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Research Article

Strong Convergence of a Projected Gradient Method

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The projected-gradient method is a powerful tool for solving constrained convex optimization problems and has extensively been studied. In the present paper, a projected-gradient method is presented for solving the minimization problem, and the strong convergence analysis of the suggested gradient projection method is given.

1. Introduction

In the present paper, our main purpose is to solve the following minimization problem:

$$\min_{x\in C} f(x), \tag{1.1}$$

where C is a nonempty closed and convex subset of a real Hilbert space H, $f : H \to R$ is a real-valued convex function.

Now it is well known that the projected-gradient method is a powerful tool for solving the above minimization problem and has extensively been studied. See, for instance, [1–8]. The classic algorithm is the following form of the projected-gradient method:

$$x_{n+1} = P_C\left(x_n - \gamma \nabla f(x_n)\right), \quad n \geq 0, \tag{1.2}$$

where $\gamma > 0$ is an any constant, P_C is the nearest point projection from H onto C, and ∇f denotes the gradient of f.

It is known [1] that if f has a Lipschitz continuous and strongly monotone gradient, then the sequence $\{x_n\}$ generated by (1.2) can be strongly convergent to a minimizer of f in C.

If the gradient of f is only assumed to be Lipschitz continuous, then $\{x_n\}$ can only be weakly convergent if H is infinite dimensional. An interesting problem is how to appropriately modify the projected gradient algorithm so as to have strong convergence? For this purpose, recently, Xu [9] introduced the following algorithm:

$$x_{n+1} = \theta_n h(x_n) + (1 - \theta_n) P_C(x_n - \gamma_n \nabla f(x_n)), \quad n \geq 0. \tag{1.3}$$

Under some additional assumptions, Xu [9] proved that the sequence $\{x_n\}$ converges strongly to a minimizer of (1.1). At the same time, Xu [9] also suggested a regularized method:

$$x_{n+1} = P_C(I - \gamma_n(\nabla f + \alpha_n I))x_n, \quad n \geq 0. \tag{1.4}$$

Consequently, Yao et al. [10] proved the strong convergence of the regularized method (1.4) under some weaker conditions.

Motivated by the above works, in this paper we will further construct a new projected gradient method for solving the minimization problem (1.1). It should be pointed out that our method also has strong convergence under some mild conditions.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. A bounded linear operator B is said to be strongly positive on H if there exists a constant $\alpha > 0$ such that

$$\langle Bx, x \rangle \geq \alpha \|x\|^2, \quad \forall x \in H. \tag{2.1}$$

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2.2}$$

A mapping $T : C \to C$ is said to be an averaged mapping, if and only if it can be written as the average of the identity I and a nonexpansive mapping; that is,

$$T = (1 - \alpha) I + \alpha R, \tag{2.3}$$

where $\alpha \in (0, 1)$ is a constant and $R : C \to C$ is a nonexpansive mappings. In this case, we call T is α-averaged.

A mapping $T : C \to C$ is said to be ν-inverse strongly monotone (ν-ism), if and only if

$$\langle x - y, Tx - Ty \rangle \geq \nu \|Tx - Ty\|^2, \quad x, y \in C. \tag{2.4}$$

The following proposition is well known, which is useful for the next section.

Proposition 2.1 (See [9]). *(1) The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^{N}$ is averaged, then so is the composite $T_1, \ldots, T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1\alpha_2 - \alpha_1\alpha_2$.*

(2) T is ν-ism, then for $\gamma > 0$, γT is (ν/γ)-ism.

Recall that the (nearest point or metric) projection from H onto C, denoted by P_C, assigns, to each $x \in H$, the unique point $P_C(x) \in C$ with the property

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}. \tag{2.5}$$

We use S to denote the solution set of (1.1). Assume that (1.1) is consistent, that is, $S \neq \emptyset$. If f is Frechet differentiable, then $x^* \in C$ solves (1.1) if and only if $x^* \in C$ satisfies the following optimality condition:

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{2.6}$$

where ∇f denotes the gradient of f. Observe that (2.6) can be rewritten as the following VI

$$\langle x^* - (x^* - \nabla f(x^*)), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{2.7}$$

(Note that the VI has been extensively studied in the literature, see, for instance [11–25].) This shows that the minimization (1.1) is equivalent to the fixed point problem

$$P_C(x^* - \gamma \nabla f(x^*)) = x^*, \tag{2.8}$$

where $\gamma > 0$ is an any constant. This relationship is very important for constructing our method.

Next we adopt the following notation:

(i) $x_n \to x$ means that x_n converges strongly to x;

(ii) $x_n \rightharpoonup x$ means that x_n converges weakly to x;

(iii) $\mathrm{Fix}(T) := \{x : Tx = x\}$ is the fixed points set of T.

Lemma 2.2 (See [26]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1. \tag{2.9}$$

Suppose

$$x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n \tag{2.10}$$

for all $n \geq 0$ *and*

$$\limsup_{n \to \infty} \left(\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \| \right) \leq 0. \tag{2.11}$$

Then, $\lim_{n \to \infty} \| y_n - x_n \| = 0$.

Lemma 2.3 (See [27] (demiclosedness principle)). *Let C be a closed and convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then*

$$(I - T)x = y. \tag{2.12}$$

In particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.

Lemma 2.4 (See [28]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \tag{2.13}$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ *and $\{\delta_n\}$ is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n \to \infty} \delta_n / \gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n \to \infty} a_n = 0$.

3. Main Results

Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to R$ be a real-valued Frechet differentiable convex function with the gradient ∇f. Let $A : C \to H$ be a ρ-contraction. Let $B : H \to H$ be a self-adjoint, strongly positive bounded linear operator with coefficient $\alpha > 0$. First, we present our algorithm for solving (1.1). Throughout, we assume $S \neq \emptyset$.

Algorithm 3.1. For given $x_0 \in C$, compute the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = P_C(I + (\sigma A - B)\theta_n) P_C(I - \gamma \nabla f) x_n, \quad n \geq 0, \tag{3.1}$$

where $\sigma > 0, \gamma > 0$ are two constants and the real number sequence $\{\theta_{\mathrm{n}}\} \subset [0, 1]$.

Remark 3.2. In (3.1), we use two projections. Now, it is well-known that the advantage of projections, which makes them successful in real-word applications, is computational.

Next, we show the convergence analysis of this Algorithm 3.1.

Theorem 3.3. *Assume that the gradient ∇f is L-Lipschitzian and $\sigma\rho < \alpha$. Let $\{x_n\}$ be a sequence generated by* (3.1)*, where $\gamma \in (0, 2/L)$ is a constant and the sequence $\{\theta_n\}$ satisfies the conditions: (i) $\lim_{n\to\infty}\theta_n = 0$ and (ii) $\sum_{n=0}^{\infty}\theta_n = \infty$. Then $\{x_n\}$ converges to a minimizer $\tilde{x}$ of* (1.1) *which solves the following variational inequality:*

$$\tilde{x} \in S \quad \textit{such that } \langle \sigma A(\tilde{x}) - B(\tilde{x}), x - \tilde{x}\rangle \leq 0, \qquad \forall x \in S. \tag{3.2}$$

By Algorithm 3.1 involved in the projection, we will use the properties of the metric projection for proving Theorem 3.3. For convenience, we list the properties of the projection as follows.

Proposition 3.4. *It is well known that the metric projection P_C of H onto C has the following basic properties:*

(i) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$, *for all $x, y \in H$;*

(ii) $\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2$, *for every $x, y \in H$;*

(iii) $\langle x - P_C(x), y - P_C(x)\rangle \leq 0$, *for all $x \in H$, $y \in C$.*

The Proof of Theorem 3.3

Let $x^* \in S$. First, from (2.8), we note that $P_C(I - \gamma\nabla f)x^* = x^*$. By (3.1), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\|P_C(I + (\sigma A - B)\theta_n)P_C(I - \gamma\nabla \mathrm{f})x_n - x^*\right\| \\
&\leq \left\|P_C(I + (\sigma A - B)\theta_n)P_C(I - \gamma\nabla f)x_n - P_C(I + (\sigma A - B)\theta_n)P_C(I - \gamma\nabla f)x^*\right\| \\
&\quad + \|P_C(I + (\sigma A - B)\theta_n)x^* - x^*\| \\
&\leq \left[1 - (\alpha - \sigma\rho)\theta_n\right]\|x_n - x^*\| + \theta_n\|\sigma A(x^*) - B(x^*)\| \\
&= \left[1 - (\alpha - \sigma\rho)\theta_n\right]\|x_n - x^*\| + (\alpha - \sigma\rho)\theta_n\frac{\|\sigma A(x^*) - B(x^*)\|}{\alpha - \sigma\rho} \\
&\leq \max\left\{\|x_n - x^*\|, \frac{\|\sigma A(x^*) - B(x^*)\|}{\alpha - \sigma\rho}\right\}.
\end{aligned} \tag{3.3}$$

Thus, by induction, we obtain

$$\|x_n - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{\|\sigma A(x^*) - B(x^*)\|}{\alpha - \sigma\rho}\right\}. \tag{3.4}$$

Note that the Lipschitz condition implies that the gradient ∇f is $(1/L)$-inverse strongly monotone (ism), which then implies that $\gamma\nabla f$ is $(1/\gamma L)$-ism. So, $I - \gamma\nabla f$ is $(\gamma L/2)$-averaged. Now since the projection P_C is $(1/2)$-averaged, we see that $P_C(I - \gamma\nabla f)$ is $((2 + \gamma L)/4)$-averaged. Hence we have that

$$P_C = \frac{1}{2}I + \frac{1}{2}R \qquad P_C(I - \gamma\nabla f) = \frac{2 - \gamma L}{4}I + \frac{2 + \gamma L}{4}T = (1 - \beta)I + \beta T, \tag{3.5}$$

where R, T are nonexpansive and $\beta = (2+\gamma L)/4 \in (0,1)$. Then we can rewrite (3.1) as

$$\begin{aligned} x_{n+1} &= \left(\frac{1}{2}I + \frac{1}{2}R\right)(I + (\sigma A - B)\theta_n)\left[(1-\beta)x_n + \beta T x_n\right] \\ &= \frac{1-\beta}{2}x_n + \frac{\beta}{2}Tx_n + \left(\frac{\theta_n}{2}(\sigma A - B) + \frac{R}{2}(I + (\sigma A - B)\theta_n)\right)\left[(1-\beta)x_n + \beta T x_n\right] \\ &= \frac{1-\beta}{2}x_n + \frac{1+\beta}{2}y_n, \end{aligned} \tag{3.6}$$

where

$$y_n = \frac{2}{1+\beta}\left(\frac{\theta_n}{2}(\sigma A - B) + \frac{R}{2}(I + (\sigma A - B)\theta_n)\right)\left[(1-\beta)x_n + \beta T x_n\right] + \frac{\beta}{1+\beta}Tx_n. \tag{3.7}$$

Set $z_n = (1-\beta)x_n + \beta T x_n$ for all n. Since $\{x_n\}$ is bounded, we deduce $\{A(x_n)\}$, $\{B(x_n)\}$, and $\{Tx_n\}$ are all bounded. Hence, there exists a constant $M > 0$ such that

$$\sup_n \{\|(\sigma A - B)z_n\|\} \le M. \tag{3.8}$$

Thus,

$$\begin{aligned} \|y_{n+1} - y_n\| &\le \frac{2}{1+\beta}\left\|\frac{\theta_{n+1}}{2}(\sigma A - B)z_{n+1} - \frac{\theta_n}{2}(\sigma A - B)z_n\right\| + \frac{\beta}{1+\beta}\|Tx_{n+1} - Tx_n\| \\ &\quad + \frac{1}{1+\beta}\|R(\theta_{n+1}\sigma A + (I - \theta_{n+1}B))z_{n+1} - R(I + (\sigma A - B)\theta_n)z_n\| \\ &\le \frac{1}{1+\beta}(\theta_{n+1} + \theta_n)M + \frac{\beta}{1+\beta}\|x_{n+1} - x_n\| + \frac{1}{1+\beta}\|z_{n+1} - z_n\| \\ &\quad + \frac{1}{1+\beta}\|\theta_{n+1}(\sigma A - B)z_{n+1} - \theta_n(\sigma A - B)z_n\| \\ &\le \frac{2}{1+\beta}(\theta_{n+1} + \theta_n)M + \|x_{n+1} - x_n\|. \end{aligned} \tag{3.9}$$

It follows that

$$\limsup_{n\to\infty}\left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\right) \le 0. \tag{3.10}$$

This together with Lemma 2.2 implies that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.11}$$

So,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\frac{1+\beta}{2}\|y_n-x_n\| = 0. \tag{3.12}$$

Since

$$\begin{aligned}\|x_n-P_C(I-\gamma\nabla f)x_n\| &\le \|x_n-x_{n+1}\| + \|x_{n+1}-P_C(I-\gamma\nabla f)x_n\| \\ &= \|x_n-x_{n+1}\| + \|P_C(I+(\sigma A-B)\theta_n)P_C(I-\gamma\nabla f)-P_C(I-\gamma\nabla f)x_n\| \\ &\le \|x_n-x_{n+1}\| + \theta_n\|(\sigma A-B)P_C(I-\gamma\nabla f)x_n\|,\end{aligned} \tag{3.13}$$

we deduce

$$\lim_{n\to\infty}\|x_n-P_C(I-\gamma\nabla f)x_n\| = 0. \tag{3.14}$$

Next we prove

$$\limsup_{k\to\infty}\langle\sigma A(x^*)-B(x^*),x_n-x^*\rangle \le 0, \tag{3.15}$$

where x^* is the unique solution of VI (3.2).

Indeed, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\sigma A(x^*)-B(x^*),x_n-x^*\rangle = \lim_{i\to\infty}\langle\sigma A(x^*)-B(x^*),x_{n_i}-x^*\rangle. \tag{3.16}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence of $\{x_{n_i}\}$ which converges weakly to a point $\tilde{x}$. Without loss of generality, we may assume that $\{x_{n_i}\}$ converges weakly to $\tilde{x}$. Since $\gamma \in (0,2/L)$, $P_C(I-\gamma\nabla f)$ is nonexpansive. Thus, from (3.14) and Lemma 2.3, we have $x_{n_i} \rightharpoonup \tilde{x} \in \mathrm{Fix}(P_C(I-\gamma\nabla f)) = S$. Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\langle\sigma A(x^*)-B(x^*),x_n-x^*\rangle &= \lim_{i\to\infty}\langle\sigma A(x^*)-B(x^*),x_{n_i}-x^*\rangle \\ &= \langle\sigma A(x^*)-B(x^*),\tilde{x}-x^*\rangle \le 0.\end{aligned} \tag{3.17}$$

Finally, we show $x_n \to \tilde{x}$. By using the property of the projection P_C, we have

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|^2 &= \|P_C(I+(\sigma A-B)\theta_n)P_C(I-\gamma\nabla f)x_n-P_C(\tilde{x})\|^2 \\ &\le \langle(I+(\sigma A-B)\theta_n)P_C(I-\gamma\nabla f)x_n-\tilde{x},x_{n+1}-\tilde{x}\rangle \\ &= \langle(I+(\sigma A-B)\theta_n)(P_C(I-\gamma\nabla f)x_n-\tilde{x}),x_{n+1}-\tilde{x}\rangle \\ &\quad + \theta_n\langle\sigma A(\tilde{x})-B(\tilde{x}),x_{n+1}-\tilde{x}\rangle\end{aligned}$$

$$\begin{aligned}
&\leq \|I+(\sigma A-B)\theta_n\|\left\|P_C(I-\gamma\nabla f)x_n - P_C(I-\gamma\nabla f)\tilde{x}\right\|\|x_{n+1}-\tilde{x}\| \\
&\quad + \theta_n\langle \sigma A(\tilde{x}) - B(\tilde{x}), x_{n+1}-\tilde{x}\rangle \\
&\leq \left[1-(\alpha-\sigma\rho)\theta_n\right]\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\| + \theta_n\langle \sigma A(\tilde{x}) - B(\tilde{x}), x_{n+1}-\tilde{x}\rangle \\
&\leq \frac{1-(\alpha-\sigma\rho)\theta_n}{2}\|x_n-\tilde{x}\|^2 + \frac{1}{2}\|x_{n+1}-\tilde{x}\|^2 + \theta_n\langle \sigma A(\tilde{x}) - B(\tilde{x}), x_{n+1}-\tilde{x}\rangle.
\end{aligned} \tag{3.18}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &\leq \left[1-(\alpha-\sigma\rho)\theta_n\right]\|x_n-\tilde{x}\|^2 + 2\theta_n\langle \sigma A(\tilde{x}) - B(\tilde{x}), x_{n+1}-\tilde{x}\rangle \\
&= \left[1-(\alpha-\sigma\rho)\theta_n\right]\|x_n-\tilde{x}\|^2 + (\alpha-\sigma\rho)\theta_n\left\{\frac{2}{\alpha-\sigma\rho}\langle \sigma A(\tilde{x}) - B(\tilde{x}), x_{n+1}-\tilde{x}\rangle\right\}.
\end{aligned} \tag{3.19}$$

It is obvious that $\limsup_{n\to\infty}((2/(\alpha-\sigma\rho))\langle \sigma A(\tilde{x}) - B(\tilde{x}), x_{n+1}-\tilde{x}\rangle \leq 0)$. Then we can apply Lemma 2.4 to the last inequality to conclude that $x_n \to \tilde{x}$. The proof is completed.

In (3.1), if we take $A = 0$ and $B = I$, then (3.1) reduces to the following.

Algorithm 3.5. For given $x_0 \in C$, compute the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = P_C(1-\theta_n)P_C(I-\gamma\nabla f)x_n, \qquad n \geq 0, \tag{3.20}$$

where $\sigma > 0, \gamma > 0$ are two constants and the real number sequence $\{\theta_n\} \subset [0,1]$.

From Theorem 3.3, we have the following result.

Theorem 3.6. *Assume that the gradient ∇f is L-Lipschitzian and $\sigma\rho < \alpha$. Let $\{x_n\}$ be a sequence generated by* (3.20)*, where $\gamma \in (0, 2/L)$ is a constant and the sequences $\{\theta_n\}$ satisfies the conditions: (i) $\lim_{n\to\infty}\theta_n = 0$ and (ii) $\sum_{n=0}^{\infty}\theta_n = \infty$. Then $\{x_n\}$ converges to a minimizer $\tilde{x}$ of* (1.1) *which is the minimum norm element in S.*

Proof. As a consequence of Theorem 3.3, we obtain that the sequence $\{x_n\}$ generated by (3.20) converges strongly to $\tilde{x}$ which satisfies

$$\tilde{x} \in S \quad \text{such that } \langle -\tilde{x}, x-\tilde{x}\rangle \leq 0, \qquad \forall x \in S. \tag{3.21}$$

This implies

$$\|\tilde{x}\|^2 \leq \langle x, \tilde{x}\rangle \leq \|x\|\|\tilde{x}\|, \quad \forall x \in S. \tag{3.22}$$

Thus,

$$\|\tilde{x}\| \leq \|x\|, \qquad \forall x \in S. \tag{3.23}$$

That is, $\tilde{x}$ is the minimum norm element in S. This completes the proof. □

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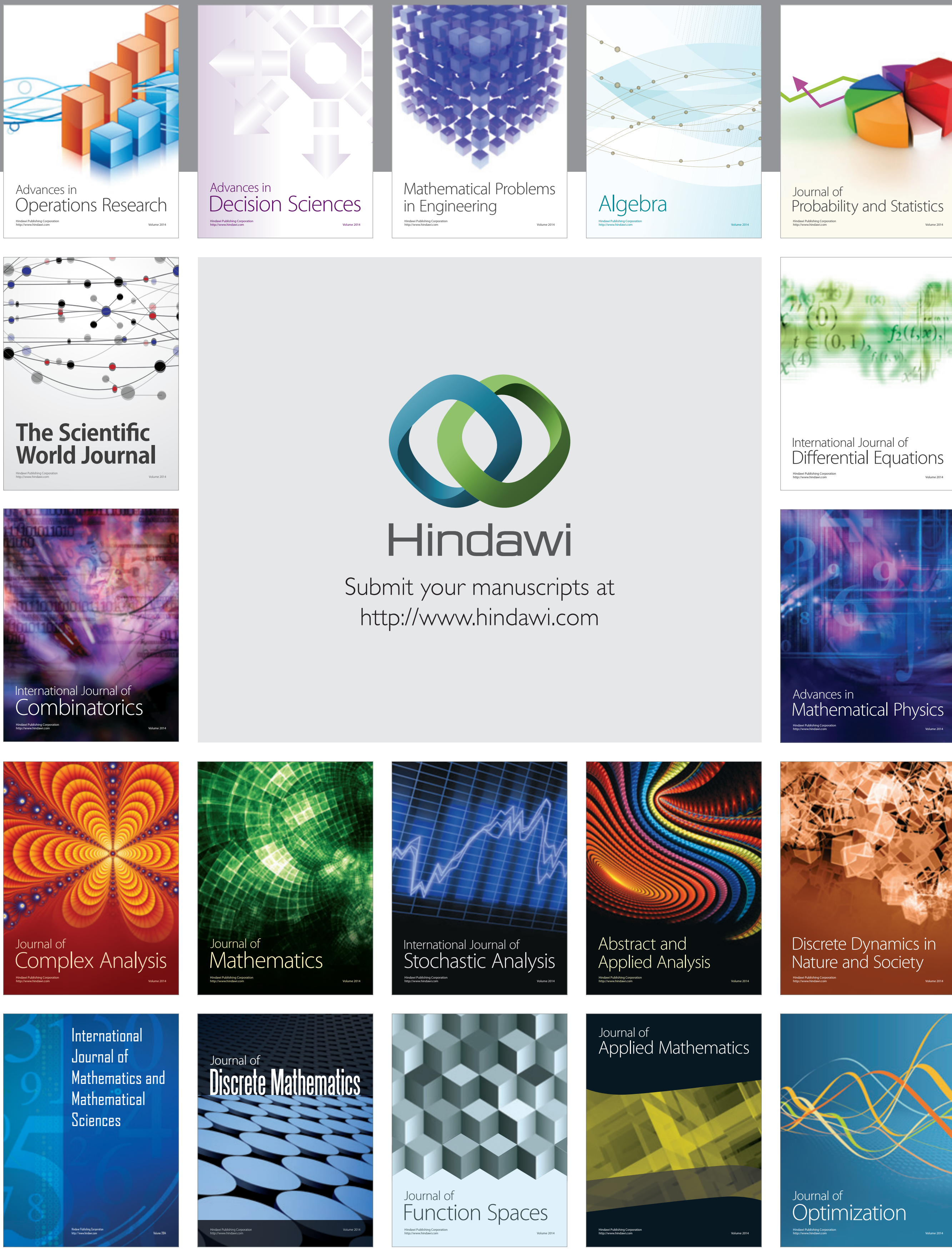
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