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Research Article

Positive Solutions to a Generalized Second-Order Difference Equation with Summation Boundary Value Problem

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By using Krasnoselskii's fixed point theorem, we study the existence of positive solutions to the three-point summation boundary value problem $\Delta^2 u(t-1) + a(t)f(u(t)) = 0$, $t \in \{1,2,\ldots,T\}$, $u(0) = \beta\sum_{s=1}^{\eta} u(s)$, $u(T+1) = \alpha\sum_{s=1}^{\eta} u(s)$, where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1,2,\ldots,T-1\}$, $0 < \alpha < (2T+2)/\eta(\eta+1)$, $0 < \beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$, and $\Delta u(t-1) = u(t) - u(t-1)$. We show the existence of at least one positive solution if f is either superlinear or sublinear.

1. Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential and difference equations was initiated by Ilin and Moiseev [1]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by many authors; one may see the text books [3, 4] and the papers [5–10]. However, all these papers are concerned with problems with three-point boundary condition restrictions on the difference of the solutions and the solutions themselves, for example,

$$\begin{aligned} u(0) &= 0, \qquad u(T+1) = 0, \\ u(0) &= 0, \qquad au(s) = u(T+1), \\ u(0) &= 0, \qquad u(T+1) - au(s) = b, \end{aligned}$$

$$u(0) - \alpha\Delta u(0) = 0, \qquad u(T+1) = \beta u(s),$$
$$u(0) - \alpha\Delta u(0) = 0, \qquad \Delta u(T+1) = 0, \tag{1.1}$$

and so forth.

In [5], Leggett-Williams developed a fixed point theorem to prove the existence of three positive solutions for Hammerstein integral equations. Since then, this theorem has been reported to be a successful technique for dealing with the existence of three solutions for the two-point boundary value problems of differential and difference equations; see [6, 7]. In [8], X. Lin and W. Liu, using the properties of the associate Green's function and Leggett-Williams fixed point theorem, studied the existence of positive solutions of the problem.

In [9], Zhang and Medina studied the existence of positive solutions for second-order boundary value problems of difference equations by applying Krasnoselskii's fixed point theorem. In [10], Henderson and Thompson used lower and upper solution methods to study the existence of multiple solutions for second-order discrete boundary value problems.

We are interested in the existence of positive solutions of the following second-order difference equation with three-point summation boundary value problem (BVP):

$$\Delta^2 u(t-1) + a(t)f(u(t)) = 0, \quad t \in \{1, 2, \ldots, T\},$$
$$u(0) = \beta\sum_{s=1}^{\eta} u(s), \qquad u(T+1) = \alpha\sum_{s=1}^{\eta} u(s), \tag{1.2}$$

where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, \ldots, T-1\}$.

The aim of this paper is to give some results for existence of positive solutions to (1.2), assuming that $0 < \alpha < (2T+2)/\eta(\eta+1)$, $0 < \beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$, and f is either superlinear or sublinear. Set

$$f_0 = \lim_{u\to 0^+}\frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to\infty}\frac{f(u)}{u}. \tag{1.3}$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. Let $\mathbb{N}$ be the nonnegative integer; we let $\mathbb{N}_{i,j} = \{k \in \mathbb{N} \mid i \leq k \leq j\}$ and $\mathbb{N}_p = \mathbb{N}_{0,p}$. By the positive solution of (1.2), we mean that a function $u(t) : \mathbb{N}_{T+1} \to [0, \infty)$ and satisfies the problem (1.2).

Recently, Sitthiwirattham [11] proved the existence of positive solutions for the boundary value problem with summation condition

$$\Delta^2 u(t-1) + a(t)f(u(t)) = 0, \quad t \in \{1, 2, \ldots, T\},$$
$$u(0) = 0, \qquad u(T+1) = \alpha\sum_{s=1}^{\eta} u(s), \tag{1.4}$$

where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, \ldots, T-1\}$, and $0 < \alpha < 2T + 2/\eta(\eta+1)$.

Throughout this paper, we suppose the following conditions hold:

$(A1)$ $f \in C([0, \infty), [0, \infty))$;

$(A2)$ $a \in C(\mathbb{N}_{T+1}, [0, \infty))$ and there exists $t_0 \in \mathbb{N}_{\eta,T+1}$ such that $a(t_0) > 0$.

The proof of the main theorem is based upon an application of the following Krasnoselskii's fixed point theorem in a cone.

Theorem 1.1 (see [12]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \longrightarrow K \tag{1.5}$$

be a completely continuous operator such that

(i) $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$, *or*

(ii) $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 2.1. *Let $\beta \neq (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$. Then, for $y \in C(\mathbb{N}_{T+1}, [0,\infty))$, the problem*

$$\Delta^2 u(t-1) + y(t) = 0, \quad t \in N_{1,T}, \tag{2.1}$$

$$u(0) = \beta\sum_{s=1}^{\eta} u(s), \qquad u(T+1) = \alpha\sum_{s=1}^{\eta} u(s), \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t) = {} & \frac{\beta\eta(\eta+1) + 2t(1-\beta\eta)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)y(s) \\ & - \frac{\beta(T+1) - (\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s-1)y(s) \\ & - \sum_{s=1}^{t-1}(t-s)y(s), \quad t \in \mathbb{N}_{T+1}. \end{aligned} \tag{2.3}$$

Proof. From (2.1), we get

$$\begin{gathered} \Delta u(t) - \Delta u(t-1) = -y(t), \\ \Delta u(t-1) - \Delta u(t-2) = -y(t-1), \\ \vdots \\ \Delta u(1) - \Delta u(0) = -y(1). \end{gathered} \tag{2.4}$$

We sum the above equations to obtain

$$\Delta u(t) = \Delta u(0) - \sum_{s=1}^{t} y(s), \quad t \in \mathbb{N}_T, \tag{2.5}$$

we denote $\sum_{s=p}^{q} y(s) = 0$, if $p > q$. Similarly, summing the above equation from $t = 0$ to $t = h$, we get

$$u(h+1) = u(0) + (h+1)\Delta u(0) - \sum_{s=1}^{h} (h+1-s) y(s), \quad h \in \mathbb{N}_T, \tag{2.6}$$

changing the variable from $h+1$ to t, we have

$$u(t) = u(0) + t\Delta u(0) - \sum_{s=1}^{t-1} (t-s) y(s) : A + Bt - \sum_{s=1}^{t-1} (t-s) y(s), \quad t \in \mathbb{N}_{T+1}. \tag{2.7}$$

We sum (2.7) from $s = 1, 2, \dots, \eta$,

$$\begin{aligned} \sum_{s=1}^{\eta} u(s) &= \eta A + \frac{\eta(\eta+1)}{2} B - \sum_{s=1}^{\eta-1} \sum_{l=1}^{\eta-s} (l-s) y(s) \\ &= \eta A + \frac{\eta(\eta+1)}{2} B - \frac{1}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s + 1) y(s). \end{aligned} \tag{2.8}$$

By (2.2) from $u(0) = \beta \sum_{s=1}^{\eta} u(s)$, we get

$$(1 - \beta\eta) A - \frac{\beta\eta(\eta+1)}{2} B = -\frac{\beta}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s + 1) y(s), \tag{2.9}$$

and from $u(T+1) = \alpha \sum_{s=1}^{\eta} u(s)$, we obtain

$$(1 - \alpha\eta) A + \left(T + 1 - \frac{\alpha\eta(\eta+1)}{2} \right) B = \sum_{s=1}^{T} (T - s + 1) y(s) - \frac{\alpha}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s + 1) y(s). \tag{2.10}$$

Therefore,

$$A = \frac{\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)y(s) - \frac{\beta(T+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s),$$
$$B = \frac{2(1-\beta\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)y(s) + \frac{\beta-\alpha}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s). \tag{2.11}$$

Hence, (2.1)-(2.2) has a unique solution

$$u(t) = \frac{\beta\eta(\eta+1)+2t(1-\beta\eta)}{(2t+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)y(s) - \frac{\beta(T+1)-(\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s) - \sum_{s=1}^{t-1}(t-s)y(s), \quad t \in \mathbb{N}_{T+1}. \tag{2.12}$$

□

Lemma 2.2. *Let* $0 < \alpha < (2T+2)/\eta(\eta+1)$, $0 < \beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$. *If* $y \in C(\mathbb{N}_{T+1}, [0,\infty))$ *and* $y(t) \geqslant 0$ *for* $t \in \mathbb{N}_{1,T}$, *then the unique solution* u *of* (2.1)-(2.2) *satisfies* $u(t) \geqslant 0$ *for* $t \in \mathbb{N}_{T+1}$.

Proof. From the fact that $\Delta^2 u(t-1) = u(t+1) - 2u(t) + u(t-1) = -y(t) \leq 0$, we know that $u(t) \geq (u(t+1)+u(t-1))/2$, so $u(t+1)/(t+1) < u(t)/t$.

Hence

$$\frac{u(T+1)-u(0)}{T+1} < \frac{u(\eta)-u(0)}{\eta+1}, \quad \eta \in \mathbb{N}_{1,T}, \tag{2.13}$$

since $u(T) \geqslant 0$ and $u(0) \geq 0$ imply that $u(t) \geqslant 0$ for $t \in \mathbb{N}_{T+1}$.

Moreover, from $u(i) > (i/\eta)u(\eta) + ((\eta - i)/\eta)u(0)$, we get

$$\begin{aligned}\sum_{s=1}^{\eta} u(s) &> \left[\frac{1}{\eta}u(\eta) + \frac{\eta-1}{\eta}u(0)\right] + \left[\frac{2}{\eta}u(\eta) + \frac{\eta-2}{\eta}u(0)\right] + \cdots + \left[\frac{\eta}{\eta}u(\eta) + \frac{\eta-\eta}{\eta}u(0)\right] \\ &= \frac{1}{\eta}u(\eta)\left[1+2+\cdots+\eta\right] + \frac{1}{\eta}u(0)\left[(\eta-1)+(\eta-2)+\cdots+0\right] \\ &= \frac{1}{\eta}u(\eta)\left[\frac{1}{2}\eta(\eta+1)\right] + \frac{1}{\eta}u(0)\left[\eta^2 - \frac{1}{2}\eta(\eta+1)\right] \\ &= \frac{1}{2}(\eta+1)u(\eta) + \frac{1}{2}(\eta-1)u(0).\end{aligned} \tag{2.14}$$

Combining (2.14) with (2.2), we can get

$$u(0) > \frac{\beta(\eta+1)}{2-\beta(\eta-1)}u(\eta), \tag{2.15}$$

again combining (2.2), (2.14), and (2.15), we obtain

$$u(T+1) > \frac{\alpha(\eta+1)}{2-\beta(\eta-1)}u(\eta), \tag{2.16}$$

such that

$$2-\beta(\eta-1) > 2-\beta\eta > 2 - \frac{2T+2-\alpha\eta(\eta+1)}{2T-\eta+1} = \frac{2(T-\eta)+\alpha\eta(\eta+1)}{2T-\eta+1} > 0. \tag{2.17}$$

By using (2.13), (2.15), and (2.16), we obtain

$$\frac{2-2\beta\eta}{\eta}u(\eta) \geqslant \frac{(\alpha-\beta)(\eta+1)}{T+1}u(\eta). \tag{2.18}$$

If $u(0) < 0$, then $u(\eta) < 0$. It implies that $(2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1) \leqslant \beta$, a contradiction to $\beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$. If $u(T) < 0$, then $u(\eta) < 0$, and the same contradiction emerges. Thus, it is true that $u(0) \geqslant 0$, $u(T) \geqslant 0$, together with (2.13), we have

$$u(t) \geqslant 0, \quad t \in \mathbb{N}_{T+1}. \tag{2.19}$$

This proof is complete. □

Lemma 2.3. *Let* $\alpha\eta(\eta+1) > 2T+2$, $\beta > \max\{(2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1), 0\}$. *If* $y \in C(\mathbb{N}_{T+1}, [0,\infty))$ *and* $y(t) \geqslant 0$ *for* $t \in \mathbb{N}_{1,T}$, *then problem* (2.1)-(2.2) *has no positive solutions.*

Proof. Suppose that problem (2.1)-(2.2) has a positive solution u satisfying $u(t) \geqslant 0$, $t \in \mathbb{N}_{T+1}$, and there is a $\tau_0 \in \mathbb{N}_{1,T}$ such that $u(\tau_0) > 0$.

If $u(T+1) > 0$, then $\sum_{s=1}^{\eta} u(s) > 0$. It implies

$$\begin{aligned} u(0) = \beta\sum_{s=1}^{\eta} u(s) &> \frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}\sum_{s=1}^{\eta} u(s) \\ &\geqslant \frac{\eta(T+1)(u(0)+u(\eta))-\eta(\eta+1)u(T+1)}{\eta(2T-\eta+1)}, \end{aligned} \tag{2.20}$$

that is,

$$\frac{u(T+1)-u(0)}{T+1} > \frac{u(\eta)-u(0)}{\eta+1}, \tag{2.21}$$

which is a contradiction to (2.13).

If $u(T+1) = 0$, then $\sum_{s=1}^{\eta} u(s)ds = 0$. When $\tau_0 \in \mathbb{N}_{1,\eta-1}$, we get $u(\tau_0) > u(T) = 0 > u(\eta)$, which contradicts to (2.13). When $\tau_0 \in \mathbb{N}_{\eta+1,T}$, we get $u(\eta) \leqslant 0 = u(0) < u(\tau_0)$, which contradicts to (2.13) again. Therefore, no positive solutions exist. □

Let $E = C(\mathbb{N}_{T+1}, [0,\infty))$, then E is a Banach space with respect to the norm

$$\|u\| = \sup_{t\in\mathbb{N}_{T+1}} |u(t)|. \tag{2.22}$$

Lemma 2.4. *Let* $0 < \alpha < (2T+2)/\eta(\eta+1)$, $0 < \beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$. *If* $y \in C(\mathbb{N}_{T+1}, [0,\infty))$ *and* $y(t) \geqslant 0$, *then the unique solution to problem* (2.1)-(2.2) *satisfies*

$$\inf_{t\in\mathbb{N}_{T+1}} u(t) \geqslant \gamma\|u\|, \tag{2.23}$$

where

$$\begin{aligned} \gamma := \min\Bigg\{ &\frac{\alpha(\eta+1)(T+1-\eta)}{(T+1)(2-\beta(\eta-1))-\alpha\eta(\eta+1)}, \frac{\alpha\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)}, \\ &\frac{\beta(\eta+1)(T+1-\eta)}{(2-\beta(\eta-1))(T+1)}, \frac{\beta\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)} \Bigg\}. \end{aligned} \tag{2.24}$$

Proof. Let $u(t)$ be maximal at $t = \tau_1$, when $\tau_1 \in \mathbb{N}_{1,T}$ and $\|u\| = u(\tau_1)$. We divide the proof into two cases.

Case i. If $u(0) \geqslant u(T+1)$ and $\inf_{t\in\mathbb{N}_{T+1}} u(t) = u(T+1)$, then either $0 \leqslant \tau_1 \leqslant \eta < T+1$ or $0 < \eta < \tau_1 < T+1$, if $0 \leqslant \tau_1 \leqslant \eta < T+1$, then

$$\begin{aligned} u(\tau_1) &\leqslant u(T+1) + \frac{u(T+1)-u(\eta)}{T+1-\eta}(\tau_1 - (T+1)) \\ &\leqslant u(T+1) + \frac{u(T+1)-u(\eta)}{T+1-\eta}(0 - (T+1)) \\ &\leqslant u(T+1)\left[1 - \left(\frac{(T+1)-(T+1)(2-\beta(\eta-1))/\alpha(\eta+1)}{T+1-\eta}\right)\right] \\ &\leqslant u(T+1)\left[\frac{(T+1)(2-\beta(\eta-1)) - \alpha\eta(\eta+1)}{\alpha(T+1)(T+1-\eta)}\right]. \end{aligned} \tag{2.25}$$

This implies

$$\inf_{t\in\mathbb{N}_{T+1}} u(t) \geqslant \frac{\alpha(T+1)(T+1-\eta)}{(T+1)(2-\beta(\eta-1)) - \alpha\eta(\eta+1)}\|u\|. \tag{2.26}$$

Similarly, if $0 < \eta < \tau_1 < T+1$, from

$$\frac{u(\eta)}{\eta} \geqslant \frac{u(\tau_1)}{\tau_1} \geqslant \frac{u(\tau_1)}{T+1}, \tag{2.27}$$

together with (2.16), we have

$$u(T+1) \geqslant \frac{\alpha\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)}\, u(\tau_1). \tag{2.28}$$

This implies

$$\inf_{t\in\mathbb{N}_{T+1}} u(t) \geqslant \frac{\alpha\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)}\, \|u\|. \tag{2.29}$$

Case ii. If $u(0) \leqslant u(T+1)$ and $\inf_{t\in\mathbb{N}_{T+1}} u(t) = u(0)$, then either $0 < \tau_1 < \eta < T+1$ or $0 < \eta \leqslant \tau_1 \leqslant T+1$, by (2.13). If $0 < \tau_1 < \eta < T+1$, from

$$\frac{u(\eta)}{T+1-\eta} \geqslant \frac{u(\tau_1)}{T+1-\tau_1} \geqslant \frac{u(\tau_1)}{T+1}, \tag{2.30}$$

together with (2.15), we have

$$u(0) > \frac{\beta(\eta+1)(T+1-\eta)}{(2-\beta(\eta-1))(T+1)}\, u(\tau_1). \tag{2.31}$$

Hence,

$$\inf_{t\in\mathbb{N}_{T+1}} u(t) > \frac{\beta(\eta+1)(T+1-\eta)}{(2-\beta(\eta-1))(T+1)} \|u\|. \tag{2.32}$$

If $0 < \eta \leqslant \tau_1 \leqslant T+1$, from

$$\frac{u(\tau_1)}{T+1} \leqslant \frac{u(\tau_1)}{\tau_1} \leqslant \frac{u(\eta)}{\eta}, \tag{2.33}$$

together with (2.15), we have

$$u(0) < \frac{\beta\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)} u(\tau_1). \tag{2.34}$$

This implies

$$\inf_{t\in N_{T+1}} u(t) < \frac{\beta\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)} \|u\|. \tag{2.35}$$

This completes the proof. □

In the rest of the paper, we assume that $0 < \alpha < (2T+2)/\eta(\eta+1)$, $T \in \mathbb{N}_{1,T}$; $0 < \beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$. It is easy to see that the BVP (1.2) has a solution $u = u(t)$ if and only if u is a solution of the operator equation

$$\begin{aligned} Au(t) \triangleq{} & \frac{\beta\eta(\eta+1)+2t(1-\beta\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)u(s)f(u(s)) \\ & - \frac{\beta(T+1)-(\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)u(s)f(u(s)) \\ & - \sum_{s=1}^{t-1}(t-s)u(s)f(u(s)). \end{aligned} \tag{2.36}$$

Denote

$$K = \left\{ u \in E : u \geqslant 0, \min_{t\in\mathbb{N}_{T+1}} u(t) \geqslant \gamma\|u\| \right\}, \tag{2.37}$$

where γ is defined in (2.24).

It is obvious that K is a cone in E. Since $Au = u$ and from Lemmas 2.2 and 2.4, then $A(K) \subset K$. It is also easy to check that $A : K \to K$ is completely continuous. In the following, for the sake of convenience, set

$$\begin{aligned}
\Lambda_1 &= \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s),\\
\Lambda_2 &= \frac{\gamma(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}sa(s).
\end{aligned} \tag{2.38}$$

3. Main Results

Now we are in the position to establish the main result.

Theorem 3.1. *The BVP* (1.2) *has at least one positive solution in the case*

(H_1) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear) or*

(H_2) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Proof. Superlinear Case

Let (H_1) hold. Since $f_0 = \lim_{u\to 0^+}(f(u)/u) = 0$ for any $\varepsilon \in (0, \Lambda_1^{-1}]$, there exists ρ_* such that

$$f(u) \leqslant \varepsilon u \quad \text{for } u \in [0, \rho_*]. \tag{3.1}$$

Let $\Omega_{\rho_*} = \{u \in E : \|u\| < \rho_*\}$ for any $u \in K \cap \partial\Omega_{\rho_*}$. From (3.1), we get

$$\begin{aligned}
Au(t) &= \frac{2t(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s))\\
&\quad - \frac{\beta(T+1)-(\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)a(s)f(u(s))\\
&\quad - \sum_{s=1}^{t-1}(t-s)a(s)f(u(s))\\
&\leqslant \frac{2t(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s))\\
&\leqslant \frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s))\\
&\leqslant \varepsilon\rho_*\frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)\\
&= \varepsilon\Lambda_1\rho_* \leqslant \rho_* = \|u\|,
\end{aligned} \tag{3.2}$$

which yields

$$\|Au\| \leqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{\rho_*}. \tag{3.3}$$

Further, since $f_\infty = \lim_{u\to\infty}(f(u)/u) = \infty$, then, for any $M^* \in [\Lambda_2^{-1}, \infty)$, there exists $\rho^* > \rho_*$ such that

$$f(u) \geqslant M^* u \quad \text{for } u \geqslant \gamma\rho^*. \tag{3.4}$$

Set $\Omega_{\rho^*} = \{u \in E : \|u\| < \rho^*\}$ for $u \in K \cap \partial\Omega_{\rho^*}$.

Since $u \in K$, $\min_{t\in N_T} u(t) \geqslant \gamma\|u\| = \gamma\rho^*$. Hence, for any $u \in K \cap \Omega_{\rho^*}$, from (3.4) and (2.23), we get

$$\begin{aligned}
Au(\eta) &= \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\
&\quad - \frac{\beta(T+1)-(\beta-\alpha)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta}(\eta-s)(\eta-s+1)a(s)f(u(s)) \\
&\quad - \sum_{s=1}^{\eta-1}(\eta-s)a(s)f(u(s)) \\
&= \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\
&\quad + \frac{1}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \\
&\quad \times \sum_{s=1}^{\eta-1}(\eta-s)\left[-(2-\beta\eta+\beta)T+(\beta(T-\eta)+\alpha\eta+1)s+(\eta-1)\beta\right]a(s)f(u(s)) \\
&= \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\
&\quad - \frac{T(2-\beta\eta+\beta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)a(s)f(u(s)) \\
&\quad + \frac{\beta(t-\eta)+\alpha\eta+1}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}\left(\eta s-s^2\right)a(s)f(u(s)) \\
&\quad + \frac{(\eta-1)\beta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)a(s)f(u(s)) \\
&\geqslant \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s))
\end{aligned}$$

$$
\begin{aligned}
&- \frac{T(2-\beta\eta+\beta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(\eta-s)a(s)f(u(s))\\
&= \frac{(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}sa(s)f(u(s))\\
&\geqslant \gamma\rho^* M^* \frac{(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}sa(s) = M^*\Lambda_2\rho^*\\
&\geqslant \rho^* = \|u\|,
\end{aligned}
\tag{3.5}
$$

which implies

$$
\|Au\| \geqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{\rho^*}. \tag{3.6}
$$

Therefore, from (3.3), (3.6), and Theorem 1.1, it follows that A has a fixed point in $K \cap (\overline{\Omega}_{\rho^*} \setminus \Omega_{\rho_*})$ such that $\rho_* \leqslant \|u\| \leqslant \rho^*$.

Sublinear Case

Let (H_2) hold. In view of $f_0 = \lim_{u\to 0^+}(f(u)/u) = \infty$ for any $M_* \in [\Lambda_2^{-1}, \infty)$, there exists $r_* > 0$ such that

$$
f(u) \geqslant M_* u \quad \text{for } 0 \leqslant u \leqslant r_*. \tag{3.7}
$$

Set $\Omega_{r_*} = \{u \in E : \|u\| < r_*\}$ for $u \in K \cap \partial\Omega_{r_*}$. Since $u \in K$, then $\min_{t\in\mathbb{N}_{T+1}} u(t) \geqslant \gamma\|u\| = \gamma r_*$. Thus, from (3.7) for any $u \in K \cap \partial\Omega_{r_*}$, we can get

$$
\begin{aligned}
Au(\eta) &= \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s))\\
&\quad - \frac{\beta(T+1)-(\beta-\alpha)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)a(s)f(u(s))\\
&\quad - \sum_{s=1}^{\eta-1}(\eta-s)a(s)f(u(s))\\
&\geqslant \gamma r_* M_* \frac{(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}sa(s) = M_*\Lambda_2 r_* \geqslant r_* = \|u\|,
\end{aligned}
\tag{3.8}
$$

which yields

$$
\|Au\| \geqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{r_*}. \tag{3.9}
$$

Since $f_\infty = \lim_{u\to\infty}(f(u)/u) = 0$, then, for any $\varepsilon_1 \in (0, \Lambda_1^{-1}]$, there exists $r_0 > r_*$ such that

$$f(u) \leqslant \varepsilon_1 u \quad \text{for } u \in [r_0, \infty). \tag{3.10}$$

We have the following two cases.

Case i. Suppose that $f(u)$ is unbounded, then, from $f \in C([0,\infty),[0,\infty))$, we know that there is $r^* > r_0$ such that

$$f(u) \leqslant f(r^*) \quad \text{for } u \in [0, r^*]. \tag{3.11}$$

Since $r^* > r_0$, then, from (3.10) and (3.11), one has

$$f(u) \leqslant f(r^*) \leqslant \varepsilon_1 r^* \quad \text{for } u \in [0, r^*]. \tag{3.12}$$

For $u \in K$, $\|u\| = r^*$, from (3.12), we obtain

$$\begin{aligned} Au(t) &\leqslant \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\ &\leqslant \varepsilon_1 r^* \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s) \\ &= \varepsilon_1\Lambda_1 r^* \leqslant r^* = \|u\|. \end{aligned} \tag{3.13}$$

Case ii. Suppose that $f(u)$ is bounded, say $f(u) \leqslant N$ for all $u \in [0,\infty)$. Taking $r^* \geqslant \max\{N/\varepsilon_1, r_*\}$, for $u \in K$, $\|u\| = r^*$, we have

$$\begin{aligned} Au(t) &\leqslant \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\ &\leqslant N\frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s) \\ &\leqslant \varepsilon_1 r^* \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s) \\ &= \varepsilon_1\Lambda_1 r^* \leqslant r^* = \|u\|. \end{aligned} \tag{3.14}$$

Hence, in either case, we may always set $\Omega_{r^*} = \{u \in E : \|u\| < r^*\}$ such that

$$\|Au\| \leqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{r^*}. \tag{3.15}$$

Hence, from (3.9), (3.15), and Theorem 1.1, it follows that A has a fixed point in $K \cap (\overline{\Omega}_{\rho^*} \setminus \Omega_{\rho_*})$ such that $r_* \leqslant \|u\| \leqslant r^*$. The proof is complete. □

4. Some Examples

In this section, in order to illustrate our result, we consider some examples.

Example 4.1. Consider the BVP

$$\Delta^2 u(\mathrm{t}-1)+t^2u^k=0, \quad t\in N_{1,4},$$
$$u(0)=\frac{1}{3}\sum_{s=1}^{2}u(s), \qquad u(5)=\frac{2}{3}\sum_{s=1}^{2}u(s). \tag{4.1}$$

Set $\alpha=2/3$, $\beta=1/3$, $\eta=2$, $T=4$, $a(t)=t^2$, and $f(u)=u^k$.

We can show that

$$0<\alpha=\frac{2}{3}<\frac{5}{3}=\frac{2T+2}{\eta(\eta+1)}, \qquad 0<\beta=\frac{1}{3}<\frac{3}{7}=\frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}. \tag{4.2}$$

Case I. $k\in(1,\infty)$. In this case, $f_0=0$, $f_\infty=\infty$, and H_1 of Theorem 3.1 holds. Then BVP (4.1) has at least one positive solution.

Case II. $k\in(0,1)$. In this case, $f_0=\infty$, $f_\infty=0$, and H_2 of Theorem 3.1 holds. Then BVP (4.1) has at least one positive solution.

Example 4.2. Consider the BVP

$$\Delta^2 u(t-1)+e^t t^e\left(\frac{\pi\sin u+2\cos u}{u^2}\right)=0, \quad t\in N_{1,4},$$
$$u(0)=\frac{1}{4}\sum_{s=1}^{3}u(s), \qquad u(5)=\frac{1}{3}\sum_{s=1}^{3}u(s). \tag{4.3}$$

Set $\alpha=1/3$, $\beta=1/4$, $\eta=3$, $T=4$, $a(t)=e^t t^e$, $f(u)=(\pi\sin u+2\cos u)/u^2$.

We can show that

$$0<\alpha=\frac{1}{3}<\frac{5}{6}=\frac{2T+2}{\eta(\eta+1)}, \qquad 0<\beta=\frac{1}{4}<\frac{1}{3}=\frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}. \tag{4.4}$$

Through a simple calculation we can get $f_0=\infty$, $f_\infty=0$. Thus, by H_2 of Theorem 3.1, we can get BVP (4.3) has at least one positive solution.

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