

## Research Article

# The Modified Rational Jacobi Elliptic Functions Method for Nonlinear Differential Difference Equations

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We modified the rational Jacobi elliptic functions method to construct some new exact solutions for nonlinear differential difference equations in mathematical physics via the lattice equation, the discrete nonlinear Schrodinger equation with a saturable nonlinearity, the discrete nonlinear Klein-Gordon equation, and the quintic discrete nonlinear Schrodinger equation. Some new types of the Jacobi elliptic solutions are obtained for some nonlinear differential difference equations in mathematical physics. The proposed method is more effective and powerful to obtain the exact solutions for nonlinear differential difference equations.

## 1. Introduction

In recent years, the study of difference equations has acquired a new significance, due in large part to their use in the formulation and analysis of discrete-time systems, the numerical integration of differential equation by finite-difference schemes and the study of deterministic chaos.

Since the work of Fermi et al. in the 1960s [1], DDEs have been the focus of many nonlinear studies. On the other hand, a considerable number of well-known analytic methods are successfully extended to nonlinear DDEs by researchers [2–17]. However, no method obeys the strength and the flexibility for finding all solutions to all types of nonlinear DDEs. Zhang et al. [18] and Aslan [19] used the  $(G'/G)$ -expansion function method to some physically important nonlinear DDEs. Qiong and Bin [12] constructed the Jacobi elliptic solutions for nonlinear DDEs. Recently, Zhang [20] and Gepreel [21] and Gepreel and

Shehata [22] have used the Jacobi elliptic function method for constructing new and more general Jacobi elliptic function solutions of the nonlinear differential difference equations. The main objective of this paper is to modify the rational Jacobi elliptic functions method to obtain the exact wave solutions for nonlinear DDEs. We use this method to calculate the exact wave solutions for some nonlinear DDEs in mathematical physics via the lattice equation, the discrete nonlinear Schrodinger equation with a saturable nonlinearity, the discrete nonlinear Klein-Gordon equation, and the quintic discrete nonlinear Schrodinger equation.

## 2. Description of the Modified Rational Jacobi Elliptic Functions Method

In this section, we would like to outline an algorithm for using the modified rational Jacobi elliptic functions method to solve nonlinear DDEs. For a given nonlinear DDEs,

$$\Delta \left( u_{n+p_1}(x), \dots, u_{n+p_k}(x), u'_{n+p_1}(x), \dots, u'_{n+p_k}(x), \dots, u_{n+p_1}^{(r)}(x), \dots, u_{n+p_k}^{(r)}(x), \right. \\ \left. v_{n+p_1}(x), \dots, v_{n+p_k}(x), v'_{n+p_1}(x), \dots, v'_{n+p_k}(x), \dots, v_{n+p_1}^{(r)}(x), \dots, v_{n+p_k}^{(r)}(x), \dots \right) = 0, \quad (2.1)$$

where  $\Delta = (\Delta_1, \dots, \Delta_g)$ ,  $x = (x_1, x_2, \dots, x_m)$ ,  $n = (n_1, \dots, n_Q)$ , and  $g, m, Q, p_1, \dots, p_k$  are integers,  $u_i^{(r)}$ ,  $v_i^{(r)}$  denotes the set of all  $r$ th order derivatives of  $u_i$ ,  $v_i$  with respect to  $x$ .

The main steps of the algorithm for the modified rational Jacobi elliptic functions method to nonlinear DDEs are outlined as follows.

*Step 1.* We seek the traveling wave solutions of the following form:

$$u_n(x) = U(\xi_n), \quad v_n(x) = V(\xi_n), \dots, \quad (2.2)$$

where

$$\xi_n = \sum_{i=1}^Q d_i n_i + \sum_{j=1}^m c_j x_j + \xi_0, \quad (2.3)$$

where  $d_i$  ( $i = 1, \dots, Q$ ),  $c_j$  ( $j = 1, \dots, m$ ) and the phase  $\xi_0$  are constants to be determined later. The transformations (2.2) are reduced (2.1) to the following ordinary differential difference equations:

$$\Omega \left( U(\xi_{n+p_1}), \dots, U(\xi_{n+p_k}), U'(\xi_{n+p_1}), \dots, U'(\xi_{n+p_k}), \dots, U^{(r)}(\xi_{n+p_1}), \dots, U^{(r)}(\xi_{n+p_k}), \right. \\ \left. V(\xi_{n+p_1}), \dots, V(\xi_{n+p_k}), V'(\xi_{n+p_1}), \dots, V'(\xi_{n+p_k}), \dots, V_{n+p_1}^{(r)}(\xi_{n+p_1}), \dots, V_{n+p_k}^{(r)}(\xi_{n+p_k}), \dots \right) = 0, \quad (2.4)$$

where  $\Omega = (\Omega_1, \dots, \Omega_g)$ .

Step 2. We suppose the modified rational Jacobi elliptic functions solutions of (2.4) in the following form:

$$\begin{aligned} U(\xi_n) &= \sum_{i=0}^K \alpha_i \left( \frac{F'(\xi_n)}{F(\xi_n)} \right)^i + \sum_{i=1}^K \beta_i \left( \frac{F(\xi_n)}{F'(\xi_n)} \right)^i, \\ V(\xi_n) &= \sum_{j=0}^L \gamma_j \left( \frac{F'(\xi_n)}{F(\xi_n)} \right)^j + \sum_{j=1}^L \lambda_j \left( \frac{F(\xi_n)}{F'(\xi_n)} \right)^j, \dots, \end{aligned} \quad (2.5)$$

where  $\alpha_i, \beta_i$  ( $i = 1, 2, \dots, K$ ),  $\gamma_j, \lambda_j$  ( $j = 1, 2, \dots, L$ ) are constants to be determined later while  $F(\xi_n)$  satisfies a discrete Jacobi elliptic differential equation:

$$F'^2(\xi_n) = e_0 + e_1 F^2(\xi_n) + e_2 F^4(\xi_n), \quad (2.6)$$

and  $e_0, e_1, e_2$  are arbitrary constants.

Step 3. Since the general solution of the proposed equation (2.6) is difficult to obtain and so the iteration relations corresponding to the general exact solutions we discuss the solutions of the proposed discrete Jacobi elliptic differential equation (2.6) at some special cases to  $e_0, e_1$ , and  $e_2$ , to cover all the Jacobi elliptic functions as follows.

Type 1. If  $e_0 = 1, e_1 = -(1 + m^2), e_2 = m^2$ . In this case, (2.6) has the solution  $F(\xi_n) = sn(\xi_n, m)$ , where  $sn(\xi_n, m)$  is the Jacobi elliptic sine function, and  $m$  is the modulus. In this case from using the properties of Jacobi elliptic functions (see [22]), the series expansion solutions (2.5) take the following form:

$$\begin{aligned} U(\xi_n) &= \sum_{i=0}^K \alpha_i \left( \frac{cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} \right)^i + \sum_{i=1}^K \beta_i \left( \frac{sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)} \right)^i, \\ V(\xi_n) &= \sum_{i=0}^L \gamma_i \left( \frac{cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} \right)^i + \sum_{i=1}^L \lambda_i \left( \frac{sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)} \right)^i, \dots \end{aligned} \quad (2.7)$$

Further using the properties of Jacobi elliptic functions, the iterative relations can be written in the following form:

$$\begin{aligned} U(\xi_{n\pm p}) &= \sum_{i=0}^K \alpha_i \left( \frac{F'(\xi_{n\pm p})}{F(\xi_{n\pm p})} \right)^i + \sum_{i=1}^K \beta_i \left( \frac{F(\xi_{n\pm p})}{F'(\xi_{n\pm p})} \right)^i, \\ V(\xi_{n\pm p}) &= \sum_{i=0}^L \gamma_i \left( \frac{F'(\xi_{n\pm p})}{F(\xi_{n\pm p})} \right)^i + \sum_{i=1}^L \lambda_i \left( \frac{F(\xi_{n\pm p})}{F'(\xi_{n\pm p})} \right)^i, \dots \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} \frac{F'(\xi_{n\pm p})}{F(\xi_{n\pm p})} &= \frac{1}{M_1} \left\{ \pm cn(d, m) cn(\xi_n, m) dn(\xi_n, m) dn(d, m) \pm m^2 sn(d, m) sn(\xi_n, m) \right. \\ &\quad \mp 2m^2 sn(d, m) sn^3(\xi_n, m) \mp 2m^2 sn^3(d, m) sn(\xi_n, m) \pm m^2 sn^3(d, m) sn^3(\xi_n, m) \\ &\quad + sn(d, m) sn(\xi_n, m) \pm m^4 sn^3(d, m) sn^3(\xi_n, m) \\ &\quad \left. \mp m^2 sn^2(d, m) sn^2(\xi_n, m) dn(\xi_n, m) dn(d, m) cn(d, m) cn(\xi_n, m) \right\}, \\ M_1 &= -cn(\phi, m) dn(\phi, m) sn(\xi_n, m) \mp sn(\phi, m) dn(\xi_n, m) (\phi) cn(\xi_n, m) \\ &\quad + m^2 sn^3(\xi_n, m) sn^2(\phi, m) cn(\phi, m) dn(\phi, m) \\ &\quad \pm m^2 sn^2(\xi_n, m) sn^3(\phi, m) cn(\xi_n, m) dn(\xi_n, m), \end{aligned} \tag{2.9}$$

$d = p_{s1}d_1 + p_{s2}d_2 + \dots + p_{sQ}d_Q$ ,  $p_{sj}$  is the  $j$ th component of shift vector  $p_s$ .

*Type 2.* If  $e_0 = 1 - m^2$ ,  $e_1 = 2m^2 - 1$ , and  $e_2 = -m^2$ . In this case, (2.6) has the solution  $F(\xi_n) = cn(\xi_n, m)$ . From using the properties of Jacobi elliptic functions, the series expansion solutions (2.5) take the following form:

$$\begin{aligned} U(\xi_n) &= \sum_{i=0}^K \alpha_i \left( -\frac{sn(\xi_n, m) dn(\xi_n, m)}{cn(\xi_n, m)} \right)^i + \sum_{i=1}^K \beta_i \left( -\frac{cn(\xi_n, m)}{sn(\xi_n, m) dn(\xi_n, m)} \right)^i, \\ V(\xi_n) &= \sum_{i=0}^L \gamma_i \left( -\frac{sn(\xi_n, m) dn(\xi_n, m)}{cn(\xi_n, m)} \right)^i + \sum_{i=1}^L \lambda_i \left( -\frac{cn(\xi_n, m)}{sn(\xi_n, m) dn(\xi_n, m)} \right)^i, \dots \end{aligned} \tag{2.10}$$

*Type 3.* If  $e_0 = m^2 - 1$ ,  $e_1 = 2 - m^2$ , and  $e_2 = -1$ . In this case, (2.6) has the solution  $F(\xi_n) = dn(\xi_n, m)$ . From using the properties of Jacobi elliptic functions, the series expansion solutions (2.5) take the following form:

$$\begin{aligned} U(\xi_n) &= \sum_{i=0}^K \alpha_i \left( -\frac{m^2 sn(\xi_n, m) cn(\xi_n, m)}{dn(\xi_n, m)} \right)^i + \sum_{i=1}^K \beta_i \left( -\frac{dn(\xi_n, m)}{m^2 sn(\xi_n, m) cn(\xi_n, m)} \right)^i, \\ V(\xi_n) &= \sum_{i=0}^L \gamma_i \left( -\frac{m^2 sn(\xi_n, m) cn(\xi_n, m)}{dn(\xi_n, m)} \right)^i + \sum_{i=1}^L \lambda_i \left( -\frac{dn(\xi_n, m)}{m^2 sn(\xi_n, m) cn(\xi_n, m)} \right)^i, \dots \end{aligned} \tag{2.11}$$

From the properties of the Jacobi elliptic functions, we can deduce the iterative relation to the above kind of solutions from Types 2 and 3 as we show in Type 1.

*Step 4.* Determining the degree  $K, L, \dots$  of (2.5) by balancing the nonlinear term(s) and the highest order derivatives of  $U(\xi_n), V(\xi_n), \dots$  in (2.4). It should be noted that the leading terms  $U(\xi_{n\pm p}), V(\xi_{n\pm p}), \dots, p \neq 0$  will not affect the balance because we are interested in balancing the terms of  $F'(\xi_n)/F(\xi_n)$ .

*Step 5.* Substituting  $U(\xi_n), V(\xi_n), \dots$  in each type form (2.1)–(2.4) and the given values of  $K, L, \dots$  into (2.4). Cleaning the denominator and collecting all terms with the same degree of  $sn(\xi_n, m), cn(\xi_n, m), dn(\xi_n, m)$  together, the left-hand side of (2.4) is converted into a polynomial in  $sn(\xi_n, m), cn(\xi_n, m), dn(\xi_n, m)$ . Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for  $\alpha_i, \beta_i, d_i, \gamma_i, \lambda_i$ , and  $c_i$ .

*Step 6.* Solving the over determined system of nonlinear algebraic equations by using Maple or Mathematica. We end up with explicit expressions for  $\alpha_i, \beta_i, d_i, \gamma_i, \lambda_i$ , and  $c_j$ .

*Step 7.* Substituting  $\alpha_i, \beta_i, d_i, \gamma_i, \lambda_i$ , and  $c_i$  into  $U(\xi_n), V(\xi_n), \dots$  in the corresponding type from (2.1)–(2.4), we can finally obtain exact solutions for (2.1).

### 3. Applications

In this section, we use the proposed method to construct the rational Jacobi elliptic wave solutions for the nonlinear DDEs via the lattice equation, the discrete nonlinear Schrodinger equation with a saturable nonlinearity, the discrete nonlinear Klein-Gordon equation, and the quintic discrete nonlinear Schrodinger equation, which are very important in mathematical physics and have been paid attention to by many researchers.

#### 3.1. Example 1. The Lattice Equation

In this section, we study the lattice equation which takes the following form [23–26]:

$$\frac{du_n(t)}{dt} = (\alpha + \beta u_n + \gamma u_n^2)(u_{n-1} - u_{n+1}), \quad (3.1)$$

where  $\alpha, \beta$ , and  $\gamma$  are nonzero constants. This equation contains hybrid lattice equation, mKdV lattice equation, modified Volterra lattice equation, and Langmuir chain equation for some special values of  $\alpha, \beta, \gamma$ . According to the above steps, to seek traveling wave solutions of (3.1), we construct the transformation

$$u_n(t) = U(\xi_n), \quad \xi_n = dn + c_1t + \xi_0, \quad (3.2)$$

where  $d, c_1$ , and  $\xi_0$  are constants. The transformation (3.2) permits us converting equation (3.1) into the following form:

$$c_1 U'(\xi_n) = \left[ \alpha + \beta U(\xi_n) + \gamma U^2(\xi_n) \right] [U(\xi_n - d) - U(\xi_n + d)], \quad (3.3)$$

where  $' = d/d\xi_n$ . Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.3), we get  $K = 1$ . Thus, the solution of (3.3) has the following form:

$$U(\xi_n) = \alpha_0 + \alpha_1 \left( \frac{F'(\xi_n)}{F(\xi_n)} \right) + \beta_1 \left( \frac{F(\xi_n)}{F'(\xi_n)} \right), \quad (3.4)$$

where  $\alpha_0$ ,  $\alpha_1$ , and  $\beta_1$  are constants to be determined later and  $F(\xi_n)$  satisfies a discrete Jacobi elliptic ordinary differential equation (2.6). When we discuss the solutions of (2.6), we get the following types.

*Type 1.* If  $e_0 = 1$ ,  $e_1 = -(1 + m^2)$ , and  $e_2 = m^2$ . In this case, the series expansion solution of (3.3) has the form:

$$U(\xi_n) = \alpha_0 + \frac{\alpha_1 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} + \frac{\beta_1 sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)}. \quad (3.5)$$

With the help of Maple, we substitute (3.5) and (2.8) into (3.3). Cleaning the denominator and collecting all terms with the same degree of  $sn(\xi_n, m)$ ,  $cn(\xi_n, m)$ ,  $dn(\xi_n, m)$  together, the left-hand side of (3.3) is converted into polynomial in  $sn(\xi_n, m)$ ,  $cn(\xi_n, m)$ ,  $dn(\xi_n, m)$ . Setting each coefficient of this polynomial to be zero, we derive a set of algebraic equations for  $\alpha_0$ ,  $\alpha_1$ ,  $d$ ,  $\beta_1$ ,  $\gamma$ ,  $\beta$ ,  $\alpha$ , and  $c_1$ . Solving the set of algebraic equations by using Maple or Mathematica software package, we have the following.

*Family 1.*

$$\begin{aligned} \alpha_0 &= \frac{-\beta}{2\gamma}, & \beta_1 &= (m^2 - 1)\alpha_1, & c_1 &= \frac{2\gamma\alpha_1^2 [cn^2(d, m) - sn^2(d, m)dn^2(d, m)]}{sn(d, m)cn(d, m)dn(d, m)}, \\ \alpha &= \frac{\beta^2}{4\gamma} + \frac{\alpha_1^2\gamma [-m^4sn^8(d, m) + 4m^2sn^6(d, m) - (4 + 2m^2)sn^4(d, m) + 4sn^2(d, m) - 1]}{sn^2(d, m)cn^2(d, m)dn^2(d, m)}, \end{aligned} \quad (3.6)$$

where  $\alpha_1$ ,  $d$ ,  $\gamma$ ,  $\beta$ , and  $m$  are arbitrary constants.

From (3.5) and (3.6), the solution of (3.3) takes the following form:

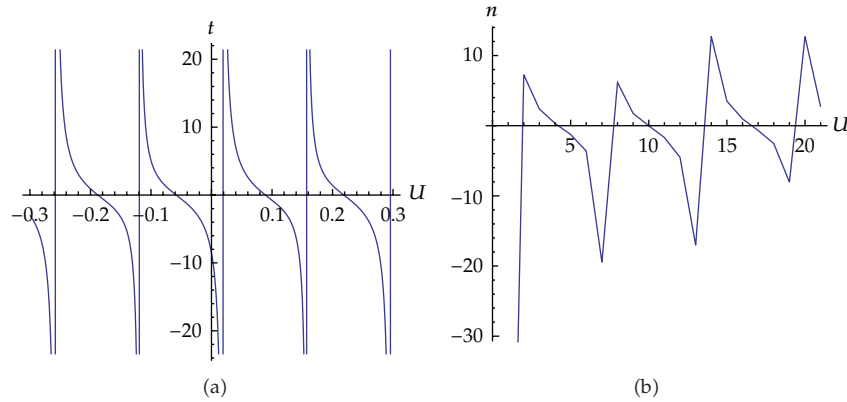
$$U(\xi_n) = -\frac{\beta}{2\gamma} + \frac{\alpha_1 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} + \frac{\alpha_1 (m^2 - 1) sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)}, \quad (3.7)$$

where  $\xi_n = dn + (2\gamma\alpha_1^2 [cn^2(d, m) - sn^2(d, m)dn^2(d, m)] / sn(d, m)cn(d, m)dn(d, m))t + \xi_0$ . Figure 1 illustrates the behavior of the exact solution (3.7).

*Family 2.*

$$\begin{aligned} \alpha_0 &= \frac{-\beta}{2\gamma}, & \beta_1 &= -(m^2 - 1)\alpha_1, & c_1 &= \frac{2\gamma\alpha_1^2 [dn^2(d, m) - m^2sn^2(d, m)cn^2(d, m)]}{sn(d, m)cn(d, m)dn(d, m)}, \\ \alpha &= \frac{\beta^2}{4\gamma} + \frac{\alpha_1^2\gamma [-m^4sn^8(d, m) + 4m^4sn^6(d, m) - (4m^4 + 2m^2)sn^4(d, m) + 4m^2sn^2(d, m) - 1]}{sn^2(d, m)cn^2(d, m)dn^2(d, m)}, \end{aligned} \quad (3.8)$$

where  $\alpha_1$ ,  $d$ ,  $\gamma$ ,  $\beta$ , and  $m$  are arbitrary constants.



**Figure 1:** (a) represents the Jacobi elliptic solution (3.7) when  $\beta = m = 0.5, \gamma = 1, \alpha_1 = 1.5, d = 0.3, n = 2$ . (b) represents the Jacobi elliptic solution (3.7) when  $\beta = m = 0.5, \gamma = 1, \alpha_1 = 1.5, d = 0.3, t = 2$ .

From (3.5) and (3.8), the solution of (3.3) has the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} + \frac{\alpha_1 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} - \frac{\alpha_1 (m^2 - 1) sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)}, \quad (3.9)$$

where  $\xi_n = dn + (2\gamma\alpha_1^2 [dn^2(d, m) - m^2 sn^2(d, m) cn^2(d, m)] / sn(d, m) cn(d, m) dn(d, m))t + \xi_0$ .

*Type 2.* If  $e_0 = 1 - m^2, e_1 = 2m^2 - 1$ , and  $e_2 = -m^2$ . In this case, the solution of (3.3) has the form:

$$U(\xi_n) = \alpha_0 - \alpha_1 \frac{sn(\xi_n, m) dn(\xi_n, m)}{cn(\xi_n, m)} - \beta_1 \frac{cn(\xi_n, m)}{sn(\xi_n, m) dn(\xi_n, m)}. \quad (3.10)$$

With the help of Maple, we substitute (3.10) into (3.3). Cleaning the denominator and collecting all terms with the same degree of  $sn(\xi_n, m), cn(\xi_n, m), dn(\xi_n, m)$  together, the left-hand side of (3.3) is converted into polynomial in  $sn(\xi_n, m), cn(\xi_n, m), dn(\xi_n, m)$ . Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations in  $\alpha_0, \alpha_1, d, \beta_1, \gamma, \beta, \alpha$ , and  $c_1$ . Solving the set of algebraic equations by using Maple or Mathematica software package, we get the following.

*Family 1.*

$$\alpha_0 = \frac{-\beta}{2\gamma}, \quad \beta_1 = -\alpha_1, \quad c_1 = \frac{2\gamma\alpha_1^2 [cn^2(d, m) - sn^2(d, m) dn^2(d, m)]}{sn(d, m) cn(d, m) dn(d, m)}, \quad (3.11)$$

$$\alpha = \frac{\beta^2}{4\gamma} + \frac{\alpha_1^2 \gamma [-m^4 sn^8(d, m) + 4m^2 sn^6(d, m) - (4 + 2m^2) sn^4(d, m) + 4sn^2(d, m) - 1]}{sn^2(d, m) cn^2(d, m) dn^2(d, m)},$$

where  $\alpha_1, d, \gamma, \beta$ , and  $m$  are arbitrary constants.

In this case, the solution of (3.3) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\alpha_1 \operatorname{sn}(\xi_n, m) \operatorname{dn}(\xi_n, m)}{\operatorname{cn}(\xi_n, m)} + \frac{\alpha_1 \operatorname{cn}(\xi_n, m)}{\operatorname{sn}(\xi_n, m) \operatorname{dn}(\xi_n, m)}, \quad (3.12)$$

where  $\xi_n = dn + (2\gamma\alpha_1^2[\operatorname{cn}^2(d, m) - \operatorname{sn}^2(d, m)\operatorname{dn}^2(d, m)]/\operatorname{sn}(d, m)\operatorname{cn}(d, m)\operatorname{dn}(d, m))t + \xi_0$ .

The Jacobi elliptic functions could be generated into the hyperbolic functions when  $m$  tends to one in the other hand, they are generated into trigonometrical functions when  $m$  tends to zero.

When  $m = 0$ , the trigonometrical solution (3.12) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} + 2\alpha_1 \cot(2\xi_n), \quad \text{where } \xi_n = dn + 4\gamma\alpha_1^2 \cot(2d)t + \xi_0. \quad (3.13)$$

Also if  $m = 1$ , the hyperbolic solution (3.12) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} + \frac{2\alpha_1}{\sinh(2\xi_n)}, \quad \text{where } \xi_n = dn + \frac{4\gamma\alpha_1^2}{\sinh(2d)}t + \xi_0. \quad (3.14)$$

*Family 2.*

$$\begin{aligned} \alpha_0 &= \frac{-\beta}{2\gamma}, & \beta_1 &= \alpha_1, & c_1 &= \frac{-2\gamma\alpha_1^2[m^2\operatorname{sn}^4(d, m) - 1]}{\operatorname{sn}(d, m)\operatorname{cn}(d, m)\operatorname{dn}(d, m)}, \\ \alpha &= \frac{\beta^2}{4\gamma} + \frac{\alpha_1^2\gamma[-m^4\operatorname{sn}^8(d, m) + 2m^2\operatorname{sn}^4(d, m) - 1]}{\operatorname{sn}^2(d, m)\operatorname{cn}^2(d, m)\operatorname{dn}^2(d, m)}, \end{aligned} \quad (3.15)$$

where  $\alpha_1, d, \gamma, \beta$ , and  $m$  are arbitrary constants.

In this case, the solution of (3.3) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\alpha_1 \operatorname{sn}(\xi_n, m) \operatorname{dn}(\xi_n, m)}{\operatorname{cn}(\xi_n, m)} - \frac{\alpha_1 \operatorname{cn}(\xi_n, m)}{\operatorname{sn}(\xi_n, m) \operatorname{dn}(\xi_n, m)}, \quad (3.16)$$

where  $\xi_n = dn - (2\gamma\alpha_1^2[m^2\operatorname{sn}^4(d, m) - 1]/\operatorname{sn}(d, m)\operatorname{cn}(d, m)\operatorname{dn}(d, m))t + \xi_0$ .

When  $m = 0$ , the trigonometrical solution (3.16) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{2\alpha_1}{\sin(2\xi_n)}, \quad \text{where } \xi_n = dn + \frac{4\gamma\alpha_1^2}{\sin(2d)}t + \xi_0. \quad (3.17)$$

When  $m = 1$ , the hyperbolic solution (3.16) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - 2\alpha_1 \coth(2\xi_n), \quad \text{where } \xi_n = dn + 4\gamma\alpha_1^2 \coth(2d)t + \xi_0. \quad (3.18)$$



Type 3. If  $e_0 = m^2 - 1$ ,  $e_1 = 2 - m^2$ , and  $e_2 = -1$ . In this case, the series expansion solution of (3.3) has the form:

$$U(\xi_n) = \alpha_0 - \frac{m^2 \alpha_1 \operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}{\operatorname{dn}(\xi_n, m)} - \frac{\beta_1 \operatorname{dn}(\xi_n, m)}{m^2 \operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}. \quad (3.19)$$

Consequently, using the Maple or Mathematica we get the following results.

Family 1.

$$\begin{aligned} \alpha_0 &= \frac{-\beta}{2\gamma}, & \beta_1 &= -m^2 \alpha_1, & c_1 &= \frac{2\gamma \alpha_1^2 [\operatorname{dn}^2(d, m) - m^2 \operatorname{sn}^2(d, m) \operatorname{cn}^2(d, m)]}{\operatorname{sn}(d, m) \operatorname{cn}(d, m) \operatorname{dn}(d, m)}, \\ \alpha &= \frac{\beta^2}{4\gamma} + \frac{\alpha_1^2 \gamma [-m^4 \operatorname{sn}^8(d, m) + 4m^4 \operatorname{sn}^6(d, m) - (4m^4 + 2m^2) \operatorname{sn}^4(d, m) + 4m^2 \operatorname{sn}^2(d, m) - 1]}{\operatorname{sn}^2(d, m) \operatorname{cn}^2(d, m) \operatorname{dn}^2(d, m)}, \end{aligned} \quad (3.20)$$

where  $\alpha_1, d, \gamma, \beta$ , and  $m$  are arbitrary constants.

In this case, the solution of (3.3) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\alpha_1 m^2 \operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}{\operatorname{dn}(\xi_n, m)} + \frac{\alpha_1 \operatorname{dn}(\xi_n, m)}{\operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}, \quad (3.21)$$

where  $\xi_n = \operatorname{dn} + (2\gamma \alpha_1^2 [\operatorname{dn}^2(d, m) - m^2 \operatorname{sn}^2(d, m) \operatorname{cn}^2(d, m)] / \operatorname{sn}(d, m) \operatorname{cn}(d, m) \operatorname{dn}(d, m))t + \xi_0$ .

Family 2.

$$\begin{aligned} \alpha_0 &= \frac{-\beta}{2\gamma}, & \beta_1 &= \alpha_1 m^2, & c_1 &= \frac{-2\gamma \alpha_1^2 [m^2 \operatorname{sn}^4(d, m) - 1]}{\operatorname{sn}(d, m) \operatorname{cn}(d, m) \operatorname{dn}(d, m)}, \\ \alpha &= \frac{\beta^2}{4\gamma} + \frac{\alpha_1^2 \gamma [-m^4 \operatorname{sn}^8(d, m) + 2m^2 \operatorname{sn}^4(d, m) - 1]}{\operatorname{sn}^2(d, m) \operatorname{cn}^2(d, m) \operatorname{dn}^2(d, m)}, \end{aligned} \quad (3.22)$$

where  $\alpha_1, d, \gamma, \beta$ , and  $m$  are arbitrary constants.

In this case, the solution of (3.3) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\alpha_1 m^2 \operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}{\operatorname{dn}(\xi_n, m)} - \frac{\alpha_1 \operatorname{dn}(\xi_n, m)}{\operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}, \quad (3.23)$$

where  $\xi_n = \operatorname{dn} - (2\gamma \alpha_1^2 [m^2 \operatorname{sn}^4(d, m) - 1] / \operatorname{sn}(d, m) \operatorname{cn}(d, m) \operatorname{dn}(d, m))t + \xi_0$ .

### 3.2. Example 2. The Discrete Nonlinear Schrodinger Equation

The discrete nonlinear Schrodinger equation (DNSE) is one of the most fundamental nonlinear lattice models [8]. Its arise in nonlinear optics as a model of infinite wave guide

arrays [27] and has been recently implemented to describe Bose-Einstein condensates in optical lattices. The class of DNSE model with saturable nonlinearity is also of particular interest in their own right, due to a feature first unveiled in [28]. In this section, we study the DNSE with a saturable nonlinearity [29, 30] form:

$$i \frac{\partial \psi_n}{\partial t} + (\psi_{n+1} + \psi_{n-1} - 2\psi_n) + \frac{\nu |\psi_n|^2}{1 + \mu |\psi_n|^2} \psi_n = 0, \quad (3.24)$$

which describes optical pulse propagations in various doped fibers,  $\psi_n$  is a complex valued wave function at sites  $n$  while  $\nu$  and  $\mu$  are real parameters. We make the transformation:

$$\psi_n = \phi(\xi_n) e^{-i(\sigma t + \rho)}, \quad \xi_n = \alpha n + \beta, \quad (3.25)$$

where  $\sigma$ ,  $\rho$ ,  $\alpha$ , and  $\beta$  are arbitrary constants. The transformation (3.25) permits us converting equation (3.24) into the following nonlinear difference equation:

$$(\sigma - 2)\phi(\xi_n) + \phi(\xi_n + \alpha) + \phi(\xi_n - \alpha) + \frac{\nu \phi^3(\xi_n)}{1 + \mu \phi^2(\xi_n)} = 0. \quad (3.26)$$

We assume that (3.26) has a solution of the form:

$$\phi(\xi_n) = \alpha_1 \left( \frac{F'(\xi_n)}{F(\xi_n)} \right) + \beta_1 \left( \frac{F(\xi_n)}{F'(\xi_n)} \right) + \alpha_0, \quad (3.27)$$

where  $\alpha_0$ ,  $\alpha_1$ , and  $\beta_1$  are constants to be determined later and  $F(\xi_n)$  satisfies a discrete Jacobi elliptic differential equation (2.6). When we discuss the solutions of (3.26), we have the following types.

*Type 1.* If  $e_0 = 1$ ,  $e_1 = -(1 + m^2)$ , and  $e_2 = m^2$ . In this case, the series expansion solution of (3.26) has the form:

$$\phi(\xi_n) = \alpha_1 \frac{cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} + \beta_1 \frac{sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)} + \alpha_0. \quad (3.28)$$

With the help of Maple, we substitute (3.28) and (2.8) into (3.26), cleaning the denominator and collecting all terms with the same order of  $sn(\xi_n, m)$ ,  $cn(\xi_n, m)$ ,  $dn(\xi_n, m)$  together, the left-hand side of (3.26) is converted into polynomial in  $sn(\xi_n, m)$ ,  $cn(\xi_n, m)$ ,  $dn(\xi_n, m)$ . Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for  $\alpha_0$ ,  $\alpha_1$ ,  $\sigma$ ,  $\beta_1$ ,  $\rho$ ,  $\alpha$ , and  $\beta$ . Solving the set of algebraic equations by using Maple or Mathematica software package, we obtain the following.

Family 1.

$$\alpha_1 = \frac{sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1]}, \quad \beta_1 = \frac{(m^2 - 1)sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1]},$$

$$\mu < 0,$$

$$v = \frac{-2\mu[m^4sn^8(\alpha, m) - 2m^4sn^6(\alpha, m) + 2m^2sn^2(\alpha, m) - 1]}{m^4sn^8(\alpha, m) - 4m^2sn^6(\alpha, m) + (2m^2 + 4)sn^4(\alpha, m) - 4sn^2(\alpha, m) + 1}, \quad \alpha_0 = 0,$$

$$\sigma = \frac{4sn^2(\alpha, m)[m^4sn^6(\alpha, m) - (m^4 + 2m^2)sn^4(\alpha, m) + (m^2 + 2)sn^2(\alpha, m) + m^2 - 2]}{m^4sn^8(\alpha, m) - 4m^2sn^6(\alpha, m) + (2m^2 + 4)sn^4(\alpha, m) - 4sn^2(\alpha, m) + 1}. \quad (3.29)$$

In this case, the solution of (3.24) takes the following form:

$$\psi_n = \left( \frac{sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)cn(\xi_n, m)dn(\xi_n, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1]sn(\xi_n, m)} \right. \\ \left. + \frac{(m^2 - 1)sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)sn(\xi_n, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1]cn(\xi_n, m)dn(\xi_n, m)} \right) \\ \times \left( \text{Exp} \left\{ -i \left[ \frac{4tsn^2(\alpha, m)\mathfrak{A}}{m^4sn^8(\alpha, m) - 4m^2sn^6(\alpha, m) + (2m^2 + 4)sn^4(\alpha, m) - 4sn^2(\alpha, m) + 1} + \rho \right] \right\} \right), \quad (3.30)$$

where  $\mathfrak{A}$  denotes  $[m^4sn^6(\alpha, m) - (m^4 + 2m^2)sn^4(\alpha, m) + (m^2 + 2)sn^2(\alpha, m) + m^2 - 2]$  and  $\xi_n = \alpha n + \beta$ . Figure 2 illustrates the behavior exact solution (3.30).

Family 2.

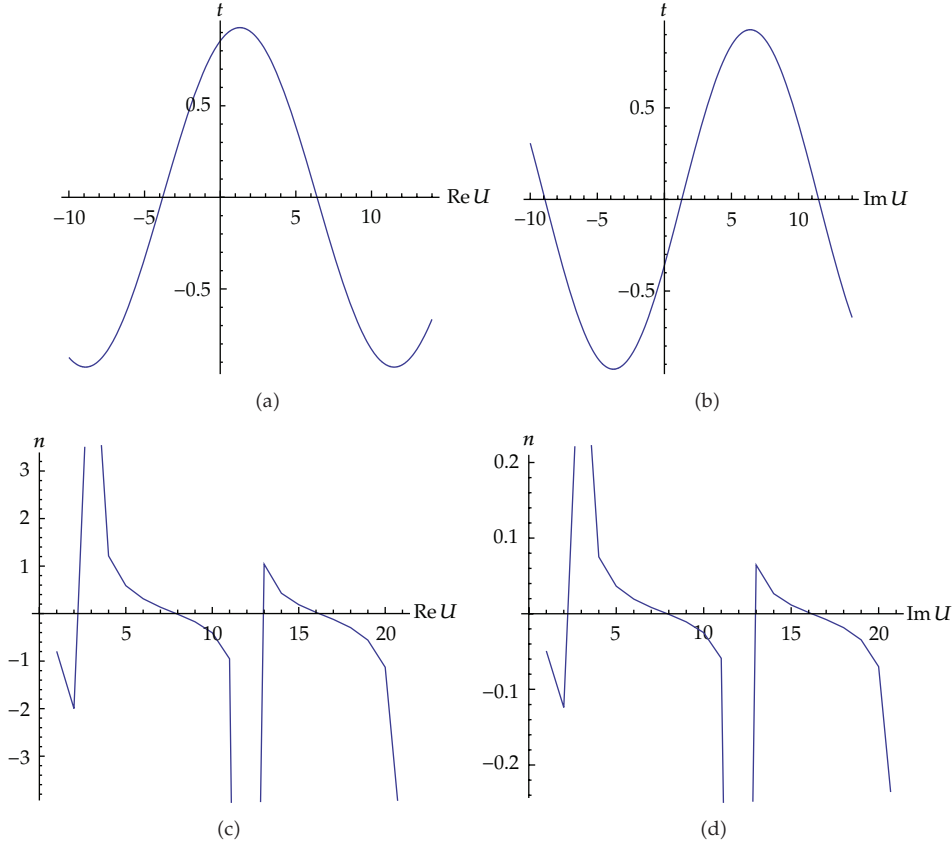
$$\alpha_1 = \frac{sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 2m^2sn^2(\alpha, m) + 1]},$$

$$\beta_1 = -\frac{(m^2 - 1)sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 2m^2sn^2(\alpha, m) + 1]},$$

$$\mu < 0,$$

$$v = \frac{-2\mu[m^4sn^8(\alpha, m) - 2m^2sn^6(\alpha, m) + 2sn^2(\alpha, m) - 1]}{m^4sn^8(\alpha, m) - 4m^4sn^6(\alpha, m) + (2m^2 + 4m^4)sn^4(\alpha, m) - 4m^2sn^2(\alpha, m) + 1},$$

$$\alpha_0 = 0,$$



**Figure 2:** (a) and (b) represent the Jacobi elliptic solution (3.30) when  $m = 0.5$ ,  $\beta = 1.5$ ,  $\alpha = 0.2$ ,  $\rho = 0.4$ ,  $n = 2$ . (c) and (d) represent the Jacobi elliptic solution (3.30) when  $m = 0.5$ ,  $\beta = 1.5$ ,  $\alpha = 0.2$ ,  $\rho = 0.4$ ,  $t = 1.5$ .

$$\sigma = \frac{4sn^2(\alpha, m) [m^4 sn^6(\alpha, m) - (2m^4 + m^2) sn^4(\alpha, m) + (2m^4 + m^2) sn^2(\alpha, m) - 2m^2 + 1]}{m^4 sn^8(\alpha, m) - 4m^4 sn^6(\alpha, m) + (2m^2 + 4m^4) sn^4(\alpha, m) - 4m^2 sn^2(\alpha, m) + 1}. \quad (3.31)$$

In this case, the solution of (3.24) takes the following form:

$$\begin{aligned} \psi_n = & \left( \frac{sn(\alpha, m) cn(\alpha, m) dn(\alpha, m) cn(\xi_n, m) dn(\xi_n, m)}{\sqrt{-\mu} [m^2 sn^4(\alpha, m) - 2m^2 sn^2(\alpha, m) + 1] sn(\xi_n, m)} \right. \\ & \left. \frac{(m^2 - 1) sn(\alpha, m) cn(\alpha, m) dn(\alpha, m) sn(\xi_n, m)}{\sqrt{-\mu} [m^2 sn^4(\alpha, m) - 2m^2 sn^2(\alpha, m) + 1] cn(\xi_n, m) dn(\xi_n, m)} \right) \\ & \times \left( \text{Exp} \left\{ -i \left[ \frac{4t sn^2(\alpha, m) \mathfrak{W}}{m^4 sn^8(\alpha, m) - 4m^4 sn^6(\alpha, m) + (2m^2 + 4m^4) sn^4(\alpha, m) - 4m^2 sn^2(\alpha, m) + 1} + \rho \right] \right\} \right), \quad (3.32) \end{aligned}$$

where  $\mathfrak{W}$  denotes  $[m^4 sn^6(\alpha, m) - (2m^4 + m^2) sn^4(\alpha, m) + (2m^4 + m^2) sn^2(\alpha, m) - 2m^2 + 1]$  and  $\xi_n = \alpha n + \beta$ .

Type 2. If  $e_0 = 1 - m^2$ ,  $e_1 = 2m^2 - 1$ , and  $e_2 = -m^2$ . In this case, the solution of (3.26) has the form:

$$\phi(\xi_n) = \alpha_0 - \alpha_1 \frac{sn(\xi_n, m) dn(\xi_n, m)}{cn(\xi_n, m)} - \beta_1 \frac{cn(\xi_n, m)}{sn(\xi_n, m) dn(\xi_n, m)}. \quad (3.33)$$

Consequently, using Maple or Mathematica we get the following results.

Family 1.

$$\alpha_1 = \frac{sn(\alpha, m) cn(\alpha, m) dn(\alpha, m)}{\sqrt{-\mu} [m^2 sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1]}, \quad \beta_1 = -\frac{sn(\alpha, m) cn(\alpha, m) dn(\alpha, m)}{\sqrt{-\mu} [m^2 sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1]},$$

$$\mu < 0,$$

$$v = \frac{-2\mu [m^4 sn^8(\alpha, m) - 2m^4 sn^6(\alpha, m) + 2m^2 sn^2(\alpha, m) - 1]}{m^4 sn^8(\alpha, m) - 4m^2 sn^6(\alpha, m) + (2m^2 + 4) sn^4(\alpha, m) - 4sn^2(\alpha, m) + 1}, \quad \alpha_0 = 0,$$

$$\sigma = \frac{4sn^2(\alpha, m) [m^4 sn^6(\alpha, m) - (m^4 + 2m^2) sn^4(\alpha, m) + (m^2 + 2) sn^2(\alpha, m) + m^2 - 2]}{m^4 sn^8(\alpha, m) - 4m^2 sn^6(\alpha, m) + (2m^2 + 4) sn^4(\alpha, m) - 4sn^2(\alpha, m) + 1}. \quad (3.34)$$

In this case, the solution of (3.24) takes the following form:

$$\psi_n = \left( \frac{-sn(\alpha, m) cn(\alpha, m) dn(\alpha, m) sn(\xi_n, m) dn(\xi_n, m)}{\sqrt{-\mu} [m^2 sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1] cn(\xi_n, m)} \right. \\ \left. + \frac{sn(\alpha, m) cn(\alpha, m) dn(\alpha, m) cn(\xi_n, m)}{\sqrt{-\mu} [m^2 sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1] sn(\xi_n, m) dn(\xi_n, m)} \right) \\ \times \left( \text{Exp} \left\{ -i \left[ \frac{4t sn^2(\alpha, m) \mathfrak{A}}{m^4 sn^8(\alpha, m) - 4m^2 sn^6(\alpha, m) + (2m^2 + 4) sn^4(\alpha, m) - 4sn^2(\alpha, m) + 1} + \rho \right] \right\} \right), \quad (3.35)$$

where  $\xi_n = \alpha n + \beta$ .

In the special case, when  $m = 0$ , the trigonometrical solution (3.35) takes the following form:

$$\psi_n = \frac{\tan(2\alpha) \cot(2\xi_n)}{\sqrt{-\mu}} \left( \text{Exp} \left\{ -i \left[ -2t \tan^2(2\alpha) + \rho \right] \right\} \right), \quad \text{where } \xi_n = \alpha n + \beta. \quad (3.36)$$

Also if  $m = 1$ , the hyperbolic solution (3.35) takes the following form:

$$\psi_n = \left( \frac{\sinh(2\alpha)}{\sqrt{-\mu} \sinh(2\xi_n)} \right) \left( \text{Exp} \left\{ i \left[ 4t \sinh^2(\alpha) - \rho \right] \right\} \right), \quad \text{where } \xi_n = \alpha n + \beta. \quad (3.37)$$

Family 2.

$$\begin{aligned}\alpha_1 &= \frac{sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 1]}, & \beta_1 &= \frac{sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 1]}, & \mu < 0, \\ \nu &= \frac{2\mu[m^4sn^8(\alpha, m) - 2(m^4 + m^2)sn^6(\alpha, m) + 6m^2sn^4(\alpha, m) - (2 + 2m^2)sn^2(\alpha, m) + 1]}{m^4sn^8(\alpha, m) - 2m^2sn^4(\alpha, m) + 1}, \\ \sigma &= \frac{4sn^2(\alpha, m)[(m^4 + m^2)sn^4(\alpha, m) - 4m^2sn^2(\alpha, m) + m^2 + 1]}{m^4sn^8(\alpha, m) - 2m^2sn^4(\alpha, m) + 1}, & \alpha_0 &= 0.\end{aligned}\tag{3.38}$$

In this case, the solution of (3.24) takes the following form:

$$\begin{aligned}\psi_n &= \left( \frac{-sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)sn(\xi_n, m)dn(\xi_n, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 1]cn(\xi_n, m)} \right. \\ &\quad \left. - \frac{sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)cn(\xi_n, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 1]sn(\xi_n, m)dn(\xi_n, m)} \right) \\ &\quad \times \left( \text{Exp} \left\{ -i \left[ \frac{4tsn^2(\alpha, m)\mathfrak{X}}{m^4sn^8(\alpha, m) - 2m^2sn^4(\alpha, m) + 1} + \rho \right] \right\} \right),\end{aligned}\tag{3.39}$$

where  $\mathfrak{X}$  denotes  $[(m^4 + m^2)sn^4(\alpha, m) - 4m^2sn^2(\alpha, m) + m^2 + 1]$  and  $\xi_n = \alpha n + \beta$ .

If  $m = 0$ , the trigonometrical solution (3.39) takes the following form:

$$\psi_n = \frac{\sin(2\alpha)}{\sqrt{-\mu} \sin(2\xi_n)} \left( \text{Exp} \left\{ -i \left[ 4t \sin^2(\alpha) + \rho \right] \right\} \right), \quad \text{where } \xi_n = \alpha n + \beta.\tag{3.40}$$

Also if  $m = 1$ , the hyperbolic solution (3.39) takes the following form:

$$\psi_n = \frac{\tanh(2\alpha) \coth(2\xi_n)}{\sqrt{-\mu}} \left( \text{Exp} \left\{ -i \left[ 2t \tanh^2(2\alpha) + \rho \right] \right\} \right), \quad \text{where } \xi_n = \alpha n + \beta.\tag{3.41}$$

Type 3. if  $e_0 = m^2 - 1$ ,  $e_1 = 2 - m^2$ , and  $e_2 = -1$ . In this case, the series expansion solution of (3.26) has the form:

$$\phi(\xi_n) = \alpha_0 - \frac{m^2\alpha_1 sn(\xi_n, m)cn(\xi_n, m)}{dn(\xi_n, m)} - \frac{\beta_1 dn(\xi_n, m)}{m^2 sn(\xi_n, m)cn(\xi_n, m)}.\tag{3.42}$$

Consequently, using Maple or Mathematica we get the following results.

Family 1.

$$\begin{aligned}
 \alpha_0 &= 0, & \alpha_1 &= \frac{sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 2m^2sn^2(\alpha, m) + 1]}, \\
 \beta_1 &= -\frac{m^2sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 2m^2sn^2(\alpha, m) + 1]}, & \mu &< 0, \\
 \nu &= \frac{-2\mu(m^4sn^8(\alpha, m) - 2m^2sn^6(\alpha, m) + 2sn^2(\alpha, m) - 1)}{m^4sn^8(\alpha, m) - 4m^4sn^6(\alpha, m) + (4m^4 + 2m^2)sn^4(\alpha, m) - 4m^2sn^2(\alpha, m) + 1}, \\
 \sigma &= \frac{4sn^2(\alpha, m)[m^4sn^6(\alpha, m) - (2m^4 + m^2)sn^4(\alpha, m) + (2m^4 + m^2)sn^2(\alpha, m) - 2m^2 + 1]}{m^4sn^8(\alpha, m) - 4m^4sn^6(\alpha, m) + (4m^4 + 2m^2)sn^4(\alpha, m) - 4m^2sn^2(\alpha, m) + 1}.
 \end{aligned} \tag{3.43}$$

In this case, the solution of (3.24) takes the following form:

$$\begin{aligned}
 \psi_n &= \left( \frac{m^2sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)sn(\xi_n, m)cn(\xi_n, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 2m^2sn^2(\alpha, m) + 1]dn(\xi_n, m)} \right. \\
 &\quad \left. + \frac{sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)dn(\xi_n, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 2m^2sn^2(\alpha, m) + 1]sn(\xi_n, m)cn(\xi_n, m)} \right) \\
 &\quad \times \left( \text{Exp} \left\{ -i \left[ \frac{4tsn^2(\alpha, m)\mathfrak{W}}{m^4sn^8(\alpha, m) - 4m^4sn^6(\alpha, m) + (4m^4 + 2m^2)sn^4(\alpha, m) - 4m^2sn^2(\alpha, m) + 1} + \rho \right] \right\} \right)
 \end{aligned} \tag{3.44}$$

where  $\xi_n = \alpha n + \beta$ .

Family 2.

$$\begin{aligned}
 \alpha_0 &= 0, & \alpha_1 &= \frac{sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 1]}, \\
 \beta_1 &= \frac{m^2sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)}{\sqrt{-\mu}[m^2sn^4(\alpha, m) - 1]}, & \mu &< 0, \\
 \nu &= \frac{2\mu[(m^4sn^8(\alpha, m) - 2(m^4 + m^2)sn^6(\alpha, m) + 6m^2sn^4(\alpha, m) - (2 + 2m^2)sn^2(\alpha, m) + 1)]}{m^4sn^8(\alpha, m) - 2m^2sn^4(\alpha, m) + 1}, \\
 \sigma &= \frac{4sn^2(\alpha, m)[(m^4 + m^2)sn^4(\alpha, m) - 4m^2sn^2(\alpha, m) + m^2 + 1]}{m^4sn^8(\alpha, m) - 2m^2sn^4(\alpha, m) + 1}.
 \end{aligned} \tag{3.45}$$

In this case, the solution of (3.24) takes the following form:

$$\begin{aligned} \psi_n = & \left( -\frac{m^2 \operatorname{sn}(\alpha, m) \operatorname{cn}(\alpha, m) \operatorname{dn}(\alpha, m) \operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}{\sqrt{-\mu} [m^2 \operatorname{sn}^4(\alpha, m) - 1] \operatorname{dn}(\xi_n, m)} \right. \\ & \left. - \frac{\operatorname{sn}(\alpha, m) \operatorname{cn}(\alpha, m) \operatorname{dn}(\alpha, m) \operatorname{dn}(\xi_n, m)}{\sqrt{-\mu} [m^2 \operatorname{sn}^4(\alpha, m) - 1] \operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)} \right) \\ & \times \left( \operatorname{Exp} \left\{ -i \left[ \frac{4t \operatorname{sn}^2(\alpha, m) \mathfrak{X}}{m^4 \operatorname{sn}^8(\alpha, m) - 2m^2 \operatorname{sn}^4(\alpha, m) + 1} + \rho \right] \right\} \right), \end{aligned} \quad (3.46)$$

where  $\xi_n = \alpha n + \beta$ .

### 3.3. Example 3. The Discrete Nonlinear Klein-Gordon Equation

In this section, we consider the following discrete nonlinear Klein-Gordon equation [31]:

$$\frac{d^2 u_n(t)}{dt^2} = g(u_n)(u_{n+1} + u_{n-1} - 2s u_n). \quad (3.47)$$

The nonconstant (in contrast to the standard models of harmonic coupling and linear dispersion [32]) function  $g(u_n)$  ensures the presence of nonlinear dispersion, which is critical for the existence of compactly supported solutions and  $s$  can take values in the interval  $[-1, 1]$ . Kevrekidis and Konotop [31] have obtained some exact compaction solutions and claim that this DDE does not have the traveling compact solution.

If we set  $g(u_n) = A + C u_n^2$  where  $A$  and  $C$  are arbitrary constants and take the traveling transformation:

$$u_n(t) = U(\xi_n), \quad \xi_n = dn + c_1 t + \xi_0, \quad (3.48)$$

where  $d$ ,  $c_1$ , and  $\xi_0$  are constants. The transformation (3.48) permits us converting equation (3.47) into the following form:

$$c_1^2 U''(\xi_n) = [A + C U^2(\xi_n)] [U(\xi_n + d) + U(\xi_n - d) - 2s U(\xi_n)], \quad (3.49)$$

where  $' = d/d\xi_n$ . Considering the homogeneous balance between the highest-order derivative and the nonlinear term in (3.49), we get  $K = 1$ . Thus, the solutions of (3.49) have the following form:

$$U(\xi_n) = \alpha_0 + \alpha_1 \left( \frac{F'(\xi_n)}{F(\xi_n)} \right) + \beta_1 \left( \frac{F(\xi_n)}{F'(\xi_n)} \right), \quad (3.50)$$

where  $\alpha_0$ ,  $\alpha_1$ , and  $\beta_1$  are constants to be determined later and  $F(\xi_n)$  satisfies a discrete Jacobi elliptic ordinary differential equation (2.6), we have the following types.



Type 1. If  $e_0 = 1$ ,  $e_1 = -(1 + m^2)$ , and  $e_2 = m^2$ . In this case, the series expansion solution of (3.49) has the form:

$$U(\xi_n) = \alpha_0 + \frac{\alpha_1 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} + \beta_1 \frac{sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)}. \quad (3.51)$$

With the help of Maple, we substitute (3.51) and (2.8) into (3.49). Cleaning the denominator and collecting all terms with the same degree of  $sn(\xi_n, m)$ ,  $cn(\xi_n, m)$ ,  $dn(\xi_n, m)$  together, the left-hand side of (3.49) is converted into polynomial in  $sn(\xi_n, m)$ ,  $cn(\xi_n, m)$ ,  $dn(\xi_n, m)$ . Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for  $\alpha_0$ ,  $\alpha_1$ ,  $d$ ,  $\beta_1$ ,  $A$ ,  $C$ ,  $s$ , and  $c_1$ . Solving the set of algebraic equations by using Maple or Mathematica software package, we have the following.

Family 1.

$$\beta_1 = (m^2 - 1)c_1 L_1, \quad \alpha_0 = 0, \quad s = \frac{-1}{CL_1^2}, \quad \alpha_1 = c_1 L_1, \quad (3.52)$$

$$A = -CL_1^2 c_1^2 \frac{[m^4 sn^8(d, m) - 4m^2 sn^6(d, m) + (2m^2 + 4) sn^4(d, m) - 4sn^2(d, m) + 1]}{sn^2(d, m) cn^2(d, m) dn^2(d, m)},$$

where  $d$ ,  $c_1$ , and  $C$  are arbitrary constants and

$$L_1 = \sqrt{\frac{m^4 sn^4(d, m) cn^4(d, m) + dn^4(d, m)}{C[m^4 sn^8(d, m) - 2m^4 sn^6(d, m) + 2m^2 sn^2(d, m) - 1]}}. \quad (3.53)$$

In this case, the solution of (3.47) takes the following form:

$$U(\xi_n) = \frac{c_1 L_1 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} + \frac{(m^2 - 1)c_1 L_1 sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)}, \quad (3.54)$$

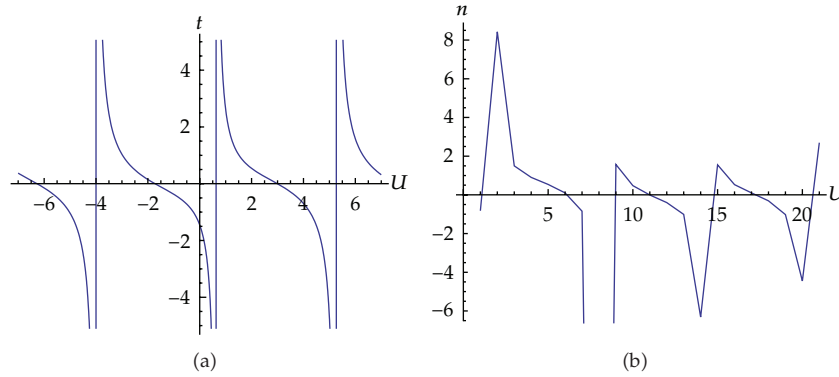
where  $\xi_n = dn + c_1 t + \xi_0$ .

Figure 3 illustrates the behavior of the exact solution (3.54).

Family 2.

$$\beta_1 = -(m^2 - 1)c_1 L_2, \quad \alpha_0 = 0, \quad s = \frac{-1}{CL_2^2}, \quad \alpha_1 = c_1 L_2, \quad (3.55)$$

$$A = -CL_2^2 c_1^2 \frac{[m^4 sn^8(d, m) - 4m^4 sn^6(d, m) + (4m^4 + 2m^2) sn^4(d, m) - 4m^2 sn^2(d, m) + 1]}{sn^2(d, m) cn^2(d, m) dn^2(d, m)},$$



**Figure 3:** (a) represents the Jacobi elliptic solution (3.54) when  $m = 0.5, c_1 = 0.4, C = -0.6, d = 0.3, n = 2$ . (b) represents the Jacobi elliptic solution (3.54) when  $m = 0.5, c_1 = 0.4, C = -0.6, d = 0.3, t = 1.5$ .

where  $d, c_1,$  and  $C$  are arbitrary constants and

$$L_2 = \sqrt{\frac{sn^4(d, m)dn^4(d, m) + cn^4(d, m)}{C[m^4sn^8(d, m) - 2m^2sn^6(d, m) + 2sn^2(d, m) - 1]}}. \tag{3.56}$$

In this case, the solution of (3.47) takes the following form:

$$U(\xi_n) = \frac{c_1L_2cn(\xi_n, m)dn(\xi_n, m)}{sn(\xi_n, m)} - \frac{(m^2 - 1)c_1L_2sn(\xi_n, m)}{cn(\xi_n, m)dn(\xi_n, m)}, \tag{3.57}$$

where  $\xi_n = dn + c_1t + \xi_0$ .

*Type 2.* If  $e_0 = 1 - m^2, e_1 = 2m^2 - 1,$  and  $e_2 = -m^2$ . In this case, the series expansion solution of (3.49) has the form:

$$U(\xi_n) = \alpha_0 - \alpha_1 \frac{sn(\xi_n, m)dn(\xi_n, m)}{cn(\xi_n, m)} - \beta_1 \frac{cn(\xi_n, m)}{sn(\xi_n, m)dn(\xi_n, m)}. \tag{3.58}$$

Consequently, using Maple or Mathematica we get the following results.

*Family 1.*

$$\beta_1 = -c_1L_1, \quad \alpha_0 = 0, \quad s = \frac{-1}{CL_1^2}, \quad \alpha_1 = c_1L_1, \tag{3.59}$$

$$A = -CL_1^2c_1^2 \frac{[m^4sn^8(d, m) - 4m^2sn^6(d, m) + (2m^2 + 4)sn^4(d, m) - 4sn^2(d, m) + 1]}{sn^2(d, m)cn^2(d, m)dn^2(d, m)},$$

where  $d, c_1,$  and  $C$  are arbitrary constants.

In this case, the solution of (3.47) takes the following form:

$$U(\xi_n) = \frac{-c_1 L_1 \operatorname{sn}(\xi_n, m) \operatorname{dn}(\xi_n, m)}{\operatorname{cn}(\xi_n, m)} + \frac{c_1 L_1 \operatorname{cn}(\xi_n, m)}{\operatorname{sn}(\xi_n, m) \operatorname{dn}(\xi_n, m)}, \quad (3.60)$$

where  $\xi_n = dn + c_1 t + \xi_0$ .

In the special case, when  $m = 0$ , the trigonometrical solution (3.60) takes the following form:

$$U(\xi_n) = \frac{2c_1 \cot(2\xi_n)}{\sqrt{-C}}, \quad \text{where } \xi_n = dn + c_1 t + \xi_0. \quad (3.61)$$

Also when  $m = 1$ , the hyperbolic solution (3.60) takes the following form:

$$U(\xi_n) = 2c_1 \operatorname{coth}(2\xi_n) \sqrt{\frac{[\tanh^4(d) + 1]}{C[\tanh^4(d) - 1]}}, \quad \text{where } \xi_n = dn + c_1 t + \xi_0. \quad (3.62)$$

Family 2.

$$\beta_1 = c_1 L_3, \quad \alpha_0 = 0, \quad s = \frac{-1}{CL_3^2}, \quad \alpha_1 = c_1 L_3, \quad (3.63)$$

$$A = -CL_3^2 c_1^2 \frac{[m^4 \operatorname{sn}^8(d, m) - 2m^2 \operatorname{sn}^4(d, m) + 1]}{\operatorname{sn}^2(d, m) \operatorname{cn}^2(d, m) \operatorname{dn}^2(d, m)},$$

where  $d$ ,  $c_1$ , and  $C$  are arbitrary constants, while

$$L_3 = \sqrt{\frac{m^4 \operatorname{sn}^8(d, m) - 2(m^4 + m^2) \operatorname{sn}^6(d, m) + 2(m^4 + m^2 + 1) \operatorname{sn}^4(d, m) - 2(m^2 + 1) \operatorname{sn}^2(d, m) + 1}{-C[m^4 \operatorname{sn}^8(d, m) - 2(m^4 + m^2) \operatorname{sn}^6(d, m) + 6m^2 \operatorname{sn}^4(d, m) - 2(m^2 + 1) \operatorname{sn}^2(d, m) + 1]}}. \quad (3.64)$$

In this case, the solution of (3.47) takes the following form:

$$U(\xi_n) = -\frac{c_1 L_3 \operatorname{sn}(\xi_n, m) \operatorname{dn}(\xi_n, m)}{\operatorname{cn}(\xi_n, m)} - \frac{c_1 L_3 \operatorname{cn}(\xi_n, m)}{\operatorname{sn}(\xi_n, m) \operatorname{dn}(\xi_n, m)}, \quad (3.65)$$

where  $\xi_n = dn + c_1 t + \xi_0$ .

When  $m = 0$ , the trigonometrical solution (3.65) takes the following form:

$$U(\xi_n) = \frac{-2c_1}{\sin(2\xi_n)} \sqrt{\frac{-\sin^2(2d) + 2}{-2C \cos(2d)}}, \quad \text{where } \xi_n = dn + c_1 t + \xi_0. \quad (3.66)$$

Also if  $m = 1$ , the hyperbolic solution (3.65) takes the following form:

$$U(\xi_n) = \frac{-2c_1 \coth(2\xi_n)}{\sqrt{-C}}, \quad \text{where } \xi_n = dn + c_1t + \xi_0. \quad (3.67)$$

Type 3. If  $e_0 = m^2 - 1$ ,  $e_1 = 2 - m^2$ , and  $e_2 = -1$ . In this case, the series expansion solution of (3.49) has the form:

$$U(\xi_n) = \alpha_0 - \frac{m^2 \alpha_1 \operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}{\operatorname{dn}(\xi_n, m)} - \frac{\beta_1 \operatorname{dn}(\xi_n, m)}{m^2 \operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}. \quad (3.68)$$

Consequently, using Maple or Mathematica we get the following results.

Family 1.

$$\beta_1 = -c_1 L_2 m^2, \quad \alpha_0 = 0, \quad s = \frac{-1}{CL_2^2}, \quad \alpha_1 = c_1 L_2,$$

$$A = -CL_2^2 c_1^2 \frac{[m^4 \operatorname{sn}^8(d, m) - 4m^4 \operatorname{sn}^6(d, m) + (4m^4 + 2m^2) \operatorname{sn}^4(d, m) - 4m^2 \operatorname{sn}^2(d, m) + 1]}{\operatorname{sn}^2(d, m) \operatorname{cn}^2(d, m) \operatorname{dn}^2(d, m)}, \quad (3.69)$$

where  $d$ ,  $c_1$ , and  $C$  are arbitrary constants and

$$L_2 = \sqrt{\frac{\operatorname{sn}^4(d, m) \operatorname{dn}^4(d, m) + \operatorname{cn}^4(d, m)}{C[m^4 \operatorname{sn}^8(d, m) - 2m^2 \operatorname{sn}^6(d, m) + 2 \operatorname{sn}^2(d, m) - 1]}}. \quad (3.70)$$

In this case, the solution of (3.47) takes the following form:

$$U(\xi_n) = \frac{-c_1 L_3 m^2 \operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}{\operatorname{dn}(\xi_n, m)} + \frac{c_1 L_3 \operatorname{dn}(\xi_n, m)}{\operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}, \quad (3.71)$$

where  $\xi_n = dn + c_1t + \xi_0$ .

Family 2.

$$\beta_1 = m^2 c_1 L_3, \quad \alpha_0 = 0, \quad s = \frac{-1}{CL_3^2}, \quad \alpha_1 = c_1 L_3,$$

$$A = -CL_3^2 c_1^2 \frac{[m^4 \operatorname{sn}^8(d, m) - 2m^2 \operatorname{sn}^4(d, m) + 1]}{\operatorname{sn}^2(d, m) \operatorname{cn}^2(d, m) \operatorname{dn}^2(d, m)}, \quad (3.72)$$

where  $d$ ,  $c_1$ , and  $C$  are arbitrary constants and

$$L_3 = \sqrt{\frac{m^4 sn^8(d, m) - 2(m^4 + m^2) sn^6(d, m) + 2(m^4 + m^2 + 1) sn^4(d, m) - 2(m^2 + 1) sn^2(d, m) + 1}{-C[m^4 sn^8(d, m) - 2(m^4 + m^2) sn^6(d, m) + 6m^2 sn^4(d, m) - 2(m^2 + 1) sn^2(d, m) + 1]}}. \quad (3.73)$$

In this case, the solution of (3.47) takes the following form:

$$U(\xi_n) = -\frac{m^2 c_1 L_3 sn(\xi_n, m) cn(\xi_n, m)}{dn(\xi_n, m)} - \frac{c_1 L_3 dn(\xi_n, m)}{sn(\xi_n, m) cn(\xi_n, m)}, \quad (3.74)$$

where  $\xi_n = dn + c_1 t + \xi_0$ .

### 3.4. Example 4. The Quintic Discrete Nonlinear Schrodinger Equation

We discuss the quintic discrete nonlinear Schrodinger (QDNLS) equation [33]:

$$\frac{id\psi_n}{dt} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \beta|\psi_n|^2\psi_n + \gamma|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) + \delta|\psi_n|^4(\psi_{n+1} + \psi_{n-1}) = 0, \quad (3.75)$$

which describes the propagation of discrete self-trapped beams in an array of weakly coupled nonlinear optical waveguides. Equation (3.75) was presented for the first time in [33], together with its localized solutions. We are looking for solutions of the form:

$$\psi_n(t) = \phi_n e^{-i\omega t}, \quad (3.76)$$

where  $\phi_n$  is a real function with respect to the discrete variable  $n$ . Substitution of equation (3.76) into (3.75) yields the corresponding symmetric equation as follows:

$$\phi_{n+1} + \phi_{n-1} = \frac{(2\alpha - \omega)\phi_n - \beta\phi_n^3}{\alpha + \gamma\phi_n^2 + \delta\phi_n^4}. \quad (3.77)$$

We take the traveling wave of transformation

$$\phi_n = U(\xi_n), \quad \xi_n = dn + k, \quad (3.78)$$

where  $d$  and  $k$  are constants. The transformation (3.78) permits us converting equation (3.77) into the following form:

$$U(\xi_n + d) + U(\xi_n - d) = \frac{(2\alpha - \omega)U(\xi_n) - \beta U^3(\xi_n)}{\alpha + \gamma U^2(\xi_n) + \delta U^4(\xi_n)}. \quad (3.79)$$

We suppose that the solutions of (3.79) have the following form:

$$U(\xi_n) = \alpha_0 + \alpha_1 \left( \frac{F'(\xi_n)}{F(\xi_n)} \right) + \beta_1 \left( \frac{F(\xi_n)}{F'(\xi_n)} \right), \quad (3.80)$$

where  $\alpha_0$ ,  $\alpha_1$ , and  $\beta_1$  are constant to be determined later and  $F(\xi_n)$  satisfies a discrete Jacobi elliptic ordinary differential equation (2.6), we have the following types.

*Type 1.* If  $e_0 = 1$ ,  $e_1 = -(1 + m^2)$ , and  $e_2 = m^2$ . In this case, the series expansion solution of (3.79) has the form:

$$U(\xi_n) = \alpha_0 + \frac{\alpha_1 \operatorname{cn}(\xi_n, m) \operatorname{dn}(\xi_n, m)}{\operatorname{sn}(\xi_n, m)} + \beta_1 \frac{\operatorname{sn}(\xi_n, m)}{\operatorname{cn}(\xi_n, m) \operatorname{dn}(\xi_n, m)}. \quad (3.81)$$

With the help of Maple, we substitute (3.81) and (2.8) into (3.79). Cleaning the denominator and collecting all terms with the same degree of  $\operatorname{sn}(\xi_n, m)$ ,  $\operatorname{cn}(\xi_n, m)$ ,  $\operatorname{dn}(\xi_n, m)$  together, the left-hand side of (3.79) is converted into polynomial in  $\operatorname{sn}(\xi_n, m)$ ,  $\operatorname{cn}(\xi_n, m)$ ,  $\operatorname{dn}(\xi_n, m)$ . Setting each coefficient of this polynomial to be zero, we derive a set of algebraic equations for  $\alpha_0$ ,  $\alpha_1$ ,  $d$ ,  $\beta_1$ ,  $\alpha$ ,  $\omega$ ,  $\beta$ ,  $\gamma$ , and  $\delta$ . Solving the set of algebraic equations by using Maple or Mathematica software package, we have the following.

*Family 1.*

$$\begin{aligned} \beta_1 &= \alpha_1 (m^2 - 1), & \alpha_0 &= 0, \\ \beta &= \frac{-2\delta\alpha_1^2 [m^4 \operatorname{sn}^8(d, m) - 2m^4 \operatorname{sn}^6(d, m) + 2m^2 \operatorname{sn}^2(d, m) - 1]}{\operatorname{sn}^2(d, m) \operatorname{cn}^2(d, m) \operatorname{dn}^2(d, m)}, \\ \omega &= \frac{4\alpha_1^2 [H_1 \delta \alpha_1^2 + H_2 \gamma]}{\operatorname{sn}^2(d, m) \operatorname{cn}^2(d, m) \operatorname{dn}^4(d, m)}, & \alpha &= \frac{-\alpha_1^2 [H_3 \delta \alpha_1^2 + H_4 \gamma]}{\operatorname{sn}^4(d, m) \operatorname{cn}^4(d, m) \operatorname{dn}^4(d, m)}, \end{aligned} \quad (3.82)$$

where  $\alpha_1$ ,  $\delta$ , and  $d$  are arbitrary constants and

$$\begin{aligned} H_1 &= m^8 \operatorname{sn}^{12}(d, m) - 6m^6 \operatorname{sn}^{10}(d, m) + (14m^4 + m^6) \operatorname{sn}^8(d, m) - (16m^2 + 4m^4) \operatorname{sn}^6(d, m) \\ &\quad + (8 - m^4 + 8m^2) \operatorname{sn}^4(d, m) + (2m^2 - 8) \operatorname{sn}^2(d, m) - m^2 + 2, \\ H_2 &= m^6 \operatorname{sn}^{10}(d, m) - (m^6 + 3m^4) \operatorname{sn}^8(d, m) + (4m^2 + 2m^4) \operatorname{sn}^6(d, m) \\ &\quad + (m^4 - 3m^2 - 2) \operatorname{sn}^4(d, m) + (2 - m^2) \operatorname{sn}^2(d, m), \\ H_3 &= m^8 \operatorname{sn}^{16}(d, m) - 8m^6 \operatorname{sn}^{14}(d, m) + (4m^6 \operatorname{sn}^{12} + 24m^4) \operatorname{sn}^{12}(d, m) \\ &\quad - (32m^2 + 24m^4) \operatorname{sn}^{10}(d, m) + (16 + 48m^2 + 6m^4) \operatorname{sn}^8(d, m) \\ &\quad - (24m^2 + 32) \operatorname{sn}^6(d, m) + (24 + 4m^2) \operatorname{sn}^4(d, m) - 8 \operatorname{sn}^2(d, m) + 1, \end{aligned}$$

$$\begin{aligned}
H_4 = & m^6 sn^{14}(d, m) - (m^6 + 5m^4) sn^{12}(d, m) + (7m^4 + 8m^2) sn^{10}(d, m) \\
& - (14m^2 + 2m^4 + 4) sn^8(d, m) + (7m^2 + 8) sn^6(d, m) - (5 + m^2) sn^4(d, m) + sn^2(d, m).
\end{aligned} \tag{3.83}$$

In this case, the solution of (3.75) takes the following form:

$$\begin{aligned}
\psi_n(t) = & \frac{\alpha_1 [cn^2(\xi_n, m) dn^2(\xi_n, m) + (m^2 - 1) sn^2(\xi_n, m)]}{sn(\xi_n, m) cn(\xi_n, m) dn(\xi_n, m)} \\
& \times \text{Exp} \left\{ -i \left[ \frac{4\alpha_1^2 [H_1 \delta \alpha_1^2 + H_2 \gamma]}{sn^2(d, m) cn^2(d, m) dn^4(d, m)} \right] t \right\},
\end{aligned} \tag{3.84}$$

where  $\xi_n = dn + k$ .

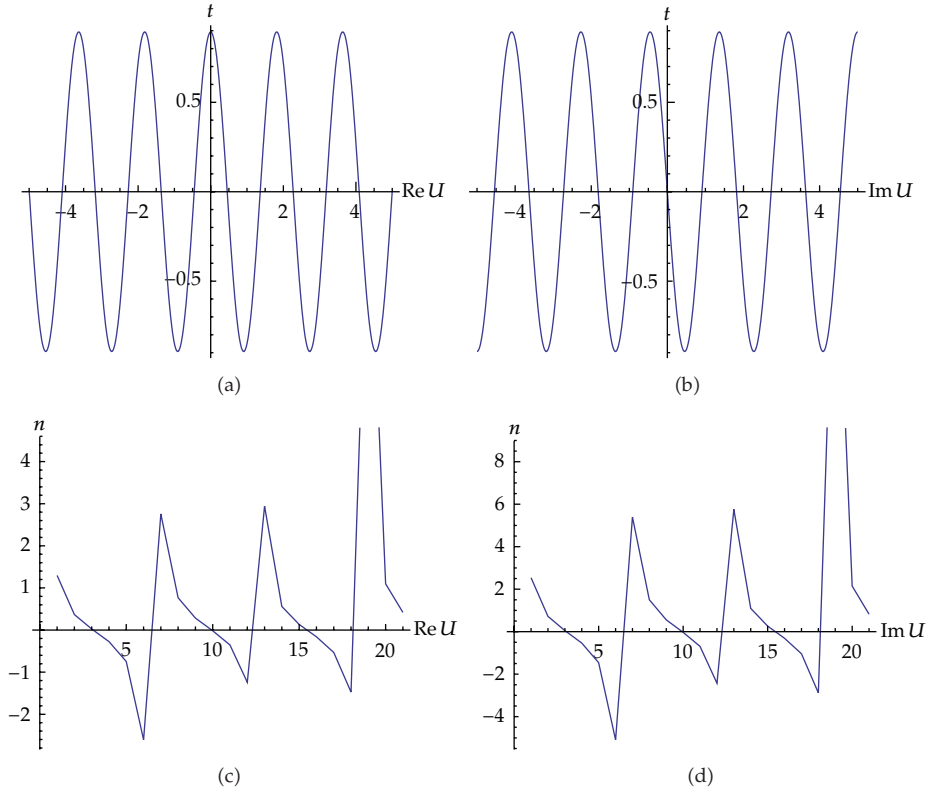
Figure 4 illustrates the behavior of the exact solution (3.84).

Family 2.

$$\begin{aligned}
\beta_1 = -\alpha_1 (m^2 - 1), \quad \alpha_0 = 0, \quad \beta = & \frac{-2\delta \alpha_1^2 [m^4 sn^8(d, m) - 2m^2 sn^6(d, m) + 2sn^2(d, m) - 1]}{sn^2(d, m) cn^2(d, m) dn^2(d, m)}, \\
\omega = & \frac{4\alpha_1^2 [H_5 \delta \alpha_1^2 + H_6 \gamma]}{sn^2(d, m) cn^4(d, m) dn^2(d, m)}, \quad \alpha = \frac{-\alpha_1^2 [H_7 \delta \alpha_1^2 + H_8 \gamma]}{sn^4(d, m) cn^4(d, m) dn^4(d, m)},
\end{aligned} \tag{3.85}$$

where  $\alpha_1, \delta$ , and  $d$  are arbitrary constants and

$$\begin{aligned}
H_5 = & m^6 sn^{12}(d, m) - 6m^6 sn^{10}(d, m) + (14m^6 + m^4) sn^8(d, m) - (16m^6 + 4m^4) sn^6(d, m) \\
& + (8m^4 - m^2 + 8m^6) sn^4(d, m) + (2m^2 - 8m^4) sn^2(d, m) + 2m^2 - 1, \\
H_6 = & m^4 sn^{10}(d, m) - (m^2 + 3m^4) sn^8(d, m) + (2m^2 + 4m^4) sn^6(d, m) \\
& + (1 - 3m^2 - 2m^4) sn^4(d, m) + (2m^2 - 1) sn^2(d, m), \\
H_7 = & m^8 sn^{16}(d, m) - 8m^8 sn^{14}(d, m) + (24m^8 + 4m^6) sn^{12}(d, m) - (24m^6 + 32m^8) sn^{10}(d, m) \\
& + (6m^4 + 48m^6 + 16m^8) sn^8(d, m) - (24m^4 + 32m^6) sn^6(d, m) + (24m^4 + 4m^2) sn^4(d, m) \\
& - 8m^2 sn^2(d, m) + 1,
\end{aligned}$$



**Figure 4:** (a) and (b) represent the Jacobi elliptic solution (3.84) when  $m = 0.5$ ,  $\alpha_1 = 0.6$ ,  $k = 0.2$ ,  $\delta = 0.4$ ,  $\gamma = 0.1$ ,  $d = 0.3$ ,  $n = 2$ . (c) and (d) represent the Jacobi elliptic solution (3.84) when  $m = 0.5$ ,  $\alpha_1 = 0.6$ ,  $k = 0.2$ ,  $\delta = 0.4$ ,  $\gamma = 0.1$ ,  $d = 0.3$ ,  $t = 1.5$ .

$$\begin{aligned}
 H_8 = & m^6 sn^{14}(d, m) - (5m^6 + m^4) sn^{12}(d, m) + (7m^4 + 8m^6) sn^{10}(d, m) \\
 & - (14m^4 + 2m^2 + 4m^6) sn^8(d, m) + (8m^4 + 7m^2) sn^6(d, m) \\
 & - (5m^2 + 1) sn^4(d, m) + sn^2(d, m).
 \end{aligned}
 \tag{3.86}$$

In this case, the solution of (3.75) takes the following form:

$$\begin{aligned}
 \psi_n(t) = & \frac{\alpha_1 [cn^2(\xi_n, m) dn^2(\xi_n, m) - (m^2 - 1) sn^2(\xi_n, m)]}{sn(\xi_n, m) cn(\xi_n, m) dn(\xi_n, m)} \\
 & \times \text{Exp} \left\{ -i \left[ \frac{4\alpha_1^2 [H_5 \delta \alpha_1^2 + H_6 \gamma]}{sn^2(d, m) cn^4(d, m) dn^2(d, m)} \right] t \right\},
 \end{aligned}
 \tag{3.87}$$

where  $\xi_n = dn + k$ .



Type 2. If  $e_0 = 1 - m^2$ ,  $e_1 = 2m^2 - 1$ , and  $e_2 = -m^2$ . In this case, the solution of (3.79) has the form:

$$U(\xi_n) = \alpha_0 - \alpha_1 \frac{sn(\xi_n, m)dn(\xi_n, m)}{cn(\xi_n, m)} - \beta_1 \frac{cn(\xi_n, m)}{sn(\xi_n, m)dn(\xi_n, m)}. \quad (3.88)$$

Consequently, using the Maple or Mathematica we get the following results.

Family 1.

$$\beta_1 = -\alpha_1, \quad \alpha_0 = 0, \quad \beta = \frac{-2\delta\alpha_1^2 [m^4 sn^8(d, m) - 2m^4 sn^6(d, m) + 2m^2 sn^2(d, m) - 1]}{sn^2(d, m)cn^2(d, m)dn^2(d, m)},$$

$$\omega = \frac{4\alpha_1^2 [H_1\delta\alpha_1^2 + H_2\gamma]}{sn^2(d, m)cn^2(d, m)dn^4(d, m)}, \quad \alpha = \frac{-\alpha_1^2 [H_3\delta\alpha_1^2 + H_4\gamma]}{sn^4(d, m)cn^4(d, m)dn^4(d, m)}, \quad (3.89)$$

where  $\alpha_1$ ,  $\delta$ , and  $d$  are arbitrary constants.

In this case, the solution of (3.75) takes the following form:

$$\psi_n(t) = \frac{\alpha_1 [cn^2(\xi_n, m) - sn^2(\xi_n, m)dn^2(\xi_n, m)]}{sn(\xi_n, m)cn(\xi_n, m)dn(\xi_n, m)} \text{Exp} \left\{ -i \left[ \frac{4\alpha_1^2 [H_1\delta\alpha_1^2 + H_2\gamma]}{sn^2(d, m)cn^2(d, m)dn^4(d, m)} \right] t \right\}, \quad (3.90)$$

where  $\xi_n = dn + k$ .

In the special case, when  $m = 0$ , the trigonometrical solution (3.90) takes the following form:

$$\psi_n(t) = 2\alpha_1 \cot(2\xi_n) \text{Exp} \left\{ -i \left[ 32\alpha_1^4\delta + \frac{32\alpha_1^4\delta}{\sin^2(2d)} + 8\alpha_1^2\gamma \right] t \right\}, \quad \text{where } \xi_n = dn + k. \quad (3.91)$$

When  $m = 1$ , the hyperbolic solution (3.90) takes the following form:

$$\psi_n(t) = \frac{2\alpha_1}{\sinh(2\xi_n)} \text{Exp} \left\{ -i \left[ \frac{4\alpha_1^2 [h_1\delta\alpha_1^2 + h_2\gamma]}{\tanh^2(d)\text{sech}^6(d)} \right] t \right\}, \quad (3.92)$$

where

$$h_1 = \tanh^{12}(d) - 6\tanh^{10}(d) + 15\tanh^8(d) - 20\tanh^6(d) + 15\tanh^4(d) - 6\tanh^2(d) + 1,$$

$$h_2 = \tanh^{10}(d) - 4\tanh^8(d) + 6\tanh^6(d) - 4\tanh^4(d) + \tanh^2(d), \quad (3.93)$$

and  $\xi_n = dn + k$ .

Family 2.

$$\beta = \frac{2\delta\alpha_1^2 [m^4 sn^8(d, m) - 2(m^4 + m^2)sn^6(d, m) + 6m^2 sn^4(d, m) - 2(m^2 + 1)sn^2(d, m) + 1]}{sn^2(d, m)cn^2(d, m)dn^2(d, m)},$$

$$\omega = \frac{-4\alpha_1^2 [H_9\delta\alpha_1^2 + H_{10}\gamma]}{sn^2(d, m)cn^4(d, m)dn^4(d, m)}, \quad \alpha = \frac{-\alpha_1^2 [H_{11}\delta\alpha_1^2 + H_{12}\gamma]}{sn^4(d, m)cn^4(d, m)dn^4(d, m)},$$

$$\beta_1 = \alpha_1, \quad \alpha_0 = 0, \tag{3.94}$$

where  $\alpha_1$ ,  $\delta$ , and  $d$  are arbitrary constants and

$$H_9 = (m^8 + m^6)sn^{12}(d, m) - 4m^6 sn^{10}(d, m) - (m^6 + m^4)sn^8(d, m) + 8m^4 sn^6(d, m) - (m^4 + m^2)sn^4(d, m) - 4m^2 sn^2(d, m) + m^2 + 1,$$

$$H_{10} = (m^6 + m^4)sn^{10}(d, m) - (m^6 + 6m^4 + m^2)sn^8(d, m) + 6(m^4 + m^2)sn^6(d, m) - (m^4 + 6m^2 + 1)sn^4(d, m) + (m^2 + 1)sn^2(d, m), \tag{3.95}$$

$$H_{11} = m^8 sn^{16}(d, m) - 4m^6 sn^{12}(d, m) + 6m^4 sn^8(d, m) - 4m^2 sn^4(d, m) + 1,$$

$$H_{12} = m^6 sn^{14}(d, m) - (m^6 + m^4)sn^{12}(d, m) - m^4 sn^{10}(d, m) + 2(m^4 + m^2)sn^8(d, m) - m^2 sn^6(d, m) - (m^2 + 1)sn^4(d, m) + sn^2(d, m).$$

In this case, the solution of (3.75) takes the following form:

$$\psi_n(t) = \frac{-\alpha_1 [cn^2(\xi_n, m) + sn^2(\xi_n, m)dn^2(\xi_n, m)]}{sn(\xi_n, m)cn(\xi_n, m)dn(\xi_n, m)} \text{Exp} \left\{ -i \left[ \frac{-4\alpha_1^2 [H_9\delta\alpha_1^2 + H_{10}\gamma]}{sn^2(d, m)cn^4(d, m)dn^4(d, m)} \right] t \right\}, \tag{3.96}$$

where  $\xi_n = dn + k$ .

When  $m = 0$ , the trigonometrical solution (3.96) takes the following form:

$$\psi_n(t) = \frac{-2\alpha_1}{\sin(2\xi_n)} \text{Exp} \left\{ -i \left[ \frac{-4\alpha_1^2 [\delta\alpha_1^2 + \sin^2(d)\cos^2(d)\gamma]}{\sin^2(d)\cos^4(d)} \right] t \right\}, \tag{3.97}$$

where  $\xi_n = dn + k$ .

When  $m = 1$ , the hyperbolic solution (3.96) takes the following form:

$$\psi_n(t) = -2\alpha_1 \coth(2\xi_n) \text{Exp} \left\{ -i \left[ \frac{-4\alpha_1^2 [h_9\delta\alpha_1^2 + h_{10}\gamma]}{\tanh^2(d)\text{sech}^8(d)} \right] t \right\}, \tag{3.98}$$

where

$$h_9 = 2\tanh^{12}(d) - 4\tanh^{10}(d) - 2\tanh^8(d) + 8\tanh^6(d) - 2\tanh^4(d) - 4\tanh^2(d) + 2, \quad (3.99)$$

$$h_{10} = 2\tanh^{10}(d) - 8\tanh^8(d) + 12\tanh^6(d) - 8\tanh^4(d) + 2\tanh^2(d),$$

and  $\xi_n = dn + k$ .

Type 3. If  $e_0 = m^2 - 1$ ,  $e_1 = 2 - m^2$ , and  $e_2 = -1$ . In this case, the series expansion solution of (3.79) has the form:

$$U(\xi_n) = \alpha_0 - \frac{m^2 \alpha_1 \operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}{dn(\xi_n, m)} - \frac{\beta_1 dn(\xi_n, m)}{m^2 \operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m)}. \quad (3.100)$$

Consequently, using the Maple or Mathematica we get the following results:

Family 1.

$$\beta_1 = -\alpha_1 m^2, \quad \alpha_0 = 0, \quad \beta = \frac{-2\delta \alpha_1^2 [m^4 \operatorname{sn}^8(d, m) - 2m^2 \operatorname{sn}^6(d, m) + 2\operatorname{sn}^2(d, m) - 1]}{\operatorname{sn}^2(d, m) \operatorname{cn}^2(d, m) dn^2(d, m)},$$

$$\omega = \frac{4\alpha_1^2 [H_5 \delta \alpha_1^2 + H_6 \gamma]}{\operatorname{sn}^2(d, m) \operatorname{cn}^4(d, m) dn^2(d, m)}, \quad \alpha = \frac{-\alpha_1^2 [H_7 \delta \alpha_1^2 + H_8 \gamma]}{\operatorname{sn}^4(d, m) \operatorname{cn}^4(d, m) dn^4(d, m)}, \quad (3.101)$$

where  $\alpha_1$ ,  $\delta$ , and  $d$  are arbitrary constants.

In this case, the solution of (3.75) takes the following form:

$$\psi_n(t) = \frac{\alpha_1 [dn^2(\xi_n, m) - m^2 \operatorname{sn}^2(\xi_n, m) \operatorname{cn}^2(\xi_n, m)]}{\operatorname{sn}(\xi_n, m) \operatorname{cn}(\xi_n, m) dn(\xi_n, m)} \operatorname{Exp} \left\{ -i \left[ \frac{4\alpha_1^2 [H_5 \delta \alpha_1^2 + H_6 \gamma]}{\operatorname{sn}^2(d, m) \operatorname{cn}^4(d, m) dn^2(d, m)} \right] t \right\}, \quad (3.102)$$

where  $\xi_n = dn + k$ .

Family 2.

$$\beta = \frac{2\delta \alpha_1^2 [m^4 \operatorname{sn}^8(d, m) - 2(m^4 + m^2) \operatorname{sn}^6(d, m) + 6m^2 \operatorname{sn}^4(d, m) - 2(m^2 + 1) \operatorname{sn}^2(d, m) + 1]}{\operatorname{sn}^2(d, m) \operatorname{cn}^2(d, m) dn^2(d, m)},$$

$$\omega = \frac{-4\alpha_1^2 [H_9 \delta \alpha_1^2 + H_{10} \gamma]}{\operatorname{sn}^2(d, m) \operatorname{cn}^4(d, m) dn^4(d, m)}, \quad \alpha = \frac{-\alpha_1^2 [H_{11} \delta \alpha_1^2 + H_{12} \gamma]}{\operatorname{sn}^4(d, m) \operatorname{cn}^4(d, m) dn^4(d, m)},$$

$$\beta_1 = m^2 \alpha_1, \quad \alpha_0 = 0, \quad (3.103)$$

where  $\alpha_1$ ,  $\delta$ , and  $d$  are arbitrary constants.

**Table 1:** The relation between the Jacobi elliptic functions and the trigonometrical functions, hyperbolic functions.

$m$	$sn(\xi, m)$	$cn(\xi, m)$	$dn(\xi, m)$	$ns(\xi, m)$	$cs(\xi, m)$	$ds(\xi, m)$
0	$\sin(\xi)$	$\cos(\xi)$	1	$\csc(\xi)$	$\cot(\xi)$	$\csc(\xi)$
1	$\tanh(\xi)$	$\operatorname{sech}(\xi)$	$\operatorname{sech}(\xi)$	$\operatorname{coth}(\xi)$	$\operatorname{csch}(\xi)$	$\operatorname{csch}(\xi)$

In this case, the solution of (3.75) takes the following form:

$$\begin{aligned} \psi_n(t) = & \frac{-\alpha_1 [m^2 sn^2(\xi_n, m) cn^2(\xi_n, m) + dn^2(\xi_n, m)]}{sn(\xi_n, m) cn(\xi_n, m) dn(\xi_n, m)} \\ & \times \operatorname{Exp} \left\{ -i \left[ \frac{-4\alpha_1^2 [H_9 \delta \alpha_1^2 + H_{10} \gamma]}{sn^2(d, m) cn^4(d, m) dn^4(d, m)} \right] t \right\}, \end{aligned} \quad (3.104)$$

where  $\xi_n = dn + k$ .

*Remark 3.1.* Wang and Ma [34] constructed the Jacobi elliptic function solution to some nonlinear DDEs in terms of Jacobi elliptic functions  $sn$  or  $cn$  or  $dn$ . In Zhang [20], an algorithm is devised to derive exact traveling wave solutions of nonlinear DDEs by means of Jacobi elliptic functions. Gepreel and Shehata [22] put a direct method to construct the rational Jacobi elliptic solution for nonlinear DDEs. In this paper, we modified the direct method which was discussed in [22]. When we are constructing the solutions of nonlinear DDEs we neglected the case  $\beta_i = 0$  because if  $\beta_i = 0$ , we get the same solution which was discussed by [22]. The Jacobi elliptic functions could be generated into the hyperbolic functions when  $m$  tends to one in the other hand, they are generated into trigonometrical functions when  $m$  tends to zero as shown in Table 1.

## 4. Conclusion

In this paper, we modified the rational Jacobi elliptic functions method to calculate some new exact solutions for the nonlinear difference differential equations via the lattice equation, the discrete nonlinear Schrodinger equation with a saturable nonlinearity, the discrete nonlinear Klein-Gordon equation, and the quintic discrete nonlinear Schrodinger equation. As a result, many new and more rational Jacobi elliptic solutions are obtained, from which hyperbolic function solutions and trigonometric function solutions are derived when the moduli  $m \rightarrow 1$  and  $m \rightarrow 0$ .

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