Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 534276, 17 pages
doi:10.1155/2012/534276

Research Article

Dynamic Complexity of an Ivlev-Type Prey-Predator System with Impulsive State Feedback Control

Chuanjun Dai,[1] Min Zhao,[2] and Lansun Chen[3]

[1] *School of Mathematics and Information Science, Wenzhou University, Zhejiang, Wenzhou 325035, China*
[2] *School of Life and Environmental Science, Wenzhou University, Zhejiang, Wenzhou 325035, China*
[3] *Institute of Mathematics, Academia Sinica, Beijing 100080, China*

Correspondence should be addressed to Min Zhao, zmcn@tom.com

Received 19 January 2012; Revised 17 April 2012; Accepted 8 May 2012

Academic Editor: Huijun Gao

The dynamic complexities of an Ivlev-type prey-predator system with impulsive state feedback control are studied analytically and numerically. Using the analogue of the Poincaré criterion, sufficient conditions for the existence and the stability of semitrivial periodic solutions can be obtained. Furthermore, the bifurcation diagrams and phase diagrams are investigated by means of numerical simulations, which illustrate the feasibility of the main results presented here.

1. Introduction

The theoretical investigation of predator-prey systems in mathematical ecology has a long history, beginning with the pioneering work of Lotka and Volterra. During this time, the theory and application of differential equations with impulsive perturbations were significantly advanced by the efforts of Lakshmikantham et al. [1]. In fact, many systems in physics, chemistry, and biology can be modeled by impulsive differential equations which can represent the abrupt jumps that occur during their evolutionary processes [2].

Many factors in the environment must be considered in predator-prey systems [3]. Impulsive perturbations are an important element because some factors, such as fires, floods, and similar disturbances, are not well suited to be considered in a continuous manner. In general, impulsive perturbations can be classified into two cases [4]. The first is perturbations caused by nature, and the second is perturbations that arise as a result of human efforts to control prey density, for instance, controlling pest outbreaks. There are many strategies to control agricultural pests, including chemical and biological controls. Chemical control methods, such as crop dusting, are useful because they quickly kill a significant portion of a pest population and sometimes provide the only feasible method for preventing economic loss. However, pesticide pollution is a major hazard to human health and the populations of natural enemies. Another important control method is biological control. Biological control is the purposeful introduction and establishment of one or more natural enemies of a pest [5, 6]. The key to successful biological pest control is to identify the pest and its natural enemy and to release the natural enemies for pest control. Proportional harvesting, for example, of fish, is also considered in this category. Consequently, it is natural to assume that these perturbations are instantaneous, that is, in the form of an impulse.

Generally speaking, there are three possible cases of impulsive perturbation: systems with impulses at fixed times, systems with impulses at variable times, and autonomous impulsive systems. In recent years, most investigations of impulsive differential equations have concentrated on systems with impulses at fixed times [7–14], while the other two kinds of impulsive differential equations have been relatively less studied. As a matter of fact, in many practical cases, impulses often occur at state-dependent times rather than at fixed times. For example, it may be desirable to control a population size by catching, crop-dusting, or releasing the predator when prey numbers reach a threshold value.

As is well known, significant developments have recently been achieved in the bifurcation theory of continuous dynamic systems [15–20]. The study of impulsive systems mainly involves the properties of their solutions, such as existence, uniqueness, stability, boundedness, and periodicity. This paper also considers bifurcation behaviors. Recently, Lakmeche and Arino [21] transformed the problem of a periodic solution into a fixed-point problem, discussed the bifurcation of periodic solutions from trivial solutions, and obtained the existence conditions for the positive period-1 solution. Tang and Chen [22] developed a complete expression for a period-1 solution and investigated the bifurcation of periodic solutions numerically using a discrete dynamic system determined by a stroboscopic map. Many papers have been devoted to the analysis of mathematical models with state-dependent impulsive effects [23]. For instance, Tang, Jiang, Zeng, Qian, Nie, and others [24–29] have studied the dynamic behaviors of predator-prey systems with impulsive state feedback control and have determined the existence and stability of positive periodic solutions using the Poincaré map and the properties of the Lambert W function.

Recently, the continuous model with Ivlev-type has been extensively studied [30–36]. The Ivlev-type functional response describes a cyrtoid or Holling II prey-dependent functional response because the feeding rate declines with increasing resource abundance until it reaches a constant rate [34]. Although a direct link between the predator and prey cannot be established unless quantitative methods are used, the precious works clearly show that the amount of two species is often related, and a change in one species can cause a change in another, especially predator. Thus, we apply Ivlev-type functional response to describe their relationship with sufficient accuracy in this paper. Using the method of impulsive perturbations, a predator-prey model with Ivlev-type and state impulsive perturbations will be considered, as follows:

$$\begin{aligned} &\left.\begin{aligned} \dot{x} &= rx\left(1-\frac{x}{k}\right)-(1-\exp(-ax))y, \\ \dot{y} &= ((1-\exp(-ax))-m)y, \end{aligned}\right\} \quad x \neq h, \\ &\left.\begin{aligned} \Delta x &= -px, \\ \Delta y &= qy+\tau, \end{aligned}\right\} \quad x = h, \end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ are functions of time representing the population densities of the prey and predator, respectively. a is the efficiency with which predators extract preys from their environment, which sometimes is called the apparency of the preys, k is the carrying capacity of prey x, m is the death rate of predator y, $p \in (0,1)$ is the average lost rate of prey x during this time the amount of prey x reaches to critical threshold $h > 0$, $q(q > 0)$ describes a released parameter for juvenile predator y, $\tau(\tau > 0)$ represents a released parameter for adult predator y, $\Delta x(t) = x(t^+) - x(t)$, and $\Delta y(t) = y(t^+) - y(t)$. When the amount of prey x reaches to critical threshold h, a control strategy is used; then the numbers of prey and predator become $(1-p)h$ and $(1+q)y(t_i(h)) + \tau$, respectively.

The rest of this paper is organized as follows. Section 2 presents certain preliminaries, important definitions, and lemmas that are frequently used in the following discussions. In Section 3, the existence and stability of a positive periodic solution of system (1.1) are stated and proved. Section 4 presents a numerical analysis to illustrate the theoretical results. Finally, conclusions and remarks are presented in Section 5.

2. Preliminaries

The dynamic behavior of system (1.1) without impulsive effects can be interpreted as follows. It has one saddle at $(0,0)$, and calculations reveal that $(0,k)$ is also a saddle, while $(-\ln(1-m)/a, -r\ln(1-m)(ak+\ln(1-m))/a^2km)$ is a stable positive focus when $ak + \ln(1-m) > 0$ and $ak + 2\ln(1-m) < 0$ hold.

Throughout this paper, it is assumed that $h < -\ln(1-m)/a$, $ak + \ln(1-m) > 0$ and $ak + 2\ln(1-m) < 0$ always hold. Only solutions with nonnegative components, continuously differentiable in the region $D = \{(x,y) : x \geq 0, y \geq 0\}$ based on the biological background of system (1.1), will be considered.

Let $R = (-\infty, \infty)$ and let $z(t) = (x(t), y(t))$ be any solution of system (1.1). The positive orbit through point $z_0 \in R_+^2 = \{(x,y) : x \geq 0, y \geq 0\}$ for $t \geq t_0 \geq 0$ is defined as

$$O^+(z_0, t_0) = \left\{ z \in R_+^2 : z = z(t),\ t \geq t_0, z(t_0) = z_0 \right\}. \tag{2.1}$$

Definition 2.1. A trajectory $O^+(z_0, t_0)$ of system (1.1) is said to be order-k periodic if there exists a positive integer $k \geq 1$ such that k is the smallest integer for which $x_0 = x_k$.

The next step is to construct the Poincaré map. To discuss the dynamics of system (1.1), consider its vector field. As shown in Figure 1, denote $S_0 = \{(x,y) \mid x = (1-p)h,\ y \geq 0\}$ and $S_1 = \{(x,y) \mid x = h,\ y \geq 0\}$. It is clear that the line $x = (1-p)h$ and the line $x = h$ intersect the isoclinal line $rx(1 - x/k) - (1 - \exp(-ax))y = 0$, or in other words, $dx/dt = 0$, at point $A((1-p)h, rh(1-p)(k-(1-p)h)/k(1-\exp(-a(1-p)h)))$, and that $B(h, rh(k-h)/k(1-\exp(-ah)))$ intersects the line $y = 0$ at point $C((1-p)h, 0)$, $D(h, 0)$. Denote $\Omega = \{(x,y) \mid 0 < y < rx(k-x)/k(1-\exp(-ax)), (1-p)h < x < h\}$, and $\Omega_1 = \Omega \cup \overline{CD}$. It is obvious that $dx/dt = 0$, $dy/dt < 0$ are satisfied at point $(x,y) \in \overset{\frown}{AB}$, where $\overset{\frown}{AB}$ is represented as $y = rx(k-x)/k(1-\exp(-ax))$ and $(1-p)h < x < h$. Any orbit passing through segment $\overset{\frown}{AB}$ and into the interior of Ω will exit Ω by passing through segment $\overline{BD}$.

Assume that point $S_n((1-p)h, y_n)$ is on section S_0. Then the trajectory $O^+(S_n, t_n)$ of system (1.1) intersects section S_1 at point $S_{n+1}(h, y_{n+1})$, where y_{n+1} is determined by y_n. Then the point $S_{n+1}(h, y_{n+1})$ jumps to point $S_{n+1}^+((1-p)h, (1+q)y_n + \tau)$ on S_0 due to the

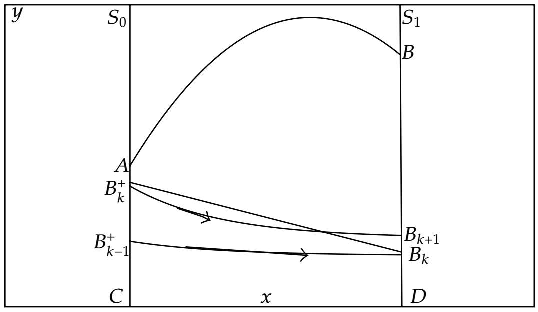

Figure 1: Poincaré map of system (1.1).

impulsive effects, and section S_0 is a Poincaré section. The following Poincaré map f can thus be obtained:

$$y_n^+ = (1+q)g(y_{n-1}^+) + \tau. \tag{2.2}$$

Now choose section S_1 as another Poincaré section. Another Poincare map f_1 can be obtained for S_1:

$$y_{n+1} = g((1+q)y_n + \tau) \equiv F(q, \tau, y_n). \tag{2.3}$$

In this discussion, y_{k+1} is determined by y_n and parameters q and τ.

Next, an autonomous system with impulsive effects will be considered:

$$\begin{aligned} &\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \quad \varphi(x,y) \neq 0, \\ &\Delta x = \xi(x,y), \qquad \Delta y = \eta(x,y), \quad \varphi(x,y) = 0, \end{aligned} \tag{2.4}$$

where $P(x,y)$ and $Q(x,y)$ are continuous differential functions and $\varphi(x,y)$ is a sufficiently smooth function with grade $\varphi(x,y) \neq 0$. Let $(\xi(t), \eta(t))$ be a positive T-periodic solution of system (2.4). The following technical lemma will now be introduced.

Lemma 2.2 (see [37]). *If the Floquet multiplier μ satisfies the condition $|\mu| < 1$, where*

$$\mu = \prod_{k=1}^{n} \Delta_k \exp\left[\int_0^T \left(\frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t))\right) dt\right], \tag{2.5}$$

with

$$\begin{aligned} \Delta_k = {} & \frac{P_+((\partial\beta/\partial y)(\partial\phi/\partial x) - (\partial\beta/\partial x)(\partial\phi/\partial y) + \partial\phi/\partial x)}{P(\partial\phi/\partial x) + Q(\partial\phi/\partial y)} \\ & + \frac{Q_+((\partial\alpha/\partial x)(\partial\phi/\partial y) - (\partial\alpha/\partial y)(\partial\phi/\partial x) + \partial\phi/\partial y)}{P(\partial\phi/\partial x) + Q(\partial\phi/\partial y)}, \end{aligned} \tag{2.6}$$

and $P, Q, \partial\alpha/\partial x, \partial\alpha/\partial y, \partial\beta/\partial x, \partial\beta/\partial y, \partial\phi/\partial x, \partial\phi/\partial y$ *are calculated at point* $(\xi(t_k^+), \eta(t_k^+))$, $P_+ = P(\xi(t_k^+), \eta(t_k^+))$, $Q_+ = Q(\xi(t_k^+), \eta(t_k^+))$ *and* $t_k (k \in N)$ *is the time of the* k*-th jump, then* $(\xi(t), \eta(t))$ *is orbitally asymptotically stable.*

Lemma 2.3 (see [38]). *Let* $F : R \times R \to R$ *be a one-parameter family of* C^2 *maps satisfying*

(i) $F(0, \mu) = 0$,

(ii) $(\partial F/\partial x)(0, 0) = 1$,

(iii) $(\partial^2 F/\partial x \partial\mu)(0, 0) > 0$,

(iv) $(\partial^2 F/\partial x^2)(0, 0) < 0$.

Then F has two branches of fixed points for μ *near zero. The first branch is* $x_1(\mu) = 0$ *for all* μ. *The second bifurcating branch* $x_2(\mu)$ *changes its value from negative to positive as* μ *increases through* $\mu = 0$ *with* $x_2(0) = 0$. *The fixed points of the first branch are stable if* $\mu < 0$ *and unstable if* $\mu > 0$, *while those of the bifurcating branch having the opposite stability.*

3. Dynamic Properties

3.1. Case $\tau = 0$

It should be stressed that the semitrivial periodic solution with $y = 0$ of system (1.1) exists if and only if $\tau = 0$. Therefore, the discussions start with $\tau = 0$.

When $\tau = 0$, system (1.1) can be stated in the following form:

$$\begin{aligned} &\left.\begin{aligned}\dot{x} &= rx\left(1 - \frac{x}{k}\right) - (1 - \exp(-ax))y,\\ \dot{y} &= ((1 - \exp(-ax)) - m)y,\end{aligned}\right\} \quad x \neq h,\\ &\left.\begin{aligned}\Delta x &= -px,\\ \Delta y &= qy.\end{aligned}\right\} \quad x = h,\end{aligned} \tag{3.1}$$

Let $y(t) = 0$ for $t \in [0, \infty)$; then from system (3.1),

$$\begin{aligned} \dot{x} &= rx\left(1 - \frac{x}{k}\right), \quad x \neq h,\\ \Delta x &= -px, \quad x = h. \end{aligned} \tag{3.2}$$

Setting $x_0 = x(0) = (1 - p)h$ leads to the solution of system (3.2), $x(t) = k(1 - p)h\exp(r(t - nT))/[k - (1 - p)h + (1 - p)h\exp(r(t - nT))]$. Let $T = \ln[k - (1 - p)h/(k - h)(1 - p)]^{1/r}$; then $x(T) = h$ and $x(T^+) = (1 - p)h$. Hence, system (3.1) has the following semitrivial periodic solution:

$$\begin{aligned} x(t) &= \frac{k(1 - p)h\exp(r(t - nT))}{k - (1 - p)h + (1 - p)h\exp(r(t - nT))},\\ y(t) &= 0, \end{aligned} \tag{3.3}$$

where $t \in (nT, (n + 1)T]$, $n \in N$, and which is denoted by $(\xi(t), 0)$.

Now the stability of this semitrivial periodic solution will be discussed.

Theorem 3.1. *The semitrivial periodic solution* (3.3) *is said to be orbitally asymptotically stable if*

$$0 < q < \left(\frac{k-(1-p)h}{(k-h)(1-p)} \right)^{(d-1)/r} \exp\left(\int_0^T \exp(-a\xi(t))dt \right) - 1. \tag{3.4}$$

Proof. In fact,

$$\begin{gathered}
P(x,y) = rx\left(1-\frac{x}{k}\right) - (1-\exp(-ax))y, \\
Q(x,y) = ((1-\exp(-ax)) - m)y, \\
\alpha(x,y) = -px, \qquad \beta(x,y) = qy, \qquad \phi(x,y) = x-h, \\
(\xi(T),\eta(T)) = (h,0), \qquad (\xi(T^+),\eta(T^+)) = ((1-p)h,0).
\end{gathered} \tag{3.5}$$

According to Lemma 2.2, a straightforward calculation yields

$$\begin{gathered}
\frac{\partial P}{\partial x} = r - \frac{2r}{k}x - ay\exp(-ax), \qquad \frac{\partial Q}{\partial y} = 1-\exp(-ax)-m, \\
\frac{\partial \alpha}{\partial x} = -p, \qquad \frac{\partial \alpha}{\partial y} = 0, \qquad \frac{\partial \beta}{\partial x} = 0, \qquad \frac{\partial \beta}{\partial y} = q, \qquad \frac{\partial \phi}{\partial x} = 1, \qquad \frac{\partial \phi}{\partial y} = 0, \\
\begin{aligned}
\Delta_1 &= \frac{P_+((\partial\beta/\partial y)(\partial\phi/\partial x) - (\partial\beta/\partial x)(\partial\phi/\partial y) + \partial\phi/\partial x)}{P(\partial\phi/\partial x) + Q(\partial\phi/\partial y)} \\
&\quad + \frac{Q_+((\partial\alpha/\partial x)(\partial\phi/\partial y) - (\partial\alpha/\partial y)(\partial\phi/\partial x) + \partial\phi/\partial y)}{P(\partial\phi/\partial x) + Q(\partial\phi/\partial y)} \\
&= \frac{P^+(\xi(T^+),\eta(T^+))(1+q)}{P(\xi(T),\eta(T))} = (1-p)(1+q)\frac{k-(1-p)h}{k-h}.
\end{aligned}
\end{gathered} \tag{3.6}$$

Furthermore,

$$\begin{aligned}
&\exp\left[\int_0^T \left(\frac{\partial P}{\partial x}(\xi(t),\eta(t)) + \frac{\partial Q}{\partial y}(\xi(t),\eta(t)) \right) dt\right] \\
&= \exp\left[\int_0^T r+1-d-\frac{2r}{k}\xi(t) - \exp(-a\xi(t))dt\right] \\
&= \left(\frac{k-(1-p)h}{(k-h)(1-p)} \right)^{(r+1-m)/r} \left(\frac{k-(1-p)h}{k-h} \right)^{-2} \exp\left[\int_0^T -\exp(-a\xi(t))dt\right].
\end{aligned} \tag{3.7}$$

Hence, the Floquet multiplier μ can be obtained by direct calculation as follows:

$$\begin{aligned}\mu &= \prod_{k=1}^{n} \Delta_k \exp\left[\int_0^T \left(\frac{\partial P}{\partial x}(\xi(t),\eta(t)) + \frac{\partial Q}{\partial y}(\xi(t),\eta(t))\right)dt\right] \\ &= (1+q)\left(\frac{k-(1-p)h}{(k-h)(1-p)}\right)^{(1-m)/r} \exp\left(-\int_0^T \exp(-a\xi(t))dt\right).\end{aligned} \tag{3.8}$$

Therefore, $|\mu| < 1$ holds if and only if (3.4) holds. This completes the proof. □

Remark 3.2. Set $q^* = ((k-(1-p)h)/((k-h)(1-p)))^{(m-1)/r} \exp(\int_0^T \exp(-a\xi(t))dt) - 1$; a bifurcation may occur at $q = q^*$ for $|\mu| = 1$, and a positive periodic solution may appear when $q > q^*$. Hence, the problem of bifurcations will now be discussed.

First, in the case $\tau = 0$, consider the Poincaré map (2.2). Set $u = y_n^+$ and $u \geq 0$ small enough. The map then takes the following form:

$$u \longmapsto (1+q)g(u) \equiv G(u,q), \tag{3.9}$$

where the function $G(u,q)$ is continuously differentiable with respect to both u and q, $g(0) = 0$; then $\lim_{u\to 0^+} g(u) = g(0) = 0$.

Second, by examining the bifurcation of map (3.9), it is possible to obtain the following theorem.

Theorem 3.3. *A transcritical bifurcation occurs when $q = q^*$. Therefore, a stable positive fixed point appears when parameter q changes through q^* from left to right. Correspondingly, system* (3.1) *has a stable positive periodic solution if $q \in (q^*, q^* + \delta)$ with $\delta > 0$.*

Proof. The values of $g'(u)$ and $g''(u)$ must be calculated at $u = 0$, where $0 \leq u \leq rh(1-p)(k-(1-p)h)/k(1-\exp(-a(1-p)h)) \equiv u_0$. From system (1.1),

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}, \tag{3.10}$$

where

$$\begin{aligned}P(x,y) &= rx\left(1-\frac{x}{k}\right) - (1-\exp(-ax))y, \\ Q(x,y) &= ((1-\exp(-ax)) - m)y.\end{aligned} \tag{3.11}$$

Let $(x, y\,(x; x_0, y_0))$ be an orbit of system (3.10), and set $x_0 = (1-p)h, y_0 = u$, $0 \leq u \leq u_0$; then

$$y(x;(1-p)h,u) \equiv y(x,u), \quad (1-p)h \leq x \leq h,\ 0 \leq u \leq u_0. \tag{3.12}$$

Using (3.12),

$$\begin{aligned}\frac{\partial y(x,u)}{\partial u} &= \exp\left[\int_{(1-p)h}^{x} \frac{\partial}{\partial y}\left(\frac{Q(s,y(s,u))}{P(s,y(s,u))}\right) ds\right],\\ \frac{\partial^2 y(x,u)}{\partial u^2} &= \frac{\partial y(x,u)}{\partial u}\int_{(1-p)h}^{x} \frac{\partial^2}{\partial y^2}\left(\frac{Q(s,y(s,u))}{P(s,y(s,u))}\right)\frac{\partial y(s,u)}{\partial u} ds.\end{aligned} \tag{3.13}$$

Clearly, it can be deduced that $\partial y(x,u)/\partial u > 0$ and

$$\begin{aligned} g'(0) = \frac{\partial y(h,0)}{\partial u} &= \exp\left(\int_{(1-p)h}^{h} \frac{\partial}{\partial y}\left(\frac{Q(s,y(s,0))}{P(s,y(s,0))}\right) ds\right)\\ &= \exp\left(\int_{(1-p)h}^{h} \frac{1-m-\exp(-as)}{rs(1-s/k)} ds\right)\\ &= \left(\frac{k-(1-p)h}{(k-h)(1-p)}\right)^{(1-r)/m} \exp\left(\int_{(1-p)h}^{h} \frac{-\exp(-as)}{rs(1-s/k)} ds\right)\\ &= \left(\frac{k-(1-p)h}{(k-h)(1-p)}\right)^{(1-r)/m} \exp\left(\int_{0}^{T} -\exp(-a\xi(t)) dt\right). \end{aligned} \tag{3.14}$$

Furthermore,

$$g''(0) = g'(0)\int_{(1-p)h}^{h} l(s)\frac{\partial y(s,0)}{\partial u} ds, \tag{3.15}$$

where

$$\begin{aligned} l(s) &= \frac{\partial^2}{\partial y^2}\left(\frac{Q(s,y(s,0))}{P(s,y(s,0))}\right)\\ &= \frac{2(1-\exp(-as)-m)(1-\exp(-as))}{(rs(1-s/k))^2}, \quad s \in [(1-p)h, h]. \end{aligned} \tag{3.16}$$

Using the previous assumption,

$$h < \frac{-\ln(1-m)}{a}. \tag{3.17}$$

It can be determined that

$$l(s) < 0, \quad s \in [(1-p)h, h). \tag{3.18}$$

Therefore,

$$g''(0) < 0. \tag{3.19}$$

The next step is to check whether the following conditions are satisfied.

(a) It is easy to see that

$$G(0, q) = 0, \quad q \in (0, \infty). \tag{3.20}$$

(b) Using (3.14),

$$\begin{aligned}\frac{\partial G(0,q)}{\partial u} &= (1+q)g'(0) \\ &= (1+q)\left(\frac{k-(1-p)h}{(k-h)(1-p)}\right)^{(1-r)/m} \exp\left(\int_0^T -\exp(-a\xi(t))dt\right),\end{aligned} \tag{3.21}$$

which yields

$$\frac{\partial G(0,q^*)}{\partial u} = 1. \tag{3.22}$$

This means that $(0, q^*)$ is a fixed point with eigenvalue 1 of map (3.9).

(c) Because (3.14) holds,

$$\frac{\partial^2 G(0,q^*)}{\partial u \partial q} = g'(0) > 0. \tag{3.23}$$

(d) Finally, (3.19) implies that

$$\frac{\partial^2 G(0,q^*)}{\partial u^2} = (1+q^*)g''(0) < 0. \tag{3.24}$$

These conditions satisfy the conditions of Lemma 2.3. This completes the proof. □

3.2. *Case* $\tau > 0$

In this subsection, the existence of a positive periodic solution with $\tau > 0$ will be discussed using the Poincaré map (2.3). Sufficient conditions will be given for the existence and stability of positive periodic solutions. The following theorem will now be proved.

Theorem 3.4. *For any $q > 0$ and $\tau > 0$, system (1.1) has a positive order-1 periodic solution.*

Proof. Let point $M_1((1-p)h, 0)$ be on section S_0. Then the trajectory $O^+(M_1, t_0)$ of system (1.1) starting from the initial point M_1 intersects section S_1 at point $N_1(h, 0)$. In state N_1, the trajectory $O^+(M_1, t_0)$ is subjected to impulsive effects, jumps to point $M_2((1-p)h, \tau)$ on section S_0, and then returns to $N_2(h, \alpha_1)$ on section S_1. Because $\tau > 0$, point M_2 is above point M_1. Furthermore, point N_2 is above point N_1, and $\alpha_1 > 0$. From (2.3), $\alpha_1 = F(q, \tau, 0) = g(\tau) > 0$, and

$$0 - F(q, \tau, 0) = 0 - \alpha_1 < 0. \tag{3.25}$$

In addition, assuming that the initial point of the trajectory $O^+(A, t_0)$ is point A, where $dy/dt < 0$ and $dx/dt = 0$, obviously, $O^+(A, t_0)$ is tangent to the line S_0, intersects S_1 at point $H(h, v_1)$, and then jumps to point $H^+((1-p)h, (1+q)v_1 + \tau)$ on S_0, and returns to point $H'(h, v_2)$ on S_1. Assume further that there exists a positive $\overline{q}$ such that $(1+\overline{q})v_1 + \tau = rh(1-p)(k-(1-p)h)/k(1-\exp(-a(1-p)h))$. Then point H^+ coincides with point A for $q = \overline{q}$, and point H^+ is above point A for $q > \overline{q}$, but below point A for $q < \overline{q}$. However, for any $q > 0$, the point H' is not above the point H in view of the geometrical structure of the phase space of system (1.1).

In conclusion, the following results can be obtained from the previous discussion:

(i) if $v_1 = v_2 (q = \overline{q})$, then system (1.1) has a positive order-1 periodic solution;

(ii) if $v_1 > v_2 (q \neq \overline{q})$, then

$$v_1 - F(q, \tau, v_1) = v_1 - v_2 > 0. \tag{3.26}$$

From (3.25) and (3.26), it follows that the Poincaré map (2.3) has a fixed point; that is, system (1.1) has a positive order-1 periodic solution. This completes the proof. □

According to the following discussion, a positive periodic solution exists when $\tau = 0, q \geq q^*$ or $\tau > 0, q > 0$. Next, the stability of a positive order-1 periodic solution of system (1.1) will be proved. This will be accomplished by means of the following theorem.

Theorem 3.5. *For any $\tau = 0, q \geq q^*$ or $\tau > 0, q > 0$, let $(\xi(t), \eta(t))$ be a positive order-1 T-periodic solution of system* (1.1) *which starts from point (h, ω). If the condition*

$$|\mu| = (1+q)\Gamma \exp\left(\int_0^T \Psi(t)dt\right) < 1 \tag{3.27}$$

holds, where

$$\begin{gathered}\Gamma = \frac{rh(1-p)(k-(1-p)h) - k(1-\exp(-ah(1-p)))[(1+q)\omega + \tau]}{rh(k-h) - k\omega(1-\exp(-ah))}, \\ \Psi(t) = \frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t)),\end{gathered} \tag{3.28}$$

then $(\xi(t), \eta(t))$ is a positive order-1 periodic solution of system (1.1) *which is orbitally asymptotically stable and has the asymptotic phase property.*

Proof. Based on the conclusion of Theorem 3.4, it is necessary only to verify the stability of the positive order-1 periodic solutions $(\xi(t), \eta(t))$ of system (1.1). In what follows, it is assumed that a periodic solution with period T passes through points $K^+((1-p)h, (1+q)\omega + \tau)$ and $K(h, \omega)$, in which $\omega \leq v_1$ holds because of the properties of the vector field of system (1.1) as outlined in the following discussion. Because the mathematical form and the period T of the solution are not known, the stability of this positive periodic solution will be discussed using Lemma 2.2. The difference between this case and that of Theorem 3.1 lies in the fact that

$$(\xi(T), \eta(T)) = (h, \omega), \qquad (\xi(T^+), \eta(T^+)) = ((1-p)h, (1+q)\omega + \tau), \tag{3.29}$$

while the others are just the same. Then

$$\begin{aligned}\Delta_1 &= \frac{P_+((\partial\beta/\partial y)(\partial\phi/\partial x) - (\partial\beta/\partial x)(\partial\phi/\partial y) + \partial\phi/\partial x)}{P(\partial\phi/\partial x) + Q(\partial\phi/\partial y)} \\ &\quad + \frac{Q_+((\partial\alpha/\partial x)(\partial\phi/\partial y) - (\partial\alpha/\partial y)(\partial\phi/\partial x) + \partial\phi/\partial y)}{P(\partial\phi/\partial x) + Q(\partial\phi/\partial y)} \\ &= \frac{P^+(\xi(T^+), \eta(T^+))(1+q)}{P(\xi(T), \eta(T))} = (1+q)\Gamma,\end{aligned} \tag{3.30}$$

where

$$\Gamma = \frac{rh(1-p)(k-(1-p)h) - k(1-\exp(-ah(1-p)))[(1+q)\omega + \tau]}{rh(k-h) - k\omega(1-\exp(-ah))}. \tag{3.31}$$

Let $\Psi(t) = (\partial P/\partial x)(\xi(t), \eta(t)) + (\partial Q/\partial y)(\xi(t), \eta(t))$; then

$$\begin{aligned}|\mu| &= \Delta_1 \exp\left(\int_0^T \left(\frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t))\right) dt\right) \\ &= (1+q)\Gamma \exp\left(\int_0^T \Psi(t) dt\right).\end{aligned} \tag{3.32}$$

If $|\mu| < 1$, that is:

$$\left|(1+q)\Gamma \exp\left(\int_0^T \Psi(t) dt\right)\right| < 1, \tag{3.33}$$

then the periodic solution is stable. This completes the proof. □

Remark 3.6. From the previously mentioned, it is known that if there exists a $q' > \overline{q}$ such that $|u| = 1$, a flip bifurcation occurs at $q = q'$. If a flip bifurcation occurs, there exists a stable positive order-2 periodic solution of system (1.1) for $q' > \overline{q}$, which may also lose its stability as q increases.

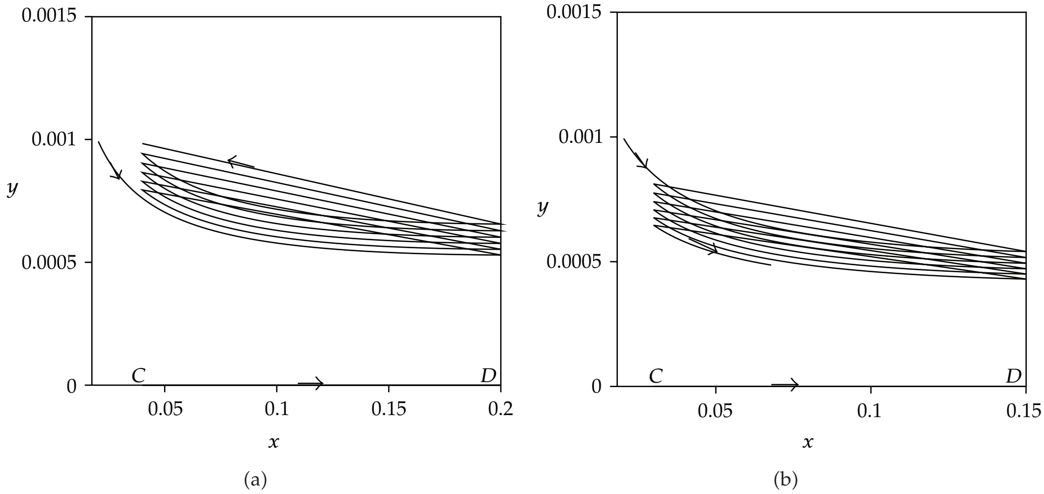

Figure 2: Trajectories with initial point (0.02, 0.001) of system (1.1) with $p = 0.8$, $q = 0.5$, $\tau = 0$: (a) $h = 0.2$, (b) $h = 0.15$.

4. Numerical Analysis

As is well known, system (1.1) cannot be solved explicitly, so it must be studied by numerical integration and the long-term dynamic behavior of the solution by numerical simulation.

To study the dynamic complexity of an Ivlev-type system with state-dependent impulsive perturbation on the predator, a semitrivial periodic solution of system (1.1) with initial conditions is first obtained numerically for a biologically feasible range of parameter values. The bifurcation diagram provides a summary of the essential dynamic behavior of system (1.1).

Next, two control parameters, q and τ, are chosen. Other parameters are set to $r = 0.95, k = 20, a = 2.8, m = 0.45$ and provide some representative values to help with the analysis.

Note that the corresponding focus $(-\ln(1-m)/a, -r\ln(1-m)(ak+\ln(1-m))/a^2km) = (0.2135, 0.4459)$, so $h \leq 0.2135$. System (1.1) has a semitrivial periodic solution when $\tau = 0$. Taking $p = 0.8$ and $h = 0.2$, from Theorem 3.1, $\mu \approx 0.7(1+q)$. Note that $\mu > 1$ is always true for any $q > 3/7$ and that the periodic semitrivial solution is unstable (Figure 1(A)).

Let $p = 0.8$ and $h = 0.15$; then $q^* \approx 0.56$ can be obtained from Remark 3.2. Setting $q = 0.5$, the solution of system (1.1) tends to a stable semitrivial periodic solution as t increases (Figure 2(b)).

When $\tau > 0$, there is no semitrivial solution of system (1.1). Figures 3(a) and 3(b) show typical bifurcation diagrams for population y in system (1.1) as p increases from 0 to 35 and τ increases from 0 to 0.16 with initial $X(0) = (0.02, 0.01)$. As p and τ increase, the bifurcation diagrams clearly show that system (1.1) has rich dynamics, including period-doubling bifurcations, periodic windows, chaotic bands, period-halving bifurcations, and crises.

In Figure 3(a), there is no fold bifurcation. The positive order-1 periodic solution is stable for $q \in (0, 3.92)$. At $q \approx 3.92$, a positive order-2 periodic solution bifurcates from the positive order-1 period solution by means of a flip bifurcation. Furthermore, order-4 and order-8 periodic solutions arise through flip bifurcation. The period-doubling bifurcation

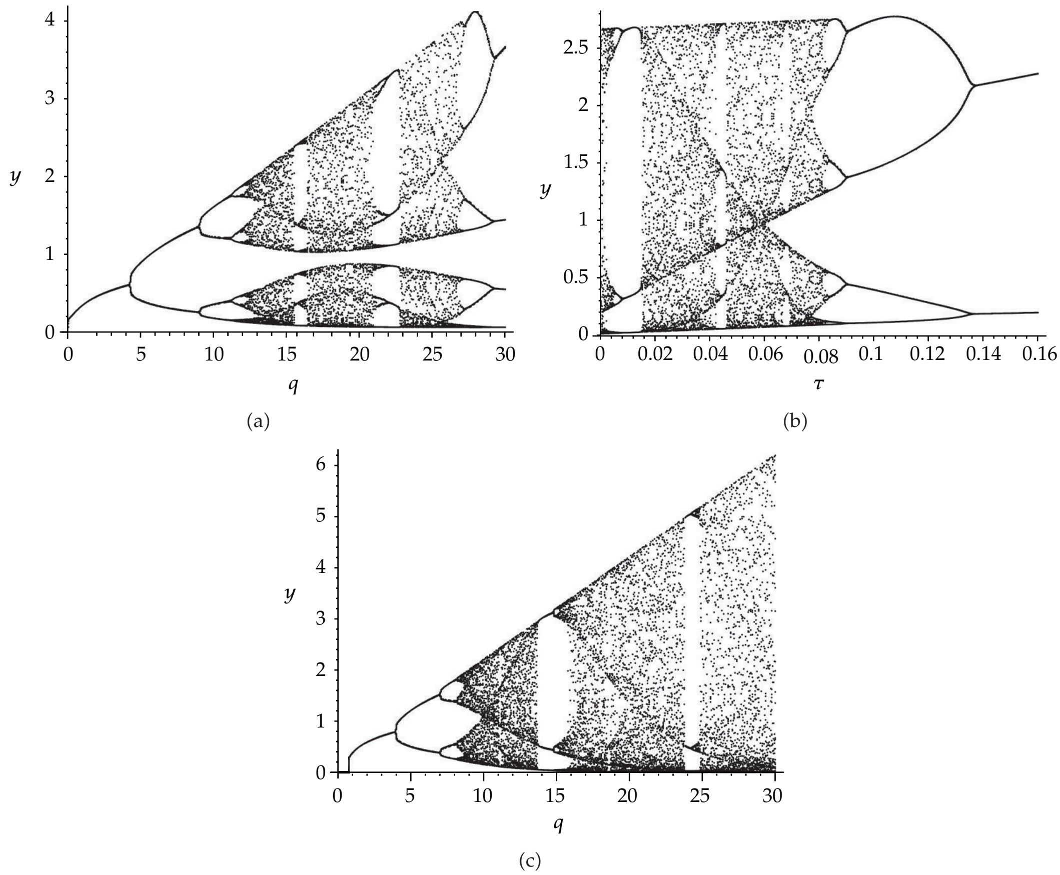

Figure 3: Bifurcation diagram of system (1.1) with initial conditions $X(0) = (0.02, 0.01)$, $h = 0.21$, $p = 0.8$: (a) $\tau = 0.065$, (b) $q = 18$, (c) $\tau = 0$.

leads to chaos. Finally, a cascade of period-halving bifurcations leads to stable order-4 periodic solutions for $q > 29.68$. Now let $q = 18$, and consider τ as a control parameter. Figure 3(b) shows a plot of the solution as a function of the bifurcation parameter τ. In this case, there is a route from chaos to a stable periodic solution via a period-halving bifurcation in which complex dynamic behaviors exist, such as periodic windows, chaotic bands, and chaotic crises (Figure 3(b)).

In Figure 3(c), q is considered as a parameter, and the bifurcation diagram of the periodic solution of system (1.1) with $\tau = 0$ is shown. It is obvious that the semitrivial periodic solution is stable for $q \in (0, 0.74)$ and unstable for $q \in (0.74, +\infty)$. A transcritical bifurcation leads to a positive order-1 periodic solution from the semitrivial periodic solution at $q \approx 0.74$. This positive order-1 periodic solution is stable for $q \in (0.74, 4.05)$ and unstable for $q \in (4.05, +\infty)$. In addition, a positive period-2 solution bifurcates from the positive order-1 periodic solution by means of a flip bifurcation at $q \approx 4.05$. Due to the period-doubling bifurcation, chaos arises, in which periodic windows, chaotic bands, and crises also exist (Figure 3(c)).

From Theorem 3.5, Remark 3.6, and analysis of the bifurcations described previously, it is known that system (1.1) has a positive order-1 periodic solution, which is shown in Figure 4(a). A flip bifurcation occurs at $q = 4.05$ according to the numerical simulations.

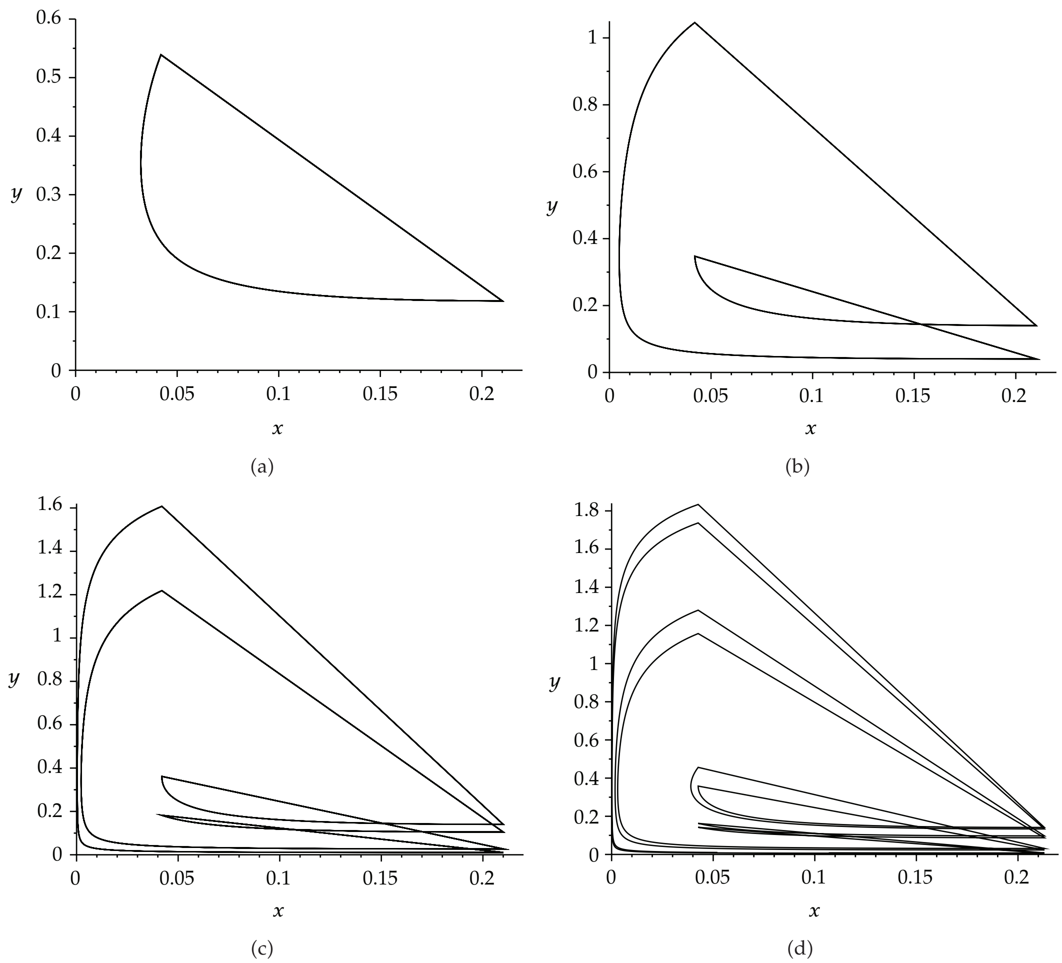

Figure 4: Periodic solutions of system (1.1): $h = 0.21$, $p = 0.8$, $\tau = 0.065$; (a) $q = 3$, (b) $q = 6$, (c) $q = 10$, (d) $q = 11.5$.

Figure 4 also shows the period-i ($i = 2, 4, 8$) solutions for different value of q. Figure 5 presents the phase diagram and time series of population y for a chaotic solution.

Based on the previous analysis, it can be seen that the impulsive state feedback control can enhance the predator y biomass level with the increasing of q, in which result is agreed with some results in reality. Further, it is also interesting to point out that the two different parameters of the impulsive state feedback control can come into rich and complex dynamical behaviors, but these dynamical behaviors are different. Moreover, the use of mathematical model with impulsive state feedback control is considered to investigate some biological problems, and the numerical simulation provides an approximation of the real biological system behaviors; hence, these results can promote the study of ecological dynamics.

5. Conclusions

In this paper, a predator-prey model with Ivlev-type function and impulsive state feedback control has been built and studied analytically and numerically. Mathematical theoretical

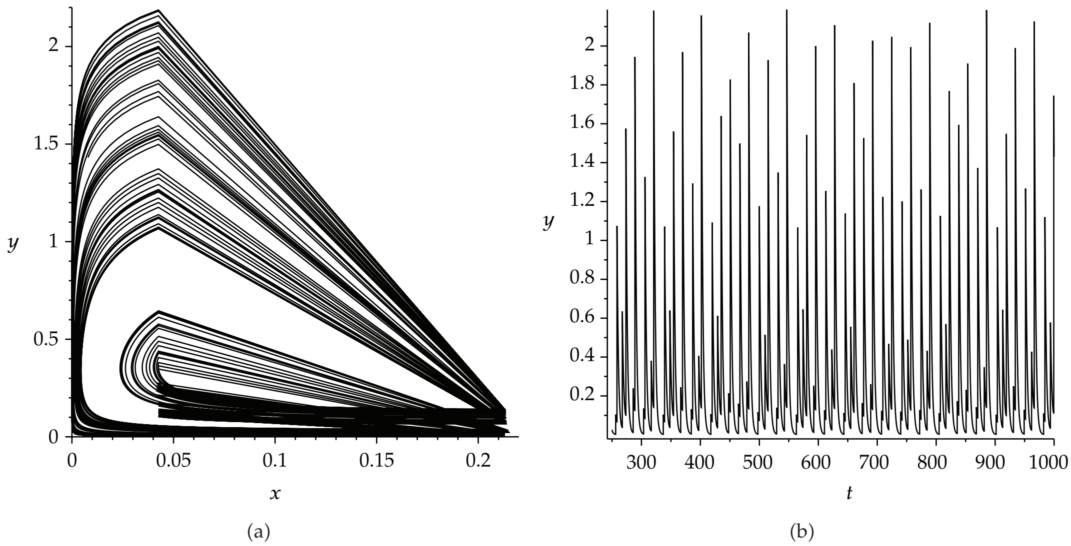

Figure 5: (a) Phase diagram; (b) time series of y for system (1.1) with $h = 0.21$, $p = 0.8$, $\tau = 0.065$, $q = 14$.

arguments have investigated the existence and stability of semitrivial periodic solutions of system (1.1) and have proved that the positive periodic solution comes into being from the semitrivial periodic solution through a transcritical bifurcation according to bifurcation theory. Numerical simulations illustrate the theory and show the complex dynamics of the impulsive system. All these results are expected to be useful in the study of the dynamic complexity of ecosystems.

Acknowledgments

The authors would like to thank the editor and the anonymous referees for their valuable comments and suggestions on this paper. This work was supported by the National Natural Science Foundation of China (Grant no. 31170338 and no. 30970305) and also by the Key Program of Zhejiang Provincial Natural Science Foundation of China (Grant no. LZ12C03001).

References

[1] V. Lakshmikantham, D. Bainov, and P. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[2] G. Jiang and Q. Lu, “Impulsive state feedback control of a predator-prey model,” *Journal of Computational and Applied Mathematics*, vol. 200, no. 1, pp. 193–207, 2007.

[3] J. M. Cushing, “Periodic time-dependent predator-prey systems,” *SIAM Journal on Applied Mathematics*, vol. 32, no. 1, pp. 82–95, 1977.

[4] H. K. Baek, S. D. Kim, and P. Kim, “Permanence and stability of an Ivlev-type predator-prey system with impulsive control strategies,” *Mathematical and Computer Modelling*, vol. 50, no. 9-10, pp. 1385–1393, 2009.

[5] R. Doutt and P. DeBach, *Biological Control of Insect Pests and Weeds*, Reinhold, New York, NY, USA, 1964.

[6] J. Grasman, O. A. Van Herwaarden, L. Hemerik, and J. C. Van Lenteren, “A two-component model of host-parasitoid interactions: determination of the size of inundative releases of parasitoids in biological pest control,” *Mathematical Biosciences*, vol. 169, no. 2, pp. 207–216, 2001.

[7] H. Yu, S. Zhong, R. P. Agarwal, and S. K. Sen, “Three-species food web model with impulsive control strategy and chaos,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 2, pp. 1002–1013, 2011.

[8] H. Baek, “Dynamic complexities of a three-species Beddington-DeAngelis system with impulsive control strategy,” *Acta Applicandae Mathematicae*, vol. 110, no. 1, pp. 23–38, 2010.

[9] W. Z. Gong, Q. F. Zhang, and X. H. Tang, “Existence of subharmonic solutions for a class of second-order p-Laplacian systems with impulsive effects,” *Journal of Applied Mathematics*, vol. 2012, Article ID 434938, 18 pages, 2012.

[10] X. Liu and L. Chen, “Complex dynamics of Holling type II Lotka-Volterra predator-prey system with impulsive perturbations on the predator,” *Chaos, Solitons & Fractals*, vol. 16, no. 2, pp. 311–320, 2003.

[11] Z. Liu, S. Zhong, C. Yin, and W. Chen, “On the dynamics of an impulsive reaction-diffusion predator-prey system with ratio-dependent functional response,” *Acta Applicandae Mathematicae*, vol. 115, no. 3, pp. 329–349, 2011.

[12] H. Yu, S. Zhong, M. Ye, and W. Chen, “Mathematical and dynamic analysis of an ecological model with an impulsive control strategy and distributed time delay,” *Mathematical and Computer Modelling*, vol. 50, no. 11-12, pp. 1622–1635, 2009.

[13] J. Hui and L. Chen, “A Single species model with impulsive diffusion,” *Acta Mathematicae Applicatae Sinica*, vol. 21, no. 1, pp. 43–48, 2005.

[14] H. Yu, S. Zhong, and M. Ye, “Dynamic analysis of an ecological model with impulsive control strategy and distributed time delay,” *Mathematics and Computers in Simulation*, vol. 80, no. 3, pp. 619–632, 2009.

[15] S. A. Hadley and L. K. Forbes, “Dynamical systems analysis of a five-dimensional trophic food web model in the Southern oceans,” *Journal of Applied Mathematics*, vol. 2009, Article ID 575047, 17 pages, 2009.

[16] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, New York, NY, USA, 1983.

[17] L. Zhang and M. Zhao, “Dynamic complexities in a hyperparasitic system with prolonged diapause for host,” *Chaos, Solitons & Fractals*, vol. 42, no. 2, pp. 1136–1142, 2009.

[18] I. M. Cabrera, H. L. B. Díaz, and M. Martínez, “Dynamical system and nonlinear regression for estimate host-parasitoid relationship,” *Journal of Applied Mathematics*, vol. 2010, Article ID 851037, 10 pages, 2010.

[19] M. Zhao, L. Zhang, and J. Zhu, “Dynamics of a host-parasitoid model with prolonged diapause for parasitoid,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 1, pp. 455–462, 2011.

[20] F. D. Chen, “Periodicity in a ratio-dependent predator-prey system with stage structure for predator,” *Journal of Applied Mathematics*, vol. 2005, no. 2, pp. 153–169, 2005.

[21] A. Lakmeche and O. Arino, “Bifurcation of non trivial periodic solutions of impulsive differential equations arising chemotherapeutic treatment,” *Dynamics of Continuous, Discrete & Impulsive Systems B*, vol. 7, no. 2, pp. 265–287, 2000.

[22] S. Tang and L. Chen, “Density-dependent birth rate, birth pulses and their population dynamic consequences,” *Journal of Mathematical Biology*, vol. 44, no. 2, pp. 185–199, 2002.

[23] L. Nie, Z. Teng, L. Hu, and J. Peng, “The dynamics of a Lotka-Volterra predator-prey model with state dependent impulsive harvest for predator,” *BioSystems*, vol. 98, no. 2, pp. 67–72, 2009.

[24] G. Zeng, L. Chen, and L. Sun, “Existence of periodic solution of order one of planar impulsive autonomous system,” *Journal of Computational and Applied Mathematics*, vol. 186, no. 2, pp. 466–481, 2006.

[25] G. Jiang, Q. Lu, and L. Qian, “Complex dynamics of a Holling type II prey-predator system with state feedback control,” *Chaos, Solitons & Fractals*, vol. 31, no. 2, pp. 448–461, 2007.

[26] L. Nie, J. Peng, Z. Teng, and L. Hu, “Existence and stability of periodic solution of a Lotka-Volterra predator-prey model with state dependent impulsive effects,” *Journal of Computational and Applied Mathematics*, vol. 224, no. 2, pp. 544–555, 2009.

[27] L. Qian, Q. Lu, Q. Meng, and Z. Feng, “Dynamical behaviors of a prey-predator system with impulsive control,” *Journal of Mathematical Analysis and Applications*, vol. 363, no. 1, pp. 345–356, 2010.

[28] S. Tang and L. Chen, “Modelling and analysis of integrated pest management strategy,” *Discrete and Continuous Dynamical Systems B*, vol. 4, no. 3, pp. 761–770, 2004.

[29] S. Tang and R. A. Cheke, “State-dependent impulsive models of integrated pest management (IPM) strategies and their dynamic consequences,” *Journal of Mathematical Biology*, vol. 50, no. 3, pp. 257–292, 2005.

[30] V. Ivlev, *Experimental Ecology of the Feeding of Fishes*, Yale University Press, New Haven, Conn, USA, 1961.

[31] R. E. Kooij and A. Zegeling, "A predator-prey model with Ivlev's functional response," *Journal of Mathematical Analysis and Applications*, vol. 198, no. 2, pp. 473–489, 1996.

[32] J. Sugie, "Two-parameter bifurcation in a predator-prey system of Ivlev type," *Journal of Mathematical Analysis and Applications*, vol. 217, no. 2, pp. 349–371, 1998.

[33] J. Feng and S. Chen, "Global asympotic behavior for the competing predators of the Ivlev types," *Mathematica Applicata*, vol. 13, no. 4, pp. 85–88, 2000.

[34] V. L. Ivlev, *Experimental Ecology of the Feeding of Fishes*, Yale University Press, New Haven, Conn, USA, 1955.

[35] H. Wang and W. Wang, "The dynamical complexity of a Ivlev-type prey-predator system with impulsive effect," *Chaos, Solitons & Fractals*, vol. 38, no. 4, pp. 1168–1176, 2008.

[36] H. Wang, "Dispersal permanence of periodic predator-prey model with Ivlev-type functional response and impulsive effects," *Applied Mathematical Modelling*, vol. 34, no. 12, pp. 3713–3725, 2010.

[37] P. Simeonov and D. Bainov, "Orbital stability of periodic solutions of autonomous systems with impulse effect," *International Journal of Systems Science*, vol. 19, no. 12, pp. 2562–2585, 1988.

[38] S. Rasband, *Chaotic Dynamics of Nonlinear Systems*, John Wiley & Sons, New York, NY, USA, 1990.