty}\|Az_n-Ay^*\| = 0$ and $\lim_{n\to\infty}\|Bu_n-Bx^*\| = 0$. From the fact that the B is β-inverse strongly monotone operator, it follows that

$$\|Bx_n-Bx^*\|\leq\|Bx_n-Bu_n\|+\|Bu_n-Bx^*\|\leq\frac{1}{\beta}\|x_n-u_n\|+\|Bu_n-Bx^*\|.\tag{3.19}$$

Applying Step 3, we have $\lim_{n\to\infty}\|Bx_n - Bx^*\| = 0$.

Step 5. $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0,\ for\ all\ i = 1, 2, \ldots$.

Noting that P_C is firmly nonexpansive, we have

$$\begin{aligned}
\left\|z_n-y^*\right\|^2&=\left\|P_C(u_n-\mu Bu_n)-P_C(x^*-\mu Bx^*)\right\|^2\\
&\leq\left\langle(u_n-\mu Bu_n)-(x^*-\mu Bx^*),z_n-y^*\right\rangle\\
&=\frac{1}{2}\Big[\left\|u_n-x^*-\mu(Bu_n-Bx^*)\right\|^2+\left\|z_n-y^*\right\|^2\\
&\qquad-\left\|(u_n-z_n)-\mu(Bu_n-Bx^*)-(x^*-y^*)\right\|^2\Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\Big[\|u_n - x^*\|^2 + \|z_n - y^*\|^2 - \|u_n - z_n - (x^* - y^*)\|^2 \\
&\qquad +2\mu\langle u_n - z_n - (x^* - y^*), Bu_n - Bx^*\rangle - \mu^2\|Bu_n - Bx^*\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|x_n - x^*\|^2 + \|z_n - y^*\|^2 - \|u_n - z_n - (x^* - y^*)\|^2 \\
&\qquad +2\mu\langle u_n - z_n - (x^* - y^*), Bu_n - Bx^*\rangle\Big], \\
\|v_n - x^*\|^2 &= \|P_C(z_n - \lambda A z_n) - P_C(y^* - \lambda A y^*)\|^2 \\
&\leq \langle (z_n - \lambda A z_n) - (y^* - \lambda A y^*), v_n - x^*\rangle \\
&= \frac{1}{2}\Big[\|z_n - y^* - \lambda(Az_n - Ay^*)\|^2 + \|v_n - x^*\|^2 \\
&\qquad -\|(z_n - v_n) - \lambda(Az_n - Ay^*) - (y^* - x^*)\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|z_n - y^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n - (y^* - x^*)\|^2 \\
&\qquad +2\lambda\langle z_n - v_n - (y^* - x^*), Az_n - Ay^*\rangle - \lambda^2\|Az_n - Ay^*\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|x_n - x^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n - (y^* - x^*)\|^2 \\
&\qquad +2\lambda\langle z_n - v_n - (y^* - x^*) Az_n - Ay^*\rangle\Big].
\end{aligned}
\tag{3.20}
$$

It follows that

$$
\begin{aligned}
\|z_n - y^*\|^2 \leq \|x_n - x^*\|^2 &- \|u_n - z_n - (x^* - y^*)\|^2 \\
&+ 2\mu\langle u_n - z_n - (x^* - y^*), Bu_n - Bx^*\rangle,
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
\|v_n - x^*\|^2 \leq \|x_n - x^*\|^2 &- \|z_n - v_n - (y^* - x^*)\|^2 \\
&+ 2\lambda\langle z_n - v_n - (y^* - x^*), Az_n - Ay^*\rangle.
\end{aligned}
\tag{3.22}
$$

By (3.3), (3.14)-(3.15), and (3.22), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1 - (1 - \beta_n)\sigma_n)\|x_n - x^*\|^2 + (1 - \beta_n)\sigma_n\|y_n - x^*\|^2 \\
&\leq (1 - (1 - \beta_n)\sigma_n)\|x_n - x^*\|^2 + (1 - \beta_n)\sigma_n\Big[\alpha_n\|f(x_n) - x^*\|^2 + \|v_n - x^*\|^2\Big] \\
&\leq (1 - (1 - \beta_n)\sigma_n)\|x_n - x^*\|^2 \\
&\quad + (1 - \beta_n)\sigma_n\Big[\alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|z_n - v_n - (y^* - x^*)\|^2 \\
&\qquad +2\lambda\langle z_n - v_n - (y^* - x^*) Az_n - Ay^*\rangle\Big] \\
&= \|x_n - x^*\|^2 + \alpha_n\|f(x_n) - x^*\|^2 + (1 - \beta_n)(1 - \sigma_n) \\
&\quad \times \Big(-\|z_n - v_n - (y^* - x^*)\|^2 + 2\lambda\langle z_n - v_n - (y^* - x^*), Az_n - Ay^*\rangle\Big).
\end{aligned}
\tag{3.23}
$$

It follows that

$$\begin{aligned}(1-\beta_n)\sigma_n\left\|v_n - z - (x^* - y^*)\right\|^2 \\ \leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + \alpha_n\left\|f(x_n) - x^*\right\|^2 \\ + 2\lambda(1-\beta_n)\sigma_n\langle z_n - v_n - (y^* - x^*), Az_n - Ay^*\rangle.\end{aligned} \tag{3.24}$$

From conditions (ii)–(iv), Steps 2 and 4, we get the following:

$$\lim_{n\longrightarrow\infty}\left\|v_n - z_n - (x^* - y^*)\right\| = 0. \tag{3.25}$$

On the other hand, from (3.14)–(3.21), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\sigma_n\left(\alpha_n\left\|f(x_n) - x^*\right\|^2 + \left\|z_n - y^*\right\|^2\right) \\ &\quad + (1-\beta_n)(1-\sigma_n)\left\|z_n - y^*\right\|^2 \\ &\leq \beta_n\|x_n - x^*\|^2 + \alpha_n\left\|f(x_n) - x^*\right\|^2 + (1-\beta_n)\left\|z_n - y^*\right\|^2 \\ &\leq \beta_n\|x_n - x^*\|^2 + \alpha_n\left\|f(x_n) - x^*\right\|^2 + (1-\beta_n) \\ &\quad \times\left[\|x_n - x^*\|^2 - \left\|u_n - z_n - (x^* - y^*)\right\|^2 + 2\mu\langle u_n - z_n - (x^* - y^*), Bu_n - Bx^*\rangle\right] \\ &= \|x_n - x^*\|^2 + \alpha_n\left\|f(x_n) - x^*\right\|^2 \\ &\quad + (1-\beta_n)\left(-\left\|u_n - z_n - (x^* - y^*)\right\|^2 + 2\mu\langle u_n - z_n - (x^* - y^*), Bu_n - Bx^*\rangle\right),\end{aligned} \tag{3.26}$$

which implies that

$$\begin{aligned}\left\|(1-\beta_n)u_n - z_n - (x^* - y^*)\right\|^2 \leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\ + \alpha_n\left\|f(x_n) - x^*\right\|^2 + 2\mu(1-\beta_n) \\ \times\langle u_n - z_n - (x^* - y^*), Bu_n - Bx^*\rangle.\end{aligned} \tag{3.27}$$

From (ii)-(iii), Steps 2 and 4, we get the following:

$$\lim_{n\longrightarrow\infty}\left\|u_n - z_n - (x^* - y^*)\right\| = 0. \tag{3.28}$$

Combining (3.25) and (3.28), we get the following:

$$\lim_{n\longrightarrow\infty}\|u_n - v_n\| = 0. \tag{3.29}$$

Using Step 3, we obtain the following:

$$\lim_{n\longrightarrow\infty}\|x_n - v_n\| = 0. \tag{3.30}$$

This together with $\|y_n - v_n\| = \alpha_n \|f(x_n) - v_n\| \rightarrow 0$ implies that

$$\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0. \tag{3.31}$$

For any $x^* \in \Omega$, we have from (3.1) that

$$\begin{aligned}
\sigma_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i)\big(\langle x_n - T_i x_n, x_n - x^*\rangle + \langle T_i x_n - T_i y_n, x_n - x^*\rangle\big)& \\
= \sigma_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i)\langle x_n - T_i y_n, x_n - x^*\rangle& \\
= \langle x_n - x_{n+1}, x_n - x^*\rangle + (1-\beta_n)(1-\sigma_n)\langle v_n - x_n, x_n - x^*\rangle.&
\end{aligned} \tag{3.32}$$

Since each T_i is nonexpansive, from (2.2), we have

$$\|T_i x_n - x_n\|^2 \leq 2\langle x_n - T_i x_n, x_n - x^*\rangle. \tag{3.33}$$

Hence, combining this inequality with (3.32), we get the following:

$$\begin{aligned}
\frac{1}{2}\sigma_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i)\|T_i x_n - x_n\|^2 \leq{}& -\sigma_n \sum_{i=1}^{n} (\beta_{i-1} - \beta_i)\langle T_i x_n - T_i y_n, x_n - x^*\rangle \\
&+ \langle x_n - x_{n+1}, x_n - x^*\rangle + (1-\beta_n)(1-\sigma_n) \\
&\times \langle v_n - x_n, x_n - x^*\rangle,
\end{aligned} \tag{3.34}$$

that is (noting that β_n is strictly decreasing),

$$\begin{aligned}
\|T_i x_n - x_n\|^2 \leq{}& \frac{2(1-\beta_n)}{\beta_{i-1} - \beta_i}\|x_n - y_n\|\|x_n - x^*\| + \frac{2}{\sigma_n(\beta_{i-1} - \beta_i)}\|x_n - x_{n+1}\|\|x_n - x^*\| \\
&+ \frac{2(1-\beta_n)(1-\sigma_n)}{\sigma_n(\beta_{i-1} - \beta_i)}\|v_n - x_n\|\|x_n - x^*\|.
\end{aligned} \tag{3.35}$$

Now from (3.30)-(3.31) and Step 2, we conclude that

$$\lim_{n\longrightarrow\infty} \|T_i x_n - x_n\| = 0, \quad \forall i = 1, 2, \ldots, \tag{3.36}$$

which completes the proof.

Step 6. $\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle \leq 0$, where $x^* = P_\Omega \cdot f(x^*)$.

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\} \rightarrow \tilde{x}$ weakly. First, it is clear from Step 5 and Lemma 2.3 that $\tilde{x} \in \bigcap_{n=1}^{\infty} F(T_n)$. Next, we prove that $\tilde{x} \in \Gamma$. From (iii), Step 3 and (3.31), we note that

$$\begin{aligned} \|y_n - G(y_n)\| &\leq \alpha_n \|f(x_n) - G(y_n)\| \\ &\quad + (1-\alpha_n)\|P_C[P_C(u_n - \mu B u_n) - \lambda A P_C(u_n - \mu B u_n)] - G(y_n)\| \\ &= \alpha_n \|f(x_n) - G(y_n)\| + (1-\alpha_n)\|G(u_n) - G(y_n)\| \\ &\leq \alpha_n \|f(x_n) - G(y_n)\| + (1-\alpha_n)\|u_n - y_n\| \\ &\longrightarrow 0. \end{aligned} \tag{3.37}$$

According to (3.31) and Lemma 2.3, we obtain that $\tilde{x} \in \Gamma$. Next, we show that $\tilde{x} \in \mathrm{EP}(F)$. Indeed, by $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.38}$$

From (A2), we have also

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq -F(u_n, y) \geq F(y, u_n), \tag{3.39}$$

and hence,

$$\left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle \geq F(y, u_{n_k}). \tag{3.40}$$

According to $r_n > r > 0$ and $u_n - x_n \rightarrow 0$, we conclude that $(u_{n_i} - x_{n_i})/r_{n_i} \rightarrow 0$ and $u_{n_k} \rightharpoonup \tilde{x}$. From (A4), we obtain the following:

$$F(y, \tilde{x}) \leq 0, \quad \forall y \in C. \tag{3.41}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)\tilde{x}$. Since $y \in C$ and $\tilde{x} \in C$ (due to $u_{n_i} \rightarrow \tilde{x}$ as $i \rightarrow \infty$), we have $y_t \in C$ and hence $F(y_t, \tilde{x}) \leq 0$. So, from (A1) and (A4), we have

$$tF(y_t, y) \geq tF(y_t, y) + (1-t)F(y_t, \tilde{x}) \geq tF(y_t, y_t) = 0 \tag{3.42}$$

and $F(y_t, y) \geq 0$, for all $t \in (0,1)$ and $y \in C$. From (A3), we obtain the following:

$$F(\tilde{x}, y) \geq 0 \quad \forall y \in C, \tag{3.43}$$

and hence $\widetilde{x} \in \mathrm{EP}(F)$. Therefore, we obtain that $\widetilde{x} \in \Omega$. Hence, it follows from Lemma 2.2 that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle &= \lim_{i\to\infty}\langle f(x^*) - x^*, x_{n_i} - x^*\rangle \\ &= \langle f(x^*) - x^*, \widetilde{x} - x^*\rangle \\ &\leq 0.\end{aligned} \tag{3.44}$$

Step 7. $\lim_{n\to\infty} x_n = x^*$.

From (3.1), (3.3)-(3.4), and the convexity of $\|\cdot\|$, we have

$$\begin{aligned}\left\|y_n - x^*\right\|^2 &= \left\|\alpha_n f(x_n) + (1-\alpha_n)v_n - x^*\right\|^2 \\ &\leq (1-\alpha_n)\|v_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - x^*, y_n - x^*\rangle \\ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - x^*, y_n - x^*\rangle.\end{aligned} \tag{3.45}$$

We can also get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 = &\left\|\beta_n(x_n - x^*) + \sigma_n\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i y_n - x^*) + (1-\beta_n)(1-\sigma_n)(y_n - x^*)\right. \\ &\left. +(1-\beta_n)(1-\sigma_n)\alpha_n(v_n - f(x_n))\right\|^2 \\ \leq &\left\|\beta_n(x_n - x^*) + \sigma_n\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i y_n - x^*) + (1-\beta_n)(1-\sigma_n)(y_n - x^*)\right\|^2 \\ &+ 2(1-\beta_n)(1-\sigma_n)\alpha_n\langle v_n - f(x_n), x_{n+1} - x^*\rangle \\ \leq &\beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left\|y_n - x^*\right\|^2 + 2(1-\beta_n)(1-\sigma_n)\alpha_n\langle v_n - x^*, x_{n+1} - x^*\rangle \\ &+ 2(1-\beta_n)(1-\sigma_n)\alpha_n\langle x^* - f(x_n), x_{n+1} - x^*\rangle.\end{aligned} \tag{3.46}$$

By (3.45), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq &\beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left[(1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - x^*, y_n - x^*\rangle\right] \\ &+ 2(1-\beta_n)(1-\sigma_n)\alpha_n\|v_n - x^*\|\|x_{n+1} - x^*\| \\ &+ 2(1-\beta_n)\alpha_n(1-\sigma_n)\langle x^* - f(x_n), x_{n+1} - x^*\rangle \\ \leq &\left[1 - (1-\beta_n)\alpha_n\right]\|x_n - x^*\|^2 + 2(1-\beta_n)(1-\sigma_n)\alpha_n\langle f(x_n) - x^*, y_n - x_{n+1}\rangle \\ &+ 2(1-\beta_n)(1-\sigma_n)\alpha_n\left\|z_n - y^*\right\|\|x_{n+1} - x^*\| \\ &+ 2(1-\beta_n)\alpha_n\sigma_n\langle f(x_n) - x^*, y_n - x^*\rangle \\ \leq &\left[1 - (1-\beta_n)\alpha_n\right]\|x_n - x^*\|^2 + 2(1-\beta_n)(1-\sigma_n)\alpha_n\left\|f(x_n) - x^*\right\|\left\|x_{n+1} - y_n\right\| \\ &+ 2(1-\beta_n)(1-\sigma_n)\alpha_n\|x_n - x^*\|\|x_{n+1} - x^*\| \\ &+ 2(1-\beta_n)\alpha_n\sigma_n\langle f(x_n) - x^*, y_n - x_n\rangle + 2(1-\beta_n)\alpha_n\sigma_n\langle f(x_n) - f(x^*), x_n - x^*\rangle \\ &+ 2(1-\beta_n)\alpha_n\sigma_n\langle f(x^*) - x^*, x_n - x^*\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq [1-(1-\beta_n)\alpha_n]\|x_n-x^*\|^2+2(1-\beta_n)(1-\sigma_n)\alpha_n\|f(x_n)-x^*\|\|x_{n+1}-y_n\| \\
&\quad +(1-\beta_n)(1-\sigma_n)\alpha_n\left(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2\right) \\
&\quad +2(1-\beta_n)\alpha_n\sigma_n\|f(x_n)-x^*\|\|y_n-x_n\|+2(1-\beta_n)\alpha_n\sigma_n\rho\|x_n-x^*\|^2 \\
&\quad +2(1-\beta_n)\alpha_n\sigma_n\langle f(x^*)-x^*,x_n-x^*\rangle,
\end{aligned}
\tag{3.47}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 \leq{}& \left[1-\frac{(2\sigma_n(1-\rho)-1)(1-\beta_n)}{1-(1-\beta_n)(1-\sigma_n)\alpha_n}\alpha_n\right]\|x_n-x^*\|^2+\frac{[(2\sigma_n(1-\rho)-1)(1-\beta_n)]\alpha_n}{1-(1-\beta_n)(1-\sigma_n)\alpha_n} \\
&\times\left[\frac{2(1-\sigma_n)}{2\sigma_n(1-\rho)-1}\|f(x_n)-x^*\|\|x_{n+1}-y_n\|+\frac{2\sigma_n}{2\sigma_n(1-\rho)-1}\|f(x_n)-x^*\|\right. \\
&\left.\times\|y_n-x_n\|+\frac{2\sigma_n}{2\sigma_n(1-\rho)-1}\langle f(x_n)-x^*,x_n-x^*\rangle\right].
\end{aligned}
\tag{3.48}
$$

From $\liminf_{n\to\infty}(2\sigma_n(1-\rho)-1)(1-\beta_n)/(1-(1-\beta_n)(1-\sigma_n))>0$, it follows that $\sum_{n=0}^{\infty}((2\sigma_n(1-\rho)-1)(1-\beta_n)/(1-(1-\beta_n)(1-\sigma_n)\alpha_n))\alpha_n=\infty$. It is clear that

$$
\begin{aligned}
\limsup_{n\to\infty}&\left[\frac{2(1-\sigma_n)}{2\sigma_n(1-\rho)-1}\|f(x_n)-x^*\|\|x_{n+1}-y_n\|+\frac{2\sigma_n}{2\sigma_n(1-\rho)-1}\|f(x_n)-x^*\|\right. \\
&\left.\times\|y_n-x_n\|+\frac{2\sigma_n}{2\sigma_n(1-\rho)-1}\langle f(x_n)-x^*,x_n-x^*\rangle\right]\leq 0.
\end{aligned}
\tag{3.49}
$$

Therefore, all conditions of Lemma 2.4 are satisfied. Therefore, we immediately deduce that $x_n \to x^*$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mappings $A,B:C\to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and $T:C\to C$ be a nonexpansive mapping such that $\Omega:=F(T)\cap EP(F)\cap\Gamma\neq\emptyset$.Let $f:C\to C$ be a contraction with coefficient $\rho\in(0,1/2)$. For given $x_0\in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and, $\{u_n\}$ be generated By*

$$
\begin{gathered}
F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in C, \\
z_n=P_C(u_n-\mu Bu_n), \\
y_n=\alpha_n f(x_n)+(1-\alpha_n)P_C(z_n-\lambda Az_n), \\
x_{n+1}=\beta_n x_n+\sigma_n(1-\beta_n)Ty_n+(1-\beta_n)(1-\sigma_n)P_C(z_n-\lambda Az_n),
\end{gathered}
\tag{3.50}
$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, *and sequences* $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [0,1]$, $\{\sigma_n\} \subset [0,1]$, *and* $\{r_n\} \subset (r, \infty)$, $r > 0$, *are such that*

(i) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$*and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $\sigma_n > 1/2(1-\rho)$, $\sum_{n=1}^{\infty}|\sigma_n - \sigma_{n-1}| < \infty$,

(iv) $\sum_{n=1}^{\infty}|r_n - r_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ *generated by* (3.50) *converges strongly to* $x^* = P_\Omega \cdot f(x^*)$, *and* (x^*, y^*) *is a solution of the general system of variational inequalities* (1.5), *where* $y^* = P_C(x^* - \mu B x^*)$.

Acknowledgments

The authors would like to express their thanks to the referee for the valuable comments and suggestions for improving this article. This paper was supported by the NSFC Tianyuan Youth Foundation of Mathematics of China (no. 11126136) and Fundamental Research Funds for the Central Universities (no. ZXH2011C002).

References

[1] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.

[2] V. Colao, G. L. Acedo, and G. Marino, "An implicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of infinite family of nonexpansive mappings," *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 2708–2715, 2009.

[3] Q. L. Dong and Y. Yao, "An iterative method for finding common solutions of systems of equilibrium problems and fixed point problems in Hilbert spaces," *Analele Stiintifice ale Universitatii Ovidius Constanta*, vol. 19, no. 3, pp. 101–116, 2011.

[4] Y. Yao, R. Chen, and Y.-C. Liou, "A unified implicit algorithm for solving the triple-hierarchical constrained optimization problem," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 1506–1515, 2012.

[5] S. Wang and B. Guo, "New iterative scheme with nonexpansive mappings for equilibrium problems and variational inequality problems in Hilbert spaces," *Journal of Computational and Applied Mathematics*, vol. 233, no. 10, pp. 2620–2630, 2010.

[6] Q.-L. Dong and B.-C. Deng, "Strong convergence theorem by hybrid method for equilibrium problems, variational inequality problems and maximal monotone operators," *Nonlinear Analysis. Hybrid Systems*, vol. 4, no. 4, pp. 689–698, 2010.

[7] R. U. Verma, "On a new system of nonlinear variational inequalities and associated iterative algorithms," *Mathematical Sciences Research Hot-Line*, vol. 3, no. 8, pp. 65–68, 1999.

[8] Y. Yao, R. Chen, and H.-K. Xu, "Strong convergence of a modified extra-gradient method to the minimum-norm solution of variational inequalities," *Abstract and Applied Analysis*, vol. 2012, Article ID 817436, 9 pages, 2012.

[9] Y. Yao and N. Shahzad, "Strong convergence of a proximal point algorithm with general errors," *Optimization Letters*. In press.

[10] R. U. Verma, "Iterative algorithms and a new system of nonlinear quasivariational inequalities," *Advances in Nonlinear Variational Inequalities*, vol. 4, no. 1, pp. 117–124, 2001.

[11] Y. Yao, Y.-C. Liou, and S. M. Kang, "Two-step projection methods for a system of variational inequality problems in Banach spaces," *Journal of Global Optimization*. In press.

[12] L.-C. Ceng, C.-Y. Wang, and J.-C. Yao, "Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities," *Mathematical Methods of Operations Research*, vol. 67, no. 3, pp. 375–390, 2008.

[13] W. Kumam and P. Kumam, "Hybrid iterative scheme by a relaxed extragradient method for solutions of equilibrium problems and a general system of variational inequalities with application to optimization," *Nonlinear Analysis. Hybrid Systems*, vol. 3, no. 4, pp. 640–656, 2009.

[14] Y. Yao, Y.-C. Liou, and S. M. Kang, "Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3472–3480, 2010.

[15] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.

[16] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[17] F. Kohsaka and W. Takahashi, "Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces," *SIAM Journal on Optimization*, vol. 19, no. 2, pp. 824–835, 2008.