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*Research Article*

# General Helices of AW(k)-Type in the Lie Group

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We study curves of AW(k)-type in the Lie group $G$ with a bi-invariant metric. Also, we characterize general helices in terms of AW(k)-type curve in the Lie group $G$.

## 1. Introduction

The geometry of curves and surfaces in a 3-dimensional Euclidean space $\mathbb{R}^3$ represented for many years a popular topic in the field of classical differential geometry. One of the important problems of the curve theory is that of Bertrand-Lancret-de Saint Venant saying that a curve in $\mathbb{R}^3$ is of constant slop; namely, its tangent makes a constant angle with a fixed direction if and only if the ratio of torsion $\tau$ and curvature $\kappa$ is a constant. These curves are said to be general helices. If both $\tau$ and $\kappa$ are nonzero constants, the curve is called cylindrical helix. Helix is one of the most fascinating curves in science and nature. Scientists have long held a fascinating, sometimes bordering on mystical obsession for helical structures in nature. Helices arise in nanosprings, carbon nanotubes, $\alpha$-helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA.

The problem of Bertrand-Lancret-de Saint Venant was generalized for curves in other 3-dimensional manifolds—in particular space forms or Sasakian manifolds. Such a curve has the property that its tangent makes a constant angle with a parallel vector field on the manifold or with a Killing vector field, respectively. For example, a curve $\alpha(s)$ in a 3-dimensional space form is called a general helix if there exists a Killing vector field $V(s)$ with constant length along $\alpha$ and such that the angle between $V$ and $\alpha'$ is a non-zero constant (see [1]). A general helix defined by a parallel vector field was studied in [2]. Moreover, in [3] it is shown that general helices in a 3-dimensional space form are extremal curvatures of a functional involving a linear combination of the curvature, the torsion, and a constant. General helices also called the Lancret curves are used in many applications (e.g., [4–7]).

The notion of AW(k)-type submanifolds was introduced by Arslan and West in [8]. In particular, many works related to curves of AW(k)-type have been done by several authors. For example, in [9, 10] the authors gave curvature conditions and charaterizations related to these curves in $\mathbb{R}^n$. Also, in [11] they investigated curves of AW(k) type in a 3-dimensional null cone and gave curvature conditions of these kinds of curves. However, to the author's knowledge, there is no article dedicated to studying the notion of AW(k)-type curves immersed in Lie group.

In this paper, we investigate curvature conditions of curves of AW(k)-type in the Lie group $G$ with a bi-invariant metric. Moreover, we characterize general helices of AW(k)-type in the Lie group $G$.

## 2. Preliminaries

Let $G$ be a Lie group with a bi-invariant metric $\langle , \rangle$ and $D$ the Levi-Civita connection of the Lie group $G$. If $\mathfrak{g}$ denotes the Lie algebra of $G$, then we know that $\mathfrak{g}$ is isomorphic to $T_eG$, where $e$ is identity of $G$. If $\langle , \rangle$ is a bi-invariant metric on $G$, then we have

$$\begin{gathered} \langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle, \\ D_X Y = \frac{1}{2}[X, Y] \end{gathered} \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{g}$.

Let $\alpha : I \subset \mathbb{R} \to G$ be a unit speed curve with parameter $s$ and $\{V_1, V_2, \ldots, V_n\}$ an orthonrmal basis of $\mathfrak{g}$. In this case, we write that any vector fields $W$ and $Z$ along the curve $\alpha$ as $W = \sum_{i=1}^{n} w_i V_i$ and $Z = \sum_{i=1}^{n} z_i V_i$, where $w_i : I \to \mathbb{R}$ and $z_i : I \to \mathbb{R}$ are smooth functions. Furthermore, the Lie bracket of two vector fields $W$ and $Z$ is given by

$$[W, Z] = \sum_{i=1}^{n} w_i z_j [V_i, V_j]. \tag{2.2}$$

Let $D_{\alpha'} W$ be the covariant derivative of $W$ along the curve $\alpha$, $V_1 = \alpha'$, and $W' = \sum_{i=1}^{n} w_i' V_i$, where $w_i' = dw_i/ds$. Then we have

$$D_{\alpha'} W = W' + \frac{1}{2}[V_1, W]. \tag{2.3}$$

A curve $\alpha$ is called a Frenet curve of osculating order $d$ if its derivatives $\alpha'(s)$, $\alpha''(s)$, $\alpha'''(s)$,…,$\alpha^{(d)}(s)$ are linearly dependent and $\alpha'(s), \alpha''(s), \alpha'''(s), \ldots, \alpha^{(d+1)}(s)$ are no longer linearly independent for all $s$. To each Frenet curve of order $d$ one can associate an orthonormal $d$-frame $V_1(s), V_2(s), V_3(s), \ldots, V_d(s)$ along $\alpha$ (such that $\alpha'(s) = V_1(s)$) called the Frenet frame and the functions $k_1, k_2, \ldots, k_{d-1} : I \to \mathbb{R}$ said to be the Frenet curvatures, such that the Frenet formulas are defined in the usual way:

$$\begin{aligned}
D_{V_1} V_1(s) &= k_1(s) V_2(s), \\
D_{V_1} V_2(s) &= -k_1(s) V_1(s) + k_2(s) V_3(s), \\
&\vdots \\
D_{V_1} V_i(s) &= -k_{i-1}(s) V_{i-1}(s) + k_i(s) V_{i+1}(s), \\
D_{V_1} V_{i+1}(s) &= -k_i(s) V_i(s).
\end{aligned} \tag{2.4}$$

If $\alpha : I \to G$ is a Frenet curve of osculating order 3 in $G$, then we define

$$\overline{k}_2(s) = \frac{1}{2}\langle [V_1, V_2], V_3 \rangle. \tag{2.5}$$

**Proposition 2.1.** *Let $\alpha$ be a Frenet curve of osculating order 3 in $G$. Then one has*

$$\begin{aligned}
[V_1, V_2] &= \langle [V_1, V_2], V_3 \rangle V_3 = 2\overline{k}_2 V_3, \\
[V_1, V_3] &= \langle [V_1, V_3], V_2 \rangle V_2 = -2\overline{k}_2 V_2, \\
[V_2, V_3] &= \langle [V_2, V_3], V_1 \rangle V_1 = 2\overline{k}_2 V_1.
\end{aligned} \tag{2.6}$$

*Proof.* Let $\alpha$ be a Frenet curve of osculating order 3 with the Frenet frame $\{V_1, V_2, V_3\}$. Since $[V_1, V_2] = a_1 V_1 + a_2 V_2 + a_3 V_3$, taking the inner product with $V_1$, $V_2$, and $V_3$, respectively, we have $a_1 = a_2 = 0$ and $\langle [V_1, V_2], V_3 \rangle = a_3$. Thus, we find

$$[V_1, V_2] = \langle [V_1, V_2], V_3 \rangle V_3. \tag{2.7}$$

From (2.5), we get

$$[V_1, V_2] = 2\overline{k}_2 V_3. \tag{2.8}$$

By using the above similar method, we can obtain $[V_1, V_3] = -2\overline{k}_2 V_2$ and $[V_2, V_3] = 2\overline{k}_2 V_1$. □

*Remark 2.2.* Let $G$ be a 3-dimensional Lie group with a bi-invariant metric. Then it is one of the Lie groups $SO(3)$, $S^3$ or a commutative group, and the following statements hold (see [6, 12]).

(i) If $G$ is $SO(3)$, then $\overline{k}_2(s) = 1/2$.

(ii) If $G$ is $S^3 \cong SU(2)$, then $\overline{k}_2(s) = 1$.

(iii) If $G$ is a commutative group, then $\overline{k}_2(s) = 0$.

**Proposition 2.3.** *Let $\alpha$ be a Frenet curve of osculating order 3 in $G$. Then one has*

$$\begin{aligned}
\alpha'(s) &= V_1(s),\\
\alpha''(s) &= k_1(s)V_2(s),\\
\alpha'''(s) &= -k_1^2(s)V_1(s) + k_1'(s)V_2(s) + k_1(s)\tau_1(s)V_3(s),\\
\alpha''''(s) &= -3k_1(s)k_1'(s)V_1(s) + \Big[k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\overline{k}_2(s)\\
&\qquad -k_1(s)\overline{k}_2^{\,2}(s)\Big]V_2(s) + \left(2k_1'(s)\tau_1(s) + k_1(s)\tau_1'(s)\right)V_3(s),
\end{aligned} \tag{2.9}$$

*where $\tau_1(s) = k_2(s) - \overline{k}_2(s)$.*

*Proof.* Let $\alpha$ be a Frenet curve of osculating order 3 in $G$. Then we have

$$\alpha''(s) = \frac{d^2\alpha}{ds^2} = V_1'(s) = D_{V_1}V_1(s) - \frac{1}{2}[V_1(s), V_1(s)] = k_1(s)V_2(s). \tag{2.10}$$

This implies that

$$\begin{aligned}
\alpha'''(s) &= k_1'(s)V_2(s) + k_1(s)V_2'(s)\\
&= k_1'(s)V_2(s) + k_1(s)\left(D_{V_1}V_2(s) - \frac{1}{2}[V_1(s), V_2(s)]\right)\\
&= k_1'(s)V_2(s) + k_1(s)\left(-k_1(s)V_1(s) + k_3(s) - \overline{k}_2(s)V_3(s)\right)\\
&= -k_1^2(s)V_1(s) + k_1'(s)V_2(s) + k_1(s)\left(k_2(s) - \overline{k}_2(s)\right)V_3(s).
\end{aligned} \tag{2.11}$$

Also, we have the following:

$$\begin{aligned}
\alpha''''(s) &= -2k_1(s)k_1'(s)V_1(s) + k_1''(s)V_2(s) + \left(k_1(s)k_2(s) - k_1(s)\overline{k}_2(s)\right)'V_3(s)\\
&\quad - k_1^2(s)V_1'(s) + k_1'(s)V_2'(s) + k_1(s)\left(k_2(s) - \overline{k}_2(s)\right)V_3'(s)\\
&= -2k_1(s)k_1'(s)V_1(s) + k_1''(s)V_2(s) + \left(k_1(s)k_2(s) - k_1(s)\overline{k}_2(s)\right)'V_3(s)\\
&\quad - k_1^2(s)\left(D_{V_1}V_1(s) - \frac{1}{2}[V_1(s), V_1(s)]\right) + k_1'(s)\left(D_{V_1}V_2(s) - \frac{1}{2}[V_1(s), V_2(s)]\right)\\
&\quad + k_1(s)\left(k_2(s) - \overline{k}_2(s)\right)\left(D_{V_1}V_3(s) - \frac{1}{2}[V_1(s), V_3(s)]\right)
\end{aligned}$$

$$= -3k_1(s)k_1'(s)V_1(s) + \left[k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\overline{k}_2(s)\right.$$
$$\left. - k_1(s)\overline{k}_2^{\,2}(s)\right]V_2(s) + \left(2k_1'(s)\tau_1(s) + k_1(s)\tau_1'(s)\right)V_3(s). \tag{2.12}$$

□

*Notation*. Let we put

$$\begin{aligned} N_1(s) &= k(s)V_2(s), \\ N_2(s) &= k_1'(s)V_2(s) + k_1(s)\tau_1(s)V_3(s), \\ N_3(s) &= \left[k_1''(s) - k_1^3 - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\overline{k}_2(s) - k_1(s)\overline{k}_2^{\,2}(s)\right]V_2(s) \\ &\quad + \left(2k_1'(s)\tau_1(s) + k_1(s)\tau_1'(s)\right)V_3(s). \end{aligned} \tag{2.13}$$

## 3. Curves of AW(k)-Type

In this section, we consider the properties of curves of AW(k)-type in the Lie group $G$.

*Definition 3.1* (see, cf. [13]). The Frenet curves of osculating order 3 are

(i) of type weak AW(2) if they satisfy

$$N_3(s) = \langle N_3(s), N_2^*(s)\rangle N_2^*(s), \tag{3.1}$$

(ii) of type weak AW(3) if they satisfy

$$N_3(s) = \langle N_3(s), N_1^*(s)\rangle N_1^*(s), \tag{3.2}$$

where

$$\begin{aligned} N_1^*(s) &= \frac{N_1(s)}{\|N_1(s)\|}, \\ N_2^*(s) &= \frac{N_2(s) - \langle N_2(s), N_1^*(s)\rangle N_1^*(s)}{\|\langle N_2(s), N_1^*(s)\rangle N_1^*(s)\|}. \end{aligned} \tag{3.3}$$

*Definition 3.2* (see [8]). The Frenet curves of osculating order 3 are

(i) of type AW(1) if they satisfy $N_3(s) = 0$,

(ii) of type AW(2) if they satisfy

$$\|N_2(s)\|^2 N_3(s) = \langle N_3(s), N_2(s)\rangle N_2(s), \tag{3.4}$$

(iii) of type AW(3) if they satisfy

$$\|N_1(s)\|^2 N_3(s) = \langle N_3(s), N_1(s)\rangle N_1(s). \tag{3.5}$$

From the definitions of type AW(k), we can obtain the following propositions.

**Proposition 3.3.** *Let $\alpha$ be a Frenet curve of osculating order 3. Then $\alpha$ is of weak $AW(2)$-type if and only if*

$$k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\overline{k}_2(s) - k_1(s)\overline{k}_2^{\,2}(s) = 0. \tag{3.6}$$

**Proposition 3.4.** *Let $\alpha$ be a Frenet curve of osculating order 3. Then $\alpha$ is of weak $AW(3)$-type if and only if*

$$2k_1'(s)\tau_1(s) + k_1(s)\tau_1'(s) = 0. \tag{3.7}$$

**Proposition 3.5.** *Let $\alpha$ be a Frenet curve of osculating order 3. Then $\alpha$ is of $AW(1)$-type if and only if*

$$\begin{gathered} k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\overline{k}_2(s) - k_1(s)\overline{k}_2^{\,2}(s) = 0, \\ k_1^2(s)\tau_1(s) = c, \end{gathered} \tag{3.8}$$

*where c is a constant.*

**Proposition 3.6.** *Let $\alpha$ be a Frenet curve of osculating order 3. Then $\alpha$ is of type $AW(2)$ if and only if*

$$\begin{aligned} &k_1'(s)\left(2k_1'(s)\tau_1(s) + k_1(s)\tau_1'(s)\right) \\ &\quad = k_1(s)\tau_1(s)\left(k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\overline{k}_2(s) - k_1(s)\overline{k}_2^{\,2}(s)\right) = 0. \end{aligned} \tag{3.9}$$

**Proposition 3.7.** *Let $\alpha$ be a Frenet curve of osculating order 3. Then $\alpha$ is of type $AW(3)$ if and only if*

$$k_1^2(s)\tau_1(s) = c, \tag{3.10}$$

*where c is a constant.*

## 4. General Helices of AW(k)-Type

In this section, we study general helices of AW(k)-type in the Lie group $G$ with a bi-invariant metric and characterize these curves.

*Definition 4.1* (see [6]). Let $\alpha : I \to G$ be a parameterized curve. Then $\alpha$ is called a general helix if it makes a constant angle with a left-invariant vector field.

Note that in the definition the left-invariant vector field may be assumed to be with unit length, and if the curve $\alpha$ is parametrized by arc-length $s$, then we have

$$\langle \alpha'(s), X \rangle = \cos\theta, \tag{4.1}$$

for $X \in \mathfrak{g}$, where $\theta$ is a constant.

If $G$ is a commutative group $\mathbb{R}^3$, then Definition 4.1 reduces to the classical definition (see [14]). Since a left-invariant vector field in $G$ is a Killing vector field, Definition 4.1 is similar to the definition given in [1].

**Theorem 4.2** (see [6]). *A curve of osculating order 3 in $G$ is a general helix if and only if*

$$\tau_1 = ck_1, \tag{4.2}$$

*where $c$ is a constant.*

From (4.2), a curve with $k_1 \neq 0$ is a general helix if and only if $(\tau_1/k_1)(s) = \text{constant}$. As a Euclidean sense, if both $k_1(s) \neq 0$ and $\tau_1(s)$ are constants, it is a cylindrical helix. We call such a curve a circular helix.

**Theorem 4.3.** *Let $\alpha$ be a Frenet curve of osculating order 3. Then $\alpha''(s)$, $\alpha'''(s)$, and $\alpha''''(s)$ are linearly dependent if and only if $\alpha(s)$ is general helix.*

*Proof.* If $\alpha''(s)$, $\alpha'''(s)$, and $\alpha''''(s)$ are linearly dependent, then the following equation holds:

$$\begin{vmatrix} 0 & k_1 & 0 \\ -k_1^2 & k_1' & k_1\tau_1 \\ -3k_1k_1' & k_1'' - k_1^3 - k_1k_2^2 + 2k_1k_2\overline{k}_2 - k_1\overline{k}_2^{\,2} & 2k_1'\tau_1 + k_1\tau_1' \end{vmatrix} = 0. \tag{4.3}$$

By a direct computation, we have

$$k_1\tau_1' - k_1'\tau_1 = 0; \tag{4.4}$$

it follows that

$$\frac{d}{ds}\left(\frac{\tau_1}{k_1}\right) = 0. \tag{4.5}$$

Thus, $\tau_1/k_1 = \text{constant}$; that is, $\alpha$ is general helix. The converse statement is trivial. □

**Theorem 4.4.** *Let $\alpha$ be a general helix of osculating order 3. Then $\alpha$ is of weak AW(3)-type if and only if $\alpha$ is a circular helix.*

*Proof.* From (3.7) and (4.2), we can obtain that $k_1 = \text{constant}$; it follows that $\tau_1 = \text{constant}$. Thus, $\alpha$ is a circular helix. The converse statement is trivial. □

**Theorem 4.5.** *A general helix of type $AW(2)$ has Frenet curvatures*

$$k_1(s) = \frac{1}{\sqrt{-(1+c^2)s^2 + d_1 s + d_2}}, \qquad \tau_1(s) = c k_1(s), \tag{4.6}$$

*where $c$, $d_1$, and $d_2$ are constants.*

*Proof.* If $\alpha$ is a general helix of type AW(2), then from (3.9) and (4.2) we have

$$\begin{aligned} &k_1'(s)\left(2k_1'(s)\tau_1(s) + k_1(s)\tau_1'(s)\right) \\ &\quad = k_1(s)\tau_1(s)\left(k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\overline{k}_2(s) - k_1(s)\overline{k}_2^{\,2}(s)\right) = 0, \end{aligned} \tag{4.7}$$

$$\frac{\tau_1(s)}{k_1(s)} = c; \tag{4.8}$$

where $c$ is a constant.

Combining (4.7) and (4.8), we have

$$k_1(s)k_1''(s) - 3\left(k_1'(s)\right)^2 - \left(1+c^2\right)k_1^4(s) = 0. \tag{4.9}$$

To solve this differential equation, we take

$$k_1(s) = x. \tag{4.10}$$

Then, (4.9) can be rewritten as the form

$$x\frac{d^2x}{ds^2} - 3\left(\frac{dx}{ds}\right)^2 = \left(1+c^2\right)x^4. \tag{4.11}$$

Let us put

$$x = y^p. \tag{4.12}$$

Then (4.11) becomes

$$py^{2p-1}\frac{d^2y}{ds^2} - p(2p+1)y^{2p-2}\left(\frac{dy}{ds}\right)^2 = \left(1+c^2\right)y^{4p}. \tag{4.13}$$

If we choose $p = -1/2$, then the above equation is

$$\frac{d^2y}{ds^2} = -2\left(1+c^2\right), \tag{4.14}$$

its general solution is given by

$$y = -\left(1 + c^2\right)s^2 + d_1 s + d_2, \tag{4.15}$$

where $d_1$ and $d_2$ are constants.

Thus, we have

$$k_1(s) = \frac{1}{\sqrt{-(1 + c^2)s^2 + d_1 s + d_2}}, \tag{4.16}$$

so, the theorem is proved. □

**Corollary 4.6.** *There exists no a circular helix of osculating order 3 of type $AW(2)$ in $G$.*

**Theorem 4.7.** *Let $\alpha$ be a general helix of osculating order 3. Then $\alpha$ is of type $AW(3)$ if and only if $\alpha$ is a circular helix.*

*Proof.* Suppose that $\alpha$ is a general helix of type AW(3). Combining (3.10) and (4.2) we find $k_1^3(s) = 1$, that is, $k_1(s) = 1$. From this $\tau_1(s) = c$. Thus, $\alpha$ is a circular helix. □

**Theorem 4.8.** *Let $\alpha$ be a curve of osculating order 3. There exists no a general helix of type $AW(1)$.*

*Proof.* We assume that $\alpha$ is a general helix of type AW(1). Then from (3.8) and (4.2) we have

$$k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) + 2k_1(s)k_2(s)\overline{k}_2(s) - k_1(s)\overline{k}_2^{\,2}(s) = 0, \tag{4.17}$$

$$k_1^2(s)\tau_1(s) = c, \tag{4.18}$$

$$\tau_1(s) = ck_1(s). \tag{4.19}$$

From (4.18) and (4.19), we have

$$k_1(s) = 1. \tag{4.20}$$

Thus, (4.17) becomes

$$k_2^2(s) - 2k_2(s)\overline{k}_2(s) + \overline{k}_2^{\,2}(s) = -1, \tag{4.21}$$

equivalently to

$$\left(k_2(s) - \overline{k}_2(s)\right)^2 = -1. \tag{4.22}$$

It is impossible, so the theorem is proved. □

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## References

[1] M. Barros, "General helices and a theorem of Lancret," *Proceedings of the American Mathematical Society*, vol. 125, no. 5, pp. 1503–1509, 1997.

[2] A. Şenol and Y. Yayli, "LC helices in space forms," *Chaos, Solitons and Fractals*, vol. 42, no. 4, pp. 2115–2119, 2009.

[3] J. Arroyo, M. Barros, and O. J. Garay, "Models of relativistic particle with curvature and torsion revisited," *General Relativity and Gravitation*, vol. 36, no. 6, pp. 1441–1451, 2004.

[4] J. V. Beltran and J. Monterde, "A characterization of quintic helices," *Journal of Computational and Applied Mathematics*, vol. 206, no. 1, pp. 116–121, 2007.

[5] C. Camcı, K. İlarslan, L. Kula, and H. H. Hacısalihoğlu, "Harmonic curvatures and generalized helices in $E^n$," *Chaos, Solitons & Fractals*, vol. 40, no. 5, pp. 2590–2596, 2009.

[6] Ü. Çiftçi, "A generalization of Lancret's theorem," *Journal of Geometry and Physics*, vol. 59, no. 12, pp. 1597–1603, 2009.

[7] R. T. Farouki, C. Y. Han, C. Manni, and A. Sestini, "Characterization and construction of helical polynomial space curves," *Journal of Computational and Applied Mathematics*, vol. 162, no. 2, pp. 365–392, 2004.

[8] K. Arslan and A. West, "Product submanifolds with pointwise 3-planar normal sections," *Glasgow Mathematical Journal*, vol. 37, no. 1, pp. 73–81, 1995.

[9] M. Külahcı, M. Bektaş, and M. Ergüt, "On harmonic curvatures of a Frenet curve in Lorentzian space," *Chaos, Solitons and Fractals*, vol. 41, no. 4, pp. 1668–1675, 2009.

[10] C. Özgür and F. Gezgin, "On some curves of $AW(k)$-type," *Differential Geometry—Dynamical Systems*, vol. 7, pp. 74–80, 2005.

[11] M. Külahci, M. Bektaş, and M. Ergüt, "Curves of $AW(k)$-type in 3-dimensional null cone," *Physics Letters A*, vol. 371, no. 4, pp. 275–277, 2007.

[12] N. do Espírito-Santo, S. Fornari, K. Frensel, and J. Ripoll, "Constant mean curvature hypersurfaces in a Lie group with a bi-invariant metric," *Manuscripta Mathematica*, vol. 111, no. 4, pp. 459–470, 2003.

[13] K. Arslan and C. Özgür, "Curves and surfaces of $AW(k)$ type," in *Geometry and Topology of Submanifolds, IX (Valenciennes/Lyon/Leuven, 1997)*, pp. 21–26, World Scientific, River Edge, NJ, USA, 1999.

[14] D. J. Struik, *Lectures on Classical Differential Geometry*, Dover, New York, NY, USA, 1988.