Research Article

Instability Induced by Cross-Diffusion in a Predator-Prey Model with Sex Structure

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In this paper, we consider a cross-diffusion predator-prey model with sex structure. We prove that cross-diffusion can destabilize a uniform positive equilibrium which is stable for the ODE system and for the weakly coupled reaction-diffusion system. As a result, we find that stationary patterns arise solely from the effect of cross-diffusion.

1. Introduction

Sex ratio means the comparison between the number of male and female in the species. The sex ratio is generally regarded as $1:1$. But for wildlife, the sex ratio of species varies with the category, environment condition, community behavior, orientation and heredity, and so forth. The animal's sex ratio in the different life history stages may vary with different animals. Take a bird as an example: the number of the older males is larger than that of the older females with the increase in age, which is contrary to the case of the mammal where the number of the older females is larger than that of the older males with the increase in age [1, 2]. Sex ratio is the basis of analyzing the dynamic state of different species, the variation of which has a huge influence on the dynamic state of the species [1–7]. Abrahams and Dill [5] have provided evidence that male and female guppies forage differently in the presence of predators and that sexual differences in the energetic equivalence of the risk of predation exist. Eubanks and Miller [6] have found that female Gladicosa pulchra (Lycosidae) wolf spiders climb trees significantly more often than males in the presence of forest floor predators. It is found that the sex of voles affects the risk of predation by mammals and female voles are more easily predated than male voles in [7].

Incorporating the sex of prey in a classical Lotka-Volterra model, Liu et al. [1] considered the following sex-structure model:

$$u'(t) = b_1 v - u(D_1 + ku + kv + c_1 w),$$

$$v'(t) = v(b_2 - D_2 - ku - kv - c_1 w), \tag{1.1}$$

$$w'(t) = w(-D_3 + c_2 u + c_2 v - c_3 w),$$

where u, v, and w are the population densities of the male prey, the female prey, and the predator species respectively. The parameters D_1, D_2, and D_3 are their mortality rates, b_1 and b_2 are the birth rates of the male prey and the female prey, c_1 and c_2 are the predation rate and the conversion rate of the predators, and k and c_3 are the intraspecific competition rates of the prey and predators. All the parameters in model (1.1) are positive.

If $\beta = b_2 - D_2 > 0$, then two obvious nonnegative equilibria of model (1.1) are $\mathbf{u}_0 = (0,0,0)$ and $\mathbf{u}_1 = (u_1, v_1, 0)$, where

$$u_1 = \frac{b_1 \beta}{k(b_1 + D_1 + \beta)}, \qquad v_1 = \frac{\beta(D_1 + \beta)}{k(b_1 + D_1 + \beta)}. \tag{1.2}$$

Moreover, model (1.1) has a positive equilibrium if and only if

$$R := c_2 \beta - kD_3 > 0. \tag{H1}$$

In this case the positive equilibrium is uniquely given by $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}, \tilde{w})$, where

$$\tilde{u} = \frac{b_1(c_3\beta + c_1 D_3)}{(b_1 + D_1 + \beta)(c_3 k + c_1 c_2)}, \qquad \tilde{v} = \frac{(D_1 + \beta)(c_3\beta + c_1 D_3)}{(b_1 + D_1 + \beta)(c_3 k + c_1 c_2)},$$

$$\tilde{w} = \frac{c_2\beta - kD_3}{c_3 k + c_1 c_2}. \tag{1.3}$$

It turns out that R plays an important role in determining the stability of $\mathbf{u}_1$ and $\tilde{\mathbf{u}}$ [1]. To be precise, $\mathbf{u}_1$ is locally asymptotically stable if $R < 0$, while $\tilde{\mathbf{u}}$ is locally asymptotically stable if $R > 0$. This shows that a uniform coexistence state exists and is stable when the intrinsic growth rate β of the female prey is larger than the critical value kD_3/c_2.

Taking account of the inhomogeneous distribution of the prey and the predator in different spatial locations within a fixed bounded domain Ω at any given time, and the natural tendency of each species to diffuse to areas of smaller population concentration, Liu and Zhou in [8] investigated the following weakly coupled reaction-diffusion system:

$$u_t - d_1 \Delta u = b_1 v - u(D_1 + ku + kv + c_1 w), \quad x \in \Omega, \ t > 0,$$

$$v_t - d_2 \Delta v = v(\beta - ku - kv - c_1 w), \quad x \in \Omega, \ t > 0,$$

$$w_t - d_3 \Delta w = w(-D_3 + c_2 u + c_2 v - c_3 w), \quad x \in \Omega, \ t > 0,$$

$$\partial_\eta u(x,t) = \partial_\eta v(x,t) = \partial_\eta w(x,t) = 0, \quad x \in \partial\Omega, \ t > 0,$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \ x \in \Omega,$$

$$(1.4)$$

where η is the outward unit normal vector of the boundary $\partial\Omega$ which is smooth, $\partial_\eta = \partial/\partial_\eta$. The homogeneous Neumann boundary condition indicates that the predator-prey system is self-contained with zero population flux across the boundary. The constants d_1, d_2, and d_3, called diffusion coefficients, are positive, and the initial values $u_0(x)$, $v_0(x)$, and $w_0(x)$ are nonnegative smooth functions which are not identically zero. Liu and Zhou in [8] found that the nonnegative constant steady states have the same stability properties as the ODE model (1.1). Therefore, Turing instability cannot occur for this reaction-diffusion system.

However, in model (1.4), only diffusion of each individual species is taken into account. In some cases, the reality is that the female prey is easily predated because of physiological factor, while the male prey can congregate and form a huge group to protect itself from the attack of the predators [7, 9]; therefore, the predators tend to keep away from their male prey. Similarly as in [10–12], we model this by the cross-diffusion term $\Delta(d_3w + d_4uw)$ for the predators, where $d_4 > 0$, called the cross-diffusion coefficient. Thus, the cross-diffusion system that we will study is the following:

$$u_t - d_1\Delta u = b_1v - u(D_1 + ku + kv + c_1w) := G_1(u,v,w), \quad x \in \Omega, \ t > 0,$$

$$v_t - d_2\Delta v = v(\beta - ku - kv - c_1w) := G_2(u,v,w), \quad x \in \Omega, \ t > 0,$$

$$w_t - \Delta(d_3w + d_4uw) = w(-D_3 + c_2u + c_2v - c_3w) := G_3(u,v,w), \quad x \in \Omega, \ t > 0, \qquad (1.5)$$

$$\partial_\eta u(x,t) = \partial_\eta v(x,t) = \partial_\eta w(x,t) = 0, \quad x \in \partial\Omega, \ t > 0,$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \quad x \in \Omega.$$

To our knowledge, only a few works investigated the effect of cross-diffusion on population structure and dynamics in the above model. Recently, H. Xu and S. Xu in [13] investigated the global existence of solutions for the corresponding full SKT model of (1.5) when the space dimension is less than ten.

An interesting feature of (1.5) is that the interaction between the predators and the male prey gives rise to a cross-diffusion term. The resulting mathematical model is a strongly coupled system of three equations which is mathematically much more complex than those considered earlier. In this paper, we will show that cross-diffusion can destabilize the uniform equilibrium $\tilde{\mathbf{u}}$ which is stable for models (1.1) and (1.4). Moreover, we will demonstrate that the nonlinear dispersive force can give rise to a spatial segregation of these species.

Our paper is organized as follows. In Section 2, we analyze the local stability of $\tilde{\mathbf{u}}$ for (1.5) and calculate the fixed point index, which is important for our later discussions on the existence of nonconstant positive steady states. In Section 3, we prove global asymptotic stability of $\tilde{\mathbf{u}}$ with $d_4 = 0$, that is, when no cross-diffusion occurs in the model. This implies that cross-diffusion has a destabilizing effect. In Section 4, we establish a priori upper and lower bounds for all possible positive steady states of (1.5). In Section 5, we study the global existence of nonconstant positive steady states of (1.5) for suitable values of the parameters. This is done by using the Leray-Schauder degree theory and the results obtained in Sections 2,

3, and 4. In Section 6, we discuss the nonexistence of nonconstant positive steady states of (1.5). In the last section, we give a brief discussion about our model.

2. Local Stability Analysis and Fixed Point Index of $\tilde{u}$

Let $\mathbf{u} = (u, v, w)^T$, $\Phi(\mathbf{u}) = (d_1 u, d_2 v, d_3 w + d_4 uw)^T$, and $\mathbf{G}(\mathbf{u}) = (G_1(\mathbf{u}), G_2(\mathbf{u}), G_3(\mathbf{u}))^T$. Then the stationary problem of (1.5) can be written as

$$-\Delta\Phi(\mathbf{u}) = \mathbf{G}(\mathbf{u}) \quad \text{in } \Omega; \qquad \partial_\eta \mathbf{u} = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

In this section, we study the linearization of (2.1) at $\tilde{\mathbf{u}}$ and calculate the fixed point index.

Similar to [14, 15], let $0 = \mu_1 < \mu_2 < \mu_3 < \mu_4 \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition, and let $E(\mu_i)$ be the eigenspace corresponding to μ_i in $H^1(\Omega)$. Let $\{\phi_{ij} : j = 1, 2, \ldots, \dim E(\mu_i)\}$ be the orthonormal basis of $E(\mu_i)$, $\mathbf{X} = [H^1(\Omega)]^3$, and $\mathbf{X}_{ij} = \{\mathbf{c}\phi_{ij} : \mathbf{c} \in \mathbb{R}^3\}$. Then

$$\mathbf{X} = \bigoplus_{i=1}^{+\infty} \mathbf{X}_i, \qquad \mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\mu_i)} \mathbf{X}_{ij}. \tag{2.2}$$

Let $\mathbf{Y} = [C^1(\overline{\Omega})]^3$, $\mathbf{Y}^+ = \{\mathbf{u} \in \mathbf{Y} : u, v, w > 0 \text{ on } \overline{\Omega}\}$, and $B(C) = \{\mathbf{u} \in \mathbf{Y} : C^{-1} < u, v, w < C \text{ on } \overline{\Omega}\}$ for $C > 0$. Since $\det \mathbf{\Phi_u}(\mathbf{u}) = d_1 d_2 (d_3 + d_4 u) > 0$ for all nonnegative $\mathbf{u}$, $\mathbf{\Phi_u^{-1}}(\mathbf{u})$ exists and $\det\{\mathbf{\Phi_u^{-1}}(\mathbf{u})\}$ is positive. Hence, $\mathbf{u}$ is a positive solution to (2.1) if and only if

$$\mathbf{F}(\mathbf{u}) := \mathbf{u} - (\mathbf{I} - \Delta)^{-1}\left\{\mathbf{\Phi_u^{-1}}(\mathbf{u})[\mathbf{G}(\mathbf{u}) + \nabla\mathbf{u}\mathbf{\Phi_{uu}}(\mathbf{u})\nabla\mathbf{u}] + \mathbf{u}\right\} = 0 \quad \text{in } \mathbf{Y}^+, \tag{2.3}$$

where $(\mathbf{I} - \Delta)^{-1}$ is the inverse of $\mathbf{I} - \Delta$ under homogeneous Neumann boundary conditions.

Further, we note that $D_\mathbf{u}\mathbf{F}(\tilde{\mathbf{u}}) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{\Phi_u^{-1}}(\tilde{\mathbf{u}})\mathbf{G_u}(\tilde{\mathbf{u}}) + \mathbf{I}\}$ and λ is an eigenvalue of $D_\mathbf{u}\mathbf{F}(\tilde{\mathbf{u}})$ if and only if, for some $i \geq 1$, it is an eigenvalue of the matrix

$$\mathbf{B}_i := \mathbf{I} - \frac{1}{1 + \mu_i}\left[\mathbf{\Phi_u^{-1}}(\tilde{\mathbf{u}})\mathbf{G_u}(\tilde{\mathbf{u}}) + \mathbf{I}\right] = \frac{1}{1 + \mu_i}\left[\mu_i \mathbf{I} - \mathbf{\Phi_u^{-1}}(\tilde{\mathbf{u}})\mathbf{G_u}(\tilde{\mathbf{u}})\right]. \tag{2.4}$$

Writing

$$H(\mu) = H(\tilde{\mathbf{u}}; \mu) := \det\left\{\mu\mathbf{I} - \mathbf{\Phi_u^{-1}}(\tilde{\mathbf{u}})\mathbf{G_u}(\tilde{\mathbf{u}})\right\}, \tag{2.5}$$

we see that if $H(\mu_i) \neq 0$, then for each integer $1 \leq j \leq \dim E(\mu_i)$, the number of negative eigenvalues of $D_\mathbf{u}\mathbf{F}(\tilde{\mathbf{u}})$ on $\mathbf{X}_{ij}$ is odd if and only if $H(\mu_i) < 0$. As a consequence, we have the following proposition.

Proposition 2.1 (see [16]). *Suppose that, for all $i \geq 1$, $H(\mu_i) \neq 0$. Then*

$$\text{index}(\mathbf{F}(\cdot), \tilde{\mathbf{u}}) = (-1)^\gamma, \tag{2.6}$$

where

$$\gamma = \sum_{i \geq 1, H(\mu_i) < 0} \dim E(\mu_i). \tag{2.7}$$

To facilitate our computation of index$(F(\cdot), \tilde{\mathbf{u}})$, we need to determine the sign of $H(\mu_i)$. In particular, as the aim of this paper is to study the existence of stationary patterns of (2.1) with respect to the cross-diffusion coefficient d_4, we will concentrate on the dependence of $H(\mu_i)$ on d_4. At this point, we note that $H(\mu) = \det\{\Phi_{\mathbf{u}}^{-1}(\tilde{\mathbf{u}})\} \det\{\mu\Phi_{\mathbf{u}}(\tilde{\mathbf{u}}) - \mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\}$. Since $\det\{\Phi_{\mathbf{u}}^{-1}(\tilde{\mathbf{u}})\}$ is positive, we will need only to consider $\det\{\mu\Phi_{\mathbf{u}}(\tilde{\mathbf{u}}) - \mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\}$. By

$$\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}}) = \begin{pmatrix} -k\tilde{u} - D_1 - \beta & b_1 - k\tilde{u} & -c_1\tilde{u} \\ -k\tilde{v} & -k\tilde{v} & -c_1\tilde{v} \\ c_2\tilde{w} & c_2\tilde{w} & -c_3\tilde{w} \end{pmatrix}, \qquad \Phi_{\mathbf{u}}(\tilde{\mathbf{u}}) = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ d_4\tilde{w} & 0 & d_3 + d_4\tilde{u} \end{pmatrix}, \tag{2.8}$$

we have

$$\det\{\mu\Phi_{\mathbf{u}}(\tilde{\mathbf{u}}) - \mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\} = C_3(d_4)\mu^3 + C_2(d_4)\mu^2 + C_1(d_4)\mu - \det\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})$$

$$:= \mathcal{C}(d_4; \mu), \tag{2.9}$$

where

$$C_3(d_4) = d_1 d_2 (d_3 + d_4\tilde{u}),$$

$$C_2(d_4) = d_1 d_2 c_3\tilde{w} + d_1 k(d_3 + d_4\tilde{u})\tilde{v} + d_2(d_3 + d_4\tilde{u})(k\tilde{u} + D_1 + \beta) - d_2 d_4 c_1\tilde{u}\tilde{w},$$

$$C_1(d_4) = d_1 k c_3\tilde{v}\tilde{w} + d_2 c_3(k\tilde{u} + D_1 + \beta)\tilde{w} + (d_3 + d_4\tilde{u})(k\tilde{u} + D_1 + \beta)k\tilde{v} + d_4 c_1\tilde{v}\tilde{w}(k\tilde{u} - b_1)$$

$$- c_1\tilde{u}(-d_2 c_2\tilde{w} + k d_4\tilde{v}\tilde{w}) + c_1 c_2 d_1\tilde{v}\tilde{w} - (d_3 + d_4\tilde{u})k\tilde{v}(k\tilde{u} - b_1),$$

$$\det\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}}) = \tilde{v}\tilde{w}\big[-(kc_3 + c_1 c_2)(k\tilde{u} + D_1 + \beta) + (c_1 c_2 + k c_3)(k\tilde{u} - b_1)\big]. \tag{2.10}$$

Notice that $k\tilde{u} - b_1 < 0$; thus $\det\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}}) < 0$. We consider the dependence of $\mathcal{C}$ on d_4. Let $\tilde{\mu}_1(d_4)$, $\tilde{\mu}_2(d_4)$, and $\tilde{\mu}_3(d_4)$ be the three roots of $\mathcal{C}(d_4; \mu) = 0$ with $\mathrm{Re}\{\tilde{\mu}_1(d_4)\} \leq \mathrm{Re}\{\tilde{\mu}_2(d_4)\} \leq \mathrm{Re}\{\tilde{\mu}_3(d_4)\}$. It follows that $\tilde{\mu}_1(d_4)\tilde{\mu}_2(d_4)\tilde{\mu}_3(d_4) < 0$ from $\det\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}}) < 0$. Thus, among $\tilde{\mu}_1(d_4)$, $\tilde{\mu}_2(d_4)$, $\tilde{\mu}_3(d_4)$ at least one is real and negative, and the product of the other two is positive.

Consider the following limits:

$$\lim_{d_4 \to \infty} \frac{C_3(d_4)}{d_4} = d_1 d_2\tilde{u} := a_3,$$

$$\lim_{d_4 \to \infty} \frac{C_2(d_4)}{d_4} = d_1 k\tilde{u}\tilde{v} + d_2\tilde{u}\left(k\tilde{u} + \frac{b_1\tilde{v}}{\tilde{u}}\right) - d_2 c_1\tilde{u}\tilde{w}$$

$$= d_1 k\tilde{u}\tilde{v} + d_2\left(k\tilde{u}^2 + b_1\tilde{v} - c_1\tilde{u}\tilde{w}\right) := a_2,$$

$$\lim_{d_4 \to \infty} \frac{C_1(d_4)}{d_4} = \tilde{u}\left(k\tilde{u} + \frac{b_1\tilde{v}}{\tilde{u}}\right)k\tilde{v} + c_1\tilde{v}\tilde{w}(k\tilde{u} - b_1) - c_1\tilde{u}k\tilde{v}\tilde{w} - \tilde{u}k\tilde{v}(k\tilde{u} - b_1)$$

$$= b_1\tilde{v}(\beta - 2c_1\tilde{w}) := a_1. \tag{2.11}$$

Therefore, $a_1 < 0$ if

$$\beta < 2c_1\tilde{w}. \tag{H2}$$

In the following, we restrict our attention to $\beta < 2c_1\tilde{w}$. In this range, $a_1 < 0$ and $C_1(d_4) < 0$ for all sufficiently large d_4. Notice that

$$\lim_{d_4 \to \infty} \frac{C(d_4; \mu)}{d_4} = a_3\mu^3 + a_2\mu^2 + a_1\mu = \mu\left(a_3\mu^2 + a_2\mu + a_1\right) \tag{2.12}$$

and $a_1 < 0 < a_3$. A continuity argument shows that, when d_4 is large, $\tilde{\mu}_1(d_4)$ is real and negative. Furthermore, as $\tilde{\mu}_2(d_4)\tilde{\mu}_3(d_4) > 0$, $\tilde{\mu}_2(d_4)$ and $\tilde{\mu}_3(d_4)$ are real and positive, and

$$\lim_{d_4 \to \infty} \tilde{\mu}_1(d_4) = \frac{-a_2 - \sqrt{a_2^2 - 4a_1a_3}}{2a_3} < 0,$$

$$\lim_{d_4 \to \infty} \tilde{\mu}_2(d_4) = 0, \tag{2.13}$$

$$\lim_{d_4 \to \infty} \tilde{\mu}_3(d_4) = \frac{-a_2 + \sqrt{a_2^2 - 4a_1a_3}}{2a_3} := \tilde{\mu} > 0.$$

Thus we have the following proposition.

Proposition 2.2. *Assume that* (H1) *and* (H2) *hold. Then there exists a positive number d_4^* such that, when $d_4 \geq d_4^*$, the three roots $\tilde{\mu}_1(d_4)$, $\tilde{\mu}_2(d_4)$, $\tilde{\mu}_3(d_4)$ of $C(d_4; \mu) = 0$ are all real and satisfy* (2.13). *Moreover, for all $d_4 \geq d_4^*$,*

$$-\infty < \tilde{\mu}_1(d_4) < 0 < \tilde{\mu}_2(d_4) < \tilde{\mu}_3(d_4),$$

$$C(d_4; \mu) < 0, \quad \mu \in (-\infty, \tilde{\mu}_1(d_4)) \cup (\tilde{\mu}_2(d_4), \tilde{\mu}_3(d_4)), \tag{2.14}$$

$$C(d_4; \mu) > 0, \quad \mu \in (\tilde{\mu}_1(d_4), \tilde{\mu}_2(d_4)) \cup (\tilde{\mu}_3(d_4), +\infty).$$

Remark 2.3. Proposition 2.2 gives a criterion for the instability of $\tilde{\mathbf{u}}$ when $\tilde{\mu} > \mu_2$ and the cross-diffusion coefficient d_4 is large enough. We further check conditions (H1) and (H2). Let the parameters d_1, d_2, d_3, b_1, D_1, k, c_1, c_2, and c_3 be fixed. Condition (H1) is equivalent to $\beta > kD_3/c_2$, and condition (H2) is equivalent to $\beta > 2kc_1D_3/\gamma_1$ for some $\gamma_1 := c_1c_2 - kc_3 > 0$. Notice that $2c_1/\gamma_1 > 1/c_2$, so there exists an unbounded region $U_1 = \{(D_3, \beta) \in \mathbb{R}_+^2 : \beta > 2kc_1D_3/\gamma_1\}$, such that for any $(D_3, \beta) \in U_1$, $\tilde{\mathbf{u}}$ is an unstable equilibrium with respect to (1.5) when $\tilde{\mu} > \mu_2$ and the cross-diffusion coefficient d_4 is sufficiently large.

3. Global Asymptotic Stability of $\tilde{u}$ for (1.4)

The aim of this section is to prove Theorem 3.2 which shows that model (1.4) has no nonconstant positive steady state no matter what the diffusion coefficients d_1, d_2, and d_3 are; in other words, diffusion alone (without cross-diffusion) cannot drive instability and cannot generate patterns for this predator-prey model. For this, we will make use of the following result.

Lemma 3.1 (see [17]). *Let a and b be positive constants. Assume that $\varphi, \psi \in C^1([a, +\infty))$, $\psi(t) \geq 0$, and φ is bounded from below. If $\varphi'(t) \leq -b\psi(t)$ and $\psi'(t)$ is bounded in $[a, +\infty)$, then $\lim_{t\to\infty}\psi(t) = 0$.*

Theorem 3.2. *Let the parameters d_1, d_2, d_3, b_1, D_1, D_3, k, c_1, c_2, c_3, and β be fixed positive constants that satisfy* (H1) *and*

$$b_1^2 < 4k\tilde{u}(D_1 + b_1). \tag{H3}$$

Let (u, v, w) be a positive solution of (1.4). *Then*

$$\|u(\cdot, t) - \tilde{u}\|_{L^2(\Omega)} \longrightarrow 0, \qquad \|v(\cdot, t) - \tilde{v}\|_{L^2(\Omega)} \longrightarrow 0,$$

$$\|w(\cdot, t) - \tilde{w}\|_{L^2(\Omega)} \longrightarrow 0 \quad as\ t \longrightarrow \infty. \tag{3.1}$$

Proof. Notice from [8] that $\tilde{u}$ is uniformly and locally asymptotically stable in the sense of [18]. We only need to prove the global stability of $\tilde{u}$. Define

$$V_1(u, v, w) = \frac{1}{2}\int_\Omega (u - \tilde{u})^2 dx + \lambda \int_\Omega \left[\left(v - \tilde{v} - \tilde{v}\ln\frac{v}{\tilde{v}}\right)dx + \rho\int_\Omega \left(w - \tilde{w} - \tilde{w}\ln\frac{w}{\tilde{w}}\right)\right]dx, \tag{3.2}$$

where $\lambda = \tilde{u}$, $\rho = c_1\tilde{u}/c_2$. Obviously, $V_1(u, v, w)$ is nonnegative and $V_1(u, v, w) = 0$ if and only if $(u, v, w) = (\tilde{u}, \tilde{v}, \tilde{w})$. The time derivative of $V_1(u, v, w)$ for the system (1.4) satisfies

$$\frac{dV_1(u, v, w)}{dt} = \int_\Omega \left((u - \tilde{u})u_t + \lambda\frac{v - \tilde{v}}{v}v_t + \rho\frac{w - \tilde{w}}{w}w_t\right)dx := -I_1(t) - I_2(t), \tag{3.3}$$

where

$$I_1(t) = \int_\Omega \left[d_1|\nabla u|^2 + \lambda\frac{d_2\tilde{v}}{v^2}|\nabla v|^2 + \rho\frac{d_3\tilde{w}}{w^2}|\nabla w|^2\right]dx,$$

$$I_2(t) = \int_\Omega \left[(D_1 + ku + k\tilde{u} + kv + c_1w)(u - \tilde{u})^2 + \lambda k(v - \tilde{v})^2 + \rho c_3(w - \tilde{w})^2\right.$$

$$\left. + (2k\tilde{u} - b_1)(u - \tilde{u})(v - \tilde{v})\right]dx,$$

$$\geq \int_\Omega \left[(D_1 + k\tilde{u})(u - \tilde{u})^2 + \lambda k(v - \tilde{v})^2 + \rho c_3(w - \tilde{w})^2 + (2k\tilde{u} - b_1)(u - \tilde{u})(v - \tilde{v})\right]dx. \tag{3.4}$$

If the matrix

$$
\begin{pmatrix}
D_1 + k\tilde{u} & \frac{1}{2}(2k\tilde{u} - b_1) & 0 \\
\frac{1}{2}(2k\tilde{u} - b_1) & \lambda k & 0 \\
0 & 0 & \rho c_3
\end{pmatrix}
\tag{3.5}
$$

is positive definite, then the quadratic form

$$
(D_1 + k\tilde{u})(u - \tilde{u})^2 + \lambda k(v - \tilde{v})^2 + \rho c_3(w - \tilde{w})^2 + (2k\tilde{u} - b_1)(u - \tilde{u})(v - \tilde{v}) \tag{3.6}
$$

is positive definite. A direct calculation shows that the matrix is positive definite if (H3) holds. Meanwhile, for every δ such that $0 < \delta < \min\{c_3\rho, (4k\tilde{u}(D_1 + b_1) - b_1^2)/(4(D_1 + 2k\tilde{u}))\}$, we have

$$
I_2(t) \geq \delta \int_\Omega \left[(u - \tilde{u})^2 + (v - \tilde{v})^2 + (w - \tilde{w})^2\right] dx. \tag{3.7}
$$

Thus,

$$
\frac{dV_1(u, v, w)}{dt} \leq -\delta \int_\Omega \left[(u - \tilde{u})^2 + (v - \tilde{v})^2 + (w - \tilde{w})^2\right] dx. \tag{3.8}
$$

Similarly to [19, Theorem 2.1], we can prove that the solution (u, v, w) is bounded, and so are the derivatives of $\int_\Omega [(u - \tilde{u})^2 + (v - \tilde{v})^2 + (w - \tilde{w})^2] dx$ by the equations in (1.4). Using Lemma 3.1, we have

$$
\|u(\cdot, t) - \tilde{u}\|_{L^2(\Omega)} \longrightarrow 0, \qquad \|v(\cdot, t) - \tilde{v}\|_{L^2(\Omega)} \longrightarrow 0,
$$
$$
\|w(\cdot, t) - \tilde{w}\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{3.9}
$$

By the fact that $V_1(u, v, w)$ is decreasing for $t \geq 0$, it is obvious that $(\tilde{u}, \tilde{v}, \tilde{w})$ is globally asymptotically stable, and the proof of Theorem 3.2 is completed. □

Remark 3.3. Notice that condition (H3) is equivalent to

$$
\gamma_2\beta > \frac{b_1(kc_3 + c_1c_2)}{4k} - c_1D_3, \qquad \gamma_2 = \frac{3kb_1c_3 + 4kc_3D_1 - b_1c_1c_2}{4k(b_1 + D_1)}. \tag{3.10}
$$

If $\gamma_2 < 0$, it is easy to verify that $-c_1/\gamma_2 > k/c_2$. Hence, there exists an unbounded region

$$
U_2 = \left\{ (D_3, \beta) \in \mathbb{R}_+^2 : \beta > \frac{kD_3}{c_2}, \gamma_2\beta > \frac{b_1(kc_3 + c_1c_2)}{4k} - c_1D_3 \right\}, \tag{3.11}
$$

such that for any $(D_3, \beta) \in U_2$, $\tilde{\mathbf{u}}$ is the unique positive steady state with respect to (1.4).

Remark 3.4. From Remarks 2.3 and 3.3, there exists an unbounded region

$$U_3 = U_1 \cap U_2 = \left\{ (D_3, \beta) \in \mathbb{R}_+^2 : \gamma_1 > 0, \beta > \frac{2kc_1}{\gamma_1} D_3, \gamma_2 \beta > \frac{b_1(kc_3 + c_1 c_2)}{4k} - c_1 D_3 \right\}, \quad (3.12)$$

such that for any $(D_3, \beta) \in U_3$, cross-diffusion can destabilize the uniform equilibrium $\tilde{\mathbf{u}}$ of (1.5) when $\tilde{\mu} > \mu_2$ and d_4 is sufficiently large.

4. A Priori Estimates

In the following, the generic constants C, C_*, and so forth, will depend on the domain Ω and the dimension N. However, as Ω and the dimension N are fixed, we will not mention the dependence explicitly. Also, for convenience, we will write Λ instead of the collective constants $(b_1, D_1, D_3, c_1, c_2, c_3, k, \beta)$. The main purpose of this section is to give a priori positive upper and lower bounds for the positive solutions to (2.1) when $R > 0$. For this, we will cite the following two results.

Lemma 4.1 (Harnack's inequality [20]). *Let $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution to $\Delta w(x) + c(x)w(x) = 0$, where $c \in C(\overline{\Omega})$, satisfying the homogeneous Neumann boundary condition. Then there exists a positive constant C_* which depends only on $\|c\|_\infty$ such that $\max_{\overline{\Omega}} w \leq C_* \min_{\overline{\Omega}} w$.*

Lemma 4.2 (maximum principle [21]). *Let $g \in C(\Omega \times \mathbb{R}^1)$ and $b_j \in C(\overline{\Omega})$, $j = 1, 2, \ldots, N$.*

(i) *If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$\Delta w(x) + \sum_{j=1}^{N} b_j(x) w_{x_j} + g(x, w(x)) \geq 0 \quad in \ \Omega, \qquad \partial_\eta w \leq 0 \quad on \ \partial\Omega, \quad (4.1)$$

and $w(x_0) = \max_{\overline{\Omega}} w(x)$, then $g(x_0, w(x_0)) \geq 0$.

(ii) *If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$\Delta w(x) + \sum_{j=1}^{N} b_j(x) w_{x_j} + g(x, w(x)) \leq 0 \quad in \ \Omega, \qquad \partial_\eta w \geq 0 \quad on \ \partial\Omega, \quad (4.2)$$

and $w(x_0) = \min_{\overline{\Omega}} w(x)$, then $g(x_0, w(x_0)) \leq 0$.

Theorem 4.3 (upper bound). *Let d and d^* be two fixed positive constants. Assume that $d_i \geq d, i = 1, 2, 3$, and $0 \leq d_4 \leq d^*$. Then every possible positive solution (u, v, w) of (2.1) satisfies*

$$\max_{\overline{\Omega}} u \leq \frac{b_1}{k}, \qquad \max_{\overline{\Omega}} v \leq \frac{\beta}{k}, \qquad \max_{\overline{\Omega}} w \leq \left(1 + \frac{d^* b_1}{dk} \right) \frac{c_2(b_1 + \beta)}{c_3 k}. \quad (4.3)$$

Proof. A direct application of the maximum principle to (2.1) gives $v \leq \beta/k$ on $\overline{\Omega}$. Let $u(x_0) = \max_{\overline{\Omega}} u$. Using the maximum principle again, we have $b_1 v(x_0) \geq u(x_0)[D_1 + ku(x_0) + kv(x_0) + c_1 w(x_0)]$. Thus, $ku(x_0)v(x_0) \leq b_1 v(x_0)$ and $u(x_0) \leq b_1/k$.

Define $\varphi = d_3 w + d_4 u w$; then, φ satisfies

$$-\Delta \varphi = w(-D_3 + c_2 u + c_2 v - c_3 w) \quad \text{in } \Omega, \qquad \partial_\eta \varphi = 0 \quad \text{on } \partial\Omega. \tag{4.4}$$

Let $\varphi(x_1) = \max_{\overline{\Omega}} \varphi$. By Lemma 4.2, we have

$$-D_3 + c_2 u(x_1) + c_2 v(x_1) - c_3 w(x_1) \geq 0. \tag{4.5}$$

It follows that

$$w(x_1) \leq \frac{c_2}{c_3}[u(x_1) + v(x_1)] \leq \frac{c_2}{c_3}\left(\frac{b_1}{k} + \frac{\beta}{k}\right) = \frac{c_2(b_1 + \beta)}{c_3 k}. \tag{4.6}$$

Hence,

$$\varphi(x_1) = [d_3 + d_4 u(x_1)] w(x_1) \leq \left(d_3 + d_4 \frac{b_1}{k}\right) \frac{c_2(b_1 + \beta)}{c_3 k},$$
$$\max_{\overline{\Omega}} w = \max_{\overline{\Omega}} \frac{\varphi}{(d_3 + d_4 u)} \leq \left(1 + \frac{d_4 b_1}{d_3 k}\right) \frac{c_2(b_1 + \beta)}{c_3 k} \leq \left(1 + \frac{d^* b_1}{dk}\right) \frac{c_2(b_1 + \beta)}{c_3 k} \tag{4.7}$$

for any $d_3 \geq d$ and $0 \leq d_4 \leq d^*$. □

Turning now to the lower bound, we first need some preliminary results.

Lemma 4.4. *Let d_{ij} be positive constants, $i = 1,2,3,4$, $j = 1,2,\ldots$, and let (u_j, v_j, w_j) be the corresponding positive solution of (2.1) with $d_i = d_{ij}$. If $(u_j, v_j, w_j) \rightarrow (u^*, v^*, w^*)$ uniformly on $\overline{\Omega}$ as $j \rightarrow \infty$ and (u^*, v^*, w^*) is a constant vector, then (u^*, v^*, w^*) must satisfy*

$$b_1 v^* - u^*(D_1 + ku^* + kv^* + c_1 w^*) = 0, \qquad \beta - ku^* - kv^* - c_1 w^* = 0,$$
$$-D_3 + c_2 u^* + c_2 v^* - c_3 w^* = 0. \tag{4.8}$$

Moreover, if u^*, v^*, and w^* are positive constants, then $(u^*, v^*, w^*) = (\tilde{u}, \tilde{v}, \tilde{w})$.

Proof. It is easy to see that for all j,

$$\int_\Omega G_1(u_j, v_j, w_j) dx = 0. \tag{4.9}$$

If $G_1(u^*, v^*, w^*) > 0$, then $G_1(u_j, v_j, w_j) > 0$ when j is large since $(u_j, v_j, w_j) \rightarrow (u^*, v^*, w^*)$. This is impossible. Similarly, $G_1(u^*, v^*, w^*) < 0$ is impossible. Therefore, $G_1(u^*, v^*, w^*) = 0$. The same argument shows that $\beta - ku^* - kv^* - c_1 w^* = 0$ and $-D_3 + c_2 u^* + c_2 v^* - c_3 w^* = 0$. Consequently, $(u^*, v^*, w^*) = (\tilde{u}, \tilde{v}, \tilde{w})$. □

Lemma 4.5. *The system*

$$u_t - d_1 \Delta u = b_1 v - u(D_1 + ku + kv), \quad x \in \Omega, \ t > 0,$$

$$v_t - d_2 \Delta v = v(\beta - ku - kv), \quad x \in \Omega, \ t > 0,$$

$$\partial_\eta u = \partial_\eta v = 0, \quad x \in \partial\Omega, \ t > 0,$$ $$(4.10)$$

$$u(x,0) \geq (\not\equiv)0, \qquad v(x,0) \geq (\not\equiv)0, \quad x \in \Omega,$$

has a unique positive constant steady state (u_1, v_1) which is globally asymptotically stable, where u_1 and v_1 are given by (1.2).

Proof. Let

$$V_2(u,v) = \frac{1}{2} \int_\Omega (u - u_1)^2 dx + \rho \int_\Omega \left(v - v_1 - v_1 \ln \frac{v}{v_1} \right) dx,$$ $$(4.11)$$

where $\rho = (b_1 - ku_1)/k = b_1(b_1 + D_1)/[k(b_1 + D_1 + \beta)] > 0$, and let (u,v) be a positive solution of (4.10). Then a direct computation gives

$$\frac{dV_2}{dt} = -\int_\Omega \left[d_1 |\nabla u|^2 + \frac{\rho d_2 v_1}{v^2} |\nabla v|^2 + (D_1 + ku + ku_1 + kv)(u - u_1)^2 + \rho k(v - v_1)^2 \right] dx \leq 0,$$ $$(4.12)$$

and $dV_2/dt = 0$ holds if and only if $(u,v) = (u_1, v_1)$. By Lemma 3.1, we can conclude that (u_1, v_1) is globally asymptotically stable. □

Theorem 4.6 (lower bound). *Let d and d^* be two fixed positive constants. Assume that $d_i \geq d$, $i = 1, 2, 3$, and $0 \leq d_4 \leq d^*$. Then there exists a positive constant $C = C(\Lambda, d, d^*)$, such that every possible positive solution (u, v, w) of (2.1) satisfies*

$$\min_{\overline{\Omega}} u, \ \min_{\overline{\Omega}} v, \ \min_{\overline{\Omega}} w \geq C.$$ $$(4.13)$$

Proof. If the conclusion does not hold, then there exists a sequence $\{d_{1j}, d_{2j}, d_{3j}, d_{4j}\}_{j=1}^\infty$ with $d_{1j}, d_{2j}, d_{3j} \geq d$ and $0 \leq d_{4j} \leq d^*$ such that the corresponding positive solution (u_j, v_j, w_j) of (2.1) satisfies

$$\min \left\{ \min_{\overline{\Omega}} u_j, \ \min_{\overline{\Omega}} v_j, \min_{\overline{\Omega}} w_j \right\} \longrightarrow 0, \quad \text{as } j \longrightarrow \infty.$$ $$(4.14)$$

Moreover, we assume that $d_{ij} \rightarrow d_i \in [d, \infty]$ for $i = 1, 2, 3$, and $d_{4j} \rightarrow d_4 \in [0, d^*]$. By Theorem 4.3 and the standard regularity theory for the elliptic equations, we may also assume that $(u_j, v_j, w_j) \rightarrow (u, v, w)$ in $[C^2(\overline{\Omega})]^3$ for some nonnegative functions u, v, w. It is easy to see that (u, v, w) also satisfies estimate (4.3), and $\min\{\min_{\overline{\Omega}} u, \ \min_{\overline{\Omega}} v, \ \min_{\overline{\Omega}} w\} = 0$. Moreover, we observe that, if $d_1, d_2, d_3 < \infty$, then (u, v, w) satisfies (2.1).

Next we derive a contradiction for all possible cases.

Firstly, we consider the case $d_1, d_2, d_3 < \infty$.

(1) In view of (2.1), $\min_{\overline{\Omega}} v = 0$ implies $v = 0$ on $\overline{\Omega}$ from the Harnack inequality. In this case, by the strong maximum principle and the Hopf boundary lemma, it follows that $u = w = 0$ on $\overline{\Omega}$. This is a contradiction to Lemma 4.4. Thus, $\min_{\overline{\Omega}} v > 0$.

(2) If $\min_{\overline{\Omega}} u = 0$, we denote $u(x_0) = \min_{\overline{\Omega}} u = 0$. By the maximum principle we have $b_1 v(x_0) \leq u(x_0)[D_1 + ku(x_0) + kv(x_0) + c_1 w(x_0)] = 0$, and so $\min_{\overline{\Omega}} v = 0$. This is a contradiction to $\min_{\overline{\Omega}} v > 0$. Thus $\min_{\overline{\Omega}} u > 0$.

(3) If $\min_{\overline{\Omega}} w = 0$, let $\varphi = d_3 w + d_4 uw$. Then $\min_{\overline{\Omega}} \varphi = 0$ and φ satisfies

$$-\Delta\varphi = \varphi(-D_3 + c_2 u + c_2 v - c_3 w)(d_3 + d_4 u)^{-1} \quad \text{in } \Omega, \qquad \partial_\eta \varphi = 0 \quad \text{on } \partial\Omega. \tag{4.15}$$

The Harnack inequality shows that $\min_{\overline{\Omega}} \varphi = 0$ implies $\varphi = 0$ on $\overline{\Omega}$. Hence, $w = 0$ on $\overline{\Omega}$. From Lemma 4.5, we have $(u, v) = (u_1, v_1)$. Define $\widehat{w}_j = w_j / \|w_j\|_\infty$; then, $(u_j, v_j, \widehat{w}_j, w_j)$ satisfies

$$-d_{1j}\Delta u_j = b_1 v_j - u_j(D_1 + ku_j + kv_j + c_1 w_j) \quad \text{in } \Omega,$$

$$-d_{2j}\Delta v_j = v_j(\beta - ku_j - kv_j - c_1 w_j) \quad \text{in } \Omega,$$

$$-\Delta(d_{3j}\widehat{w}_j + d_{4j} u_j \widehat{w}_j) = \widehat{w}_j(-D_3 + c_2 u_j + c_2 v_j - c_3 w_j) \quad \text{in } \Omega, \tag{4.16}$$

$$\partial_\eta u_j = \partial_\eta v_j = \partial_\eta \widehat{w}_j = 0 \quad \text{on } \partial\Omega.$$

Similarly to the above, we can prove that there exists a subsequence of $\{\widehat{w}_j\}$, denoted by itself, and a nonnegative function $\widehat{w}$, such that $\widehat{w}_j \to \widehat{w}$ in $C^2(\overline{\Omega})$ and $\|\widehat{w}\|_\infty = 1$. Moreover, $\widehat{w}$ satisfies

$$-(d_3 + d_4 u_1)\Delta\widehat{w} = \widehat{w}(-D_3 + c_2 u_1 + c_2 v_1) \quad \text{in } \Omega, \qquad \partial_\eta \widehat{w} = 0 \quad \text{on } \partial\Omega. \tag{4.17}$$

Since $\|\widehat{w}\|_\infty = 1$, by the strong maximum principle and the Hopf boundary lemma, we find that $\widehat{w} > 0$ on $\overline{\Omega}$. Applying the maximum principle again, we have $-D_3 + c_2 u_1 + c_2 v_1 = 0$. Thus, $u_1 + v_1 = D_3/c_2$. Noting that $u_1 + v_1 = \beta/k$ in (1.2), it follows that $c_2\beta - kD_3 = 0$, which is a contradiction to the condition $R = c_2\beta - kD_3 > 0$.

Next, we consider the remaining cases.

Integrating by parts, we obtain that

$$b_1 \int_\Omega v_j dx = \int_\Omega u_j(D_1 + ku_j + kv_j + c_1 w_j)dx,$$

$$\int_\Omega v_j(\beta - ku_j - kv_j - c_1 w_j)dx = 0, \tag{4.18}$$

$$\int_\Omega w_j(-D_3 + c_2 u_j + c_2 v_j - c_3 w_j)dx = 0,$$

for $j = 1, 2, \ldots$. Moreover, (u, v, w) satisfies

$$b_1 \int_\Omega v \, dx = \int_\Omega u(D_1 + ku + kv + c_1 w) \, dx,$$

$$\int_\Omega v(\beta - ku - kv - c_1 w) \, dx = 0, \tag{4.19}$$

$$\int_\Omega w(-D_3 + c_2 u + c_2 v - c_3 w) \, dx = 0.$$

If $d_1 = \infty$, then u satisfies

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad \partial_\eta u = 0 \quad \text{on } \partial\Omega. \tag{4.20}$$

Hence, u is a constant. If $u = 0$, from (4.19), we have in turn that $v = w = 0$. This contradicts Lemma 4.4. So, u is a positive constant and either $\min_{\overline{\Omega}} v = 0$ or $\min_{\overline{\Omega}} w = 0$.

If $d_2 < \infty$. In the case of $\min_{\overline{\Omega}} v = 0$, similarly to the arguments of (1), we have $v = 0$ on $\overline{\Omega}$. This contradicts the first equation of (4.19). Thus $\min_{\overline{\Omega}} v > 0$ and $\min_{\overline{\Omega}} w = 0$. Note that w satisfies

$$-(d_3 + d_4 u)\Delta w = w(-D_3 + c_2 u - c_3 w) \quad \text{in } \Omega, \qquad \partial_\eta w = 0 \quad \text{on } \partial\Omega. \tag{4.21}$$

If $d_3 < \infty$, the Harnack inequality implies that $w = 0$ on $\overline{\Omega}$. If $d_3 = \infty$, then w ia a constant. Since $\min_{\overline{\Omega}} w = 0$, so $w = 0$ on $\overline{\Omega}$. Therefore, by Lemma 4.5, $(u, v, w) = (u_1, v_1, 0)$. Similarly to the arguments of (3), we arrive at $c_2\beta - kD_3 = 0$, which is a contradiction.

Similarly, we can derive contradictions for all the other cases. □

5. Existence of Stationary Patterns for the Model (1.5)

In this section we discuss the existence of nonconstant positive solutions to (2.1). These solutions are obtained for large cross-diffusion coefficient d_4, with the other parameters d_1, d_2, d_3, b_1, D_1, D_3, k, c_1, c_2, c_3, and β suitably fixed. Our main result is as follows.

Theorem 5.1. *Let the parameters d_1, d_2, d_3, b_1, D_1, D_3, k, c_1, c_2, c_3, and β be fixed such that* (H1), (H2), *and* (H3) *hold. Let $\tilde{\mu}$ be given by the limit* (2.13). *If $\tilde{\mu} \in (\mu_n, \mu_{n+1})$ for some $n \geq 2$ and the sum $\sum_{i=2}^n \dim E(\mu_i)$ is odd, then there exists a positive constant d_4^* such that* (2.1) *has at least one nonconstant positive solution for $d_4 > d_4^*$.*

Proof. By Proposition 2.2 and our assumption on $\tilde{\mu}$, there exists a positive constant d_4^* such that (2.14) holds if $d_4 > d_4^*$, and

$$\tilde{\mu}_1(d_4) < 0 = \mu_1 < \tilde{\mu}_2(d_4) < \mu_2, \qquad \tilde{\mu}_3(d_4) \in (\mu_n, \mu_{n+1}). \tag{5.1}$$

We will prove that for any $d_4 > d_4^*$, (2.1) has at least one nonconstant positive solution. The proof, which is by contradiction, is based on the homotopy invariance of the topological degree.

Suppose on the contrary that the assertion is not true for some $d_4 = \tilde{d}_4 > d_4^*$. In the following we fix $d_4 = \tilde{d}_4$.

For $\theta \in [0, 1]$, define $\boldsymbol{\Phi}(\theta; \mathbf{u}) = (d_1 u, d_2 v, d_3 w + \theta d_4 u w)^T$ and consider the problem

$$-\Delta \boldsymbol{\Phi}(\theta; \mathbf{u}) = \mathbf{G}(\mathbf{u}) \quad \text{in } \Omega, \qquad \partial_\eta \mathbf{u} = 0, \quad \text{on } \partial\Omega. \tag{5.2}$$

Then $\mathbf{u}$ is a positive nonconstant solution of (2.1) if and only if it is such a solution of (5.2) for $\theta = 1$. It is obvious that $\tilde{\mathbf{u}}$ is the unique constant positive solution of (5.2) for any $0 \le \theta \le 1$. As we observed in Section 2, for any $0 \le \theta \le 1$, $\mathbf{u}$ is a positive solution of (5.2) if and only if

$$\mathbf{F}(\boldsymbol{\theta}; \mathbf{u}) := \mathbf{u} - (\mathbf{I} - \Delta)^{-1} \left\{ \boldsymbol{\Phi_u}^{-1}(\theta; \mathbf{u})[\mathbf{G}(\mathbf{u}) + \nabla \mathbf{u} \boldsymbol{\Phi_{uu}}(\theta; \mathbf{u}) \nabla \mathbf{u}] + \mathbf{u} \right\} = 0 \quad \text{in } \mathbf{Y}^+. \tag{5.3}$$

It is obvious that $\mathbf{F}(1; \mathbf{u}) = \mathbf{F}(\mathbf{u})$. Theorem 3.2 shows that $\mathbf{F}(0; \mathbf{u}) = 0$ has only the positive solution $\tilde{\mathbf{u}}$ in $\mathbf{Y}^+$. By a direct computation,

$$D_{\mathbf{u}}\mathbf{F}(\theta; \tilde{\mathbf{u}}) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1} \left\{ \boldsymbol{\Phi_u}^{-1}(\theta; \tilde{\mathbf{u}}) \mathbf{G_u}(\tilde{\mathbf{u}}) + \mathbf{I} \right\}. \tag{5.4}$$

In particular,

$$D_{\mathbf{u}}\mathbf{F}(0; \tilde{\mathbf{u}}) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1} \left\{ \mathbf{D}^{-1} \mathbf{G_u}(\tilde{\mathbf{u}}) + \mathbf{I} \right\}, \tag{5.5}$$

where $\mathbf{D} = \operatorname{diag}(d_1, d_2, d_3)$ and

$$D_{\mathbf{u}}F(1; \tilde{\mathbf{u}}) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1} \left\{ \boldsymbol{\Phi_u}^{-1} \mathbf{G_u}(\tilde{\mathbf{u}}) + \mathbf{I} \right\} = D_{\mathbf{u}}\mathbf{F}(\tilde{\mathbf{u}}). \tag{5.6}$$

From (2.5) and (2.9) we see that

$$H(\mu) = \det\left\{ \boldsymbol{\Phi_u}^{-1}(\tilde{\mathbf{u}}) \right\} C(d_4; \mu). \tag{5.7}$$

In view of (2.14) and (5.1), it follows that

$$\begin{aligned} H(\mu_1) &= H(0) > 0, \\ H(\mu_i) &< 0, \quad 2 \le i \le n, \\ H(\mu_i) &> 0, \quad i \ge n + 1. \end{aligned} \tag{5.8}$$

Therefore, zero is not an eigenvalue of the matrix $\mu_i \mathbf{I} - \boldsymbol{\Phi_u}^{-1}(\tilde{\mathbf{u}}) \mathbf{G_u}(\tilde{\mathbf{u}})$ for all $i \ge 1$, and

$$\sum_{i \ge 1, H(\mu_i) < 0} \dim E(\mu_i) = \sum_{i=2}^{n} \dim E(\mu_i) = \sigma_n, \quad \text{which is odd.} \tag{5.9}$$

Thanks to Proposition 2.1, we have

$$\mathrm{index}(\mathbf{F}(1;\cdot),\tilde{\mathbf{u}}) = (-1)^{\gamma} = (-1)^{\sigma_n} = -1. \tag{5.10}$$

Similarly, we can easily show that

$$\mathrm{index}(F(0;\cdot),\tilde{\mathbf{u}}) = (-1)^0 = 1. \tag{5.11}$$

Now, by Theorems 4.3 and 4.6, there exists a positive constant C such that, for all $0 \leq \theta \leq 1$, the positive solutions of (2.1) satisfy $1/C < u, v, w < C$. Therefore, $\mathbf{F}(\theta;\mathbf{u}) \neq 0$ on $\partial B(C)$ for all $0 \leq \theta \leq 1$. By the homotopy invariance of the topological degree,

$$\deg(\mathbf{F}(1;\cdot),0,B(C)) = \deg(\mathbf{F}(0;\cdot),0,B(C)). \tag{5.12}$$

On the other hand, by our supposition, both equations $\mathbf{F}(1;\mathbf{u}) = 0$ and $\mathbf{F}(0;\mathbf{u}) = 0$ have only the positive solution $\tilde{\mathbf{u}}$ in $B(C)$. Hence, by (5.10) and (5.11), we have

$$\deg(\mathbf{F}(1;\cdot),0,B(C)) = \mathrm{index}(\mathbf{F}(1;\cdot),\tilde{\mathbf{u}}) = -1,$$

$$\deg(\mathbf{F}(0;\cdot),0,B(C)) = \mathrm{index}(\mathbf{F}(0;\cdot),\tilde{\mathbf{u}}) = 1. \tag{5.13}$$

This contradicts (5.12), and thus we complete the proof of Theorem 5.1. $\square$

Remark 5.2. Assume that all the conditions hold in Theorem 5.1. Theorem 3.2 shows that $\tilde{\mathbf{u}}$ is a globally asymptotically stable equilibrium for the system (1.4). However, Theorem 5.1 implies that the cross-diffusion system (1.5) has at least one nonconstant positive steady state. Our results demonstrate that stationary patterns can be found due to the emergence of cross-diffusion.

6. Nonexistence of Nonconstant Positive Solution of (2.1)

In this section, we discuss the nonexistence of nonconstant positive solution of (2.1) when the cross-diffusion coefficient $d_4 > 0$ is small.

Theorem 6.1. *If the parameters d_1, d_3, d_4, b_1, D_1, D_3, k, c_1, c_2, c_3, and β satisfy* (H1), (H3), *and*

$$\frac{c_1 d_4^2 \tilde{u}\tilde{w}}{c_2} < 4d_1 d_3, \tag{H4}$$

then the problem (2.1) *has no nonconstant positive solution.*

Proof. Assume that (u, v, w) is a positive solution of (2.1). Let $\lambda = \tilde{u}$, $\rho = c_1\tilde{u}/c_2$. Multiplying the equations of (2.1) by $(u - \tilde{u})$, $\lambda(v - \tilde{v})/v$, and $\rho(w - \tilde{w})/w$, respectively, and integrating by parts, as in the proof of Theorem 3.2, we obtain $0 = -I_3 - I_4$, where

$$
\begin{aligned}
I_3 &= \int_\Omega \left[d_1|\nabla u|^2 + \lambda \frac{d_2\tilde{v}}{v^2}|\nabla v|^2 + \rho\frac{\tilde{w}(d_3 + d_4 u)}{w^2}|\nabla w|^2 + \rho\frac{d_4\tilde{w}}{w}\nabla u \cdot \nabla w \right] dx, \\
I_4 &= \int_\Omega \left[(D_1 + ku + k\tilde{u} + kv + c_1 w)(u - \tilde{u})^2 + \lambda k(v - \tilde{v})^2 \right. \\
&\qquad\qquad \left. + \rho c_3(w - \tilde{w})^2 + (2k\tilde{u} - b_1)(u - \tilde{u})(v - \tilde{v}) \right] dx.
\end{aligned}
\tag{6.1}
$$

Applying (H3) and (H4), it is easy to prove that $I_3 \geq 0$ and $I_4 \geq 0$. This implies that $(u, v, w) = (\tilde{u}, \tilde{v}, \tilde{w})$ on $\overline{\Omega}$ and the proof is complete. $\qquad\square$

Remark 6.2. Theorem 6.1 shows that the problem (2.1) has no nonconstant positive solutions if one of d_1 and d_3 is sufficiently large; that is, unlimitedly increasing one of the diffusion rates d_1 and d_3 will eventually wipe out all nonconstant solutions of (2.1). However, Theorem 6.1 does not tell us the effect of the diffusion rate d_2 on the stationary problem (2.1). Using the similar arguments in Section 2, we can find that d_2 does not cause instability of $\tilde{\mathbf{u}}$. Therefore, we conjecture that the problem (2.1) has no nonconstant positive solutions if d_2 is sufficiently large.

Remark 6.3. Theorems 5.1 and 6.1 seem to indicate that diffusion tends to suppress pattern formation, while cross-diffusion seems to help create patterns.

7. Discussion

In this paper, we have introduced a more realistic mathematical model for a diffusive predator-prey system where the prey has a sex structure comprising male and female members. In this model, we model the tendency of the predators to keep away from the male prey by a cross-diffusion. As a result, our model is a strongly coupled cross-diffusion system, which is mathematically more complex than systems used to model sex-structured predator-prey behavior hitherto [1, 8]. What is noteworthy about this model is that, as the cross-diffusion term arises, it is precisely this cross-diffusion that destabilizes the uniform positive equilibrium and gives rise to stationary patterns for the model. Indeed, stationary patterns do not arise for the ODE (spatially independent) model, nor the PDE model without cross-diffusion.

In fact, one can see that this particular cross-diffusion term is also significant from the mathematical point of view. The following system represents the general form of SKT-type cross-diffusion [22] in the predator-prey model

$$
\begin{aligned}
u_t - \Delta(d_1 u + d_{12} uv + d_{13} uw) &= G_1(u, v, w), \quad x \in \Omega, \ t > 0, \\
v_t - \Delta(d_2 v + d_{21} uv + d_{23} vw) &= G_2(u, v, w), \quad x \in \Omega, \ t > 0, \\
w_t - \Delta(d_3 w + d_{31} uw + d_{32} vw) &= G_3(u, v, w), \quad x \in \Omega, \ t > 0, \\
\partial_\eta u(x, t) = \partial_\eta v(x, t) &= \partial_\eta w(x, t) = 0, \quad x \in \partial\Omega, \ t > 0, \\
u(x, 0) = u_0(x), \qquad v(x, 0) &= v_0(x), \qquad w(x, 0) = w_0(x), \quad x \in \Omega,
\end{aligned}
\tag{7.1}
$$

where d_{ij}, $i, j = 1$, 2, 3, $i \neq j$, are cross-diffusion coefficients. Using the method in Section 2 of this paper, we have investigated the stability of $\tilde{\mathbf{u}}$ for each d_{ij}, respectively. We found that either d_{13} or d_{23} does not cause instability of $\tilde{\mathbf{u}}$, while each of the other d_{ij} can induce the instability of $\tilde{\mathbf{u}}$. For $d_{32} \neq 0$, we can obtain the similar conclusions as Theorem 5.1 by the same mathematical treatment in this paper. Unfortunately, the existence of nonconstant positive steady states has not been obtained when $d_{12} \neq 0$ or $d_{21} \neq 0$, because we cannot establish a priori lower bounds for all possible positive steady states of (7.1).

On the other hand, as pointed out in [23], a Lotka-Volterra-type model can be regarded as a local approximation to a nonlinear system. In the present paper, we only consider the case that the interaction terms on the right-hand side of (1.5) are linear. For the nonlinear case, it will be extremely difficult to analyze positive steady states.

In this paper, we do not discuss the stability and the number of the nonconstant positive solutions. We will consider them in the coming papers.

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