

## Research Article

# He-Laplace Method for Linear and Nonlinear Partial Differential Equations

Hradyesh Kumar Mishra<sup>1</sup> and Atulya K. Nagar<sup>2</sup>

<sup>1</sup> Department of Mathematics, Jaypee University of Engineering & Technology, Guna 473226, India

<sup>2</sup> Department of Mathematics and Computer Science, Liverpool Hope University, Liverpool L16 9JD, UK

Correspondence should be addressed to Atulya K. Nagar, nagara@hope.ac.uk

Received 9 May 2012; Accepted 11 June 2012

Academic Editor: Alfredo Bellen

Copyright © 2012 H. Kumar Mishra and A. K. Nagar. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A new treatment for homotopy perturbation method is introduced. The new treatment is called He-Laplace method which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. The nonlinear terms can be easily handled by the use of He's polynomials. The method is implemented on linear and nonlinear partial differential equations. It is found that the proposed scheme provides the solution without any discretization or restrictive assumptions and avoids the round-off errors.

## 1. Introduction

Many important phenomena occurring in various field of engineering and science are frequently modeled through linear and nonlinear differential equations. However, it is still very difficult to obtain closed-form solutions for most models of real-life problems. A broad class of analytical methods and numerical methods were used to handle such problems. In recent years, various methods have been proposed such as finite difference method [1, 2] adomian decomposition method [3–8], variational iteration method [9–12], integral transform [13], weighted finite difference techniques [14], Laplace decomposition method [15–17], but all these methods have some limitations.

The homotopy perturbation method was first introduced by Chinese mathematician He [18–25]. The essential idea of this method is to introduce a homotopy parameter  $p$ , say which takes the values from 0 to 1. When  $p = 0$ , the system of equations usually reduces to a simplified form which normally admits a rather simple solution. As  $p$  gradually increases to 1, the system goes through a sequence of deformation and the solution of each of which is close to that at the previous stage of deformation. Eventually at  $p = 1$ , the system takes the

original form of equation, and final stage of deformation gives the desired result. One of the most remarkable features of the HPM is that only a few perturbation terms are sufficient to obtain a reasonably accurate solution.

The HPM has been employed to solve a large variety of linear and nonlinear problems. This technique was used by He [23, 25] to find the solution of nonlinear boundary value problems, the Blasius differential equation. Sharma and Methi [26] apply HPM for solution of equation to unsteady flow of a polytropic gas. Ganji and Rafei [27] implemented HPM for solution of nonlinear Hirota-Satsuma coupled KdV partial differential equations. Biazar and Ghazvini [13] presented solution of systems of Volterra integral equations. Abbasbandy [28] employed He's homotopy perturbation technique to solve functional integral equations, and obtained results were compared with the Lagrange interpolation formula. Ganji and Sadighi [29] considered the nonlinear coupled system of reaction-diffusion equations using HPM. They reported that the HPM is a powerful and efficient scheme to find analytical solutions for a wide class of nonlinear engineering problems and presents a rapid convergence for the solutions. The solution obtained by HPM shows that the results are in excellent agreement with those obtained by Adomian decomposition method. A comparison between the HPM and the Adomian decomposition shows that the former is more effective than the latter as the HPM can overcome the difficulties arising in calculating Adomian polynomials.

In the present paper, we use the homotopy perturbation method coupled with the Laplace transformation for solving the linear and nonlinear PDEs. It is worth mentioning that the proposed method is an elegant combination of the Laplace transformation, the homotopy perturbation method, and He's polynomials. The use of He's polynomials in the nonlinear term was first introduced by Ghorbani and Saberi-Nadjafi [30], Ghorbani [31]. The proposed algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for linear and nonlinear partial differential equations.

## 2. Basic Idea of Homotopy Perturbation Method

Consider the following nonlinear differential equation

$$A(\mathbf{y}) - f(r) = 0, \quad r \in \Omega, \quad (2.1)$$

with the boundary conditions of

$$B\left(\mathbf{y}, \frac{\partial \mathbf{y}}{\partial n}\right) = 0, \quad r \in \Gamma, \quad (2.2)$$

where  $A$ ,  $B$ ,  $f(r)$ , and  $\Gamma$  are a general differential operator, a boundary operator, a known analytic function and the boundary of the domain  $\Omega$ , respectively.

The operator  $A$  can generally be divided into a linear part  $L$  and a nonlinear part  $N$ . Equation (2.1) may therefore be written as.

$$L(\mathbf{y}) + N(\mathbf{y}) - f(r) = 0, \quad (2.3)$$

By the homotopy technique, we construct a homotopy  $v(r, p) : \Omega \times [0, 1] \rightarrow R$  which satisfies

$$H(v, p) = (1 - p)[L(v) - L(y_0)] + p[A(v) - f(r)] = 0 \quad (2.4)$$

or

$$H(v, p) = L(v) - L(y_0) + pL(y_0) + p[N(v) - f(r)] = 0, \quad (2.5)$$

where  $p \in [0, 1]$  is an embedding parameter, while  $y_0$  is an initial approximation of (2.1), which satisfies the boundary conditions. Obviously, from (2.4) and (2.5) we will have

$$H(v, 0) = L(v) - L(y_0) = 0, \quad (2.6)$$

$$H(v, 1) = A(v) - f(r) = 0.$$

The changing process of  $p$  from zero to unity is just that of  $v(r, p)$  from  $y_0$  to  $y(r)$ . In topology, this is called deformation, while  $L(v) - L(y_0)$  and  $A(v) - f(r)$  are called homotopy. If the embedding parameter  $p$  is considered as a small parameter, applying the classical perturbation technique, we can assume that the solution of (2.4) and (2.5) can be written as a power series in  $p$ :

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots \infty. \quad (2.7)$$

Setting  $p = 1$  in (2.7), we have

$$y = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots. \quad (2.8)$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages. The series (2.8) is convergent for most cases. However, the convergent rate depends on the nonlinear operator  $A(v)$ . Moreover, He [21] made the following suggestions.

- (1) The second derivative of  $N(v)$  with respect to  $v$  must be small because the parameter may be relatively large; that is,  $p \rightarrow 1$ .
- (2) The norm of  $L^{-1}(\partial N/\partial v)$  must be smaller than one so that the series converges.

### 3. He-Laplace Method

To illustrate the basic idea of this method, we consider a general nonlinear nonhomogeneous partial differential equation with initial conditions of the form

$$\frac{\partial^2 y}{\partial t^2} + R_1 y(x, t) + R_2 y(x, t) + N y(x, t) = f(x, t), \quad (3.1)$$

$$y(x, 0) = \alpha(x), \quad \frac{\partial y}{\partial t}(x, 0) = \beta(x),$$

where  $R_1 = \partial^2/\partial x^2$  and  $R_2 = \partial/\partial x$  are the linear differential operators,  $N$  represents the general nonlinear differential operator and  $f(x, t)$  is the source term. Taking the Laplace transform (denoted by  $L$ ) on both sides of (3.1)

$$\begin{aligned} L\left[\frac{\partial^2 y}{\partial t^2}\right] + L[R_1 y(x, t) + R_2 y(x, t)] + L[Ny(x, t)] &= L[f(x, t)] \\ \implies s^2 L[y(x, t)] - sy(x, 0) - \frac{\partial y}{\partial t}(x, 0) & \\ = -L[R_1 y(x, t) + R_2 y(x, t)] - L[Ny(x, t)] + L[f(x, t)]. & \end{aligned} \quad (3.2)$$

Applying the initial conditions given in (3.1), we have

$$L[y(x, t)] = \frac{\alpha(x)}{s} + \frac{\beta(x)}{s^2} - \frac{1}{s^2} (L[R_1 y(x, t) + R_2 y(x, t)] + L[Ny(x, t)]) + \frac{1}{s^2} (L[f(x, t)]). \quad (3.3)$$

Operating the inverse Laplace transform on both sides of (3.3), we have

$$y(x, t) = F(x, t) - L^{-1} \left[ \frac{1}{s^2} (L[R_1 y(x, t) + R_2 y(x, t)] + L[Ny(x, t)]) \right], \quad (3.4)$$

where  $F(x, t)$  represents the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method:

$$y(x, t) = \sum_{n=0}^{\infty} p^n y_n(x, t), \quad (3.5)$$

and the nonlinear term can be decomposed as

$$Ny(x, t) = \sum_{n=0}^{\infty} p^n H_n(y). \quad (3.6)$$

For some He's polynomials  $H_n$  (see [31, 32]) with the coupling of the Laplace transform and the homotopy perturbation method are given by

$$\sum_{n=0}^{\infty} p^n y_n(x, t) = F(x, t) - p \left( L^{-1} \left[ \frac{1}{s^2} L \left[ (R_1 + R_2) \sum_{n=0}^{\infty} p^n y_n(x, t) + \sum_{n=0}^{\infty} p^n H_n(y) \right] \right] \right). \quad (3.7)$$

Comparing the coefficients of like powers of  $p$ , we have the following approximations:

$$\begin{aligned}
 p^0: y_0(x, t) &= F(x, t), \\
 p^1: y_1(x, t) &= -L^{-1}\left(\frac{1}{s^2}L[(R_1 + R_2)y_0(x, t) + H_0(y)]\right), \\
 p^2: y_2(x, t) &= -L^{-1}\left(\frac{1}{s^2}L[(R_1 + R_2)y_1(x, t) + H_1(y)]\right), \\
 p^3: y_3(x, t) &= -L^{-1}\left(\frac{1}{s^2}L[(R_1 + R_2)y_2(x, t) + H_2(y)]\right), \\
 &\vdots
 \end{aligned} \tag{3.8}$$

#### 4. Application

To demonstrate the applicability of the above-presented method, we have applied it to two linear and three nonlinear partial differential equations. These examples have been chosen because they have been widely discussed in literature.

*Example 4.1.* Consider the following homogeneous linear PDE [33]:

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2} = 0 \tag{4.1}$$

with the following conditions:

$$y(x, 0) = e^x - x, \quad y(0, t) = 1 + t, \quad \frac{\partial y}{\partial x}(1, t) = e - 1. \tag{4.2}$$

By applying the aforesaid method subject to the initial condition, we have

$$y(x, s) = \frac{e^x - x}{s} - \frac{1}{s}L\left[\frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2}\right] \tag{4.3}$$

The inverse of the Laplace transform implies that

$$y(x, t) = e^x - x - L^{-1}\left[\frac{1}{s}L\left[\frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2}\right]\right] \tag{4.4}$$

Now, we apply the homotopy perturbation method; we have

$$\sum_{n=0}^{\infty} p^n y_n(x, t) = e^x - x - p\left(L^{-1}\left[\frac{1}{s}L\left[\frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2}\right]\right]\right), \tag{4.5}$$

Comparing the coefficient of like powers of  $p$ , we have

$$\begin{aligned} p^0 : y_0(x, t) &= e^x - x, \\ p^1 : y_1(x, t) &= -L^{-1} \left[ \frac{1}{s} L \left[ \frac{\partial y_0}{\partial x} - \frac{\partial^2 y_0}{\partial x^2} \right] \right] = t, \\ p^2 : y_2(x, t) &= -L^{-1} \left[ \frac{1}{s} L \left[ \frac{\partial y_1}{\partial x} - \frac{\partial^2 y_1}{\partial x^2} \right] \right] = 0. \end{aligned} \quad (4.6)$$

Proceeding in a similar manner, we have

$$\begin{aligned} p^3 : y_3(x, t) &= 0, \\ p^4 : y_4(x, t) &= 0, \\ &\vdots \end{aligned} \quad (4.7)$$

so the solution  $y(x, t)$  is given by

$$\begin{aligned} y(x, t) &= e^x - x + t + 0 + 0 \cdots \\ &= e^x - x + t, \end{aligned} \quad (4.8)$$

which is the exact solution of the problem.

*Example 4.2.* Consider the following homogeneous linear PDE (Klein-Gordon equation) [33]:

$$\frac{\partial^2 y}{\partial t^2} + y - \frac{\partial^2 y}{\partial x^2} = 0, \quad (4.9)$$

with the following conditions:

$$y(x, 0) = e^{-x} + x, \quad \frac{\partial y}{\partial t}(x, 0) = 0. \quad (4.10)$$

By applying the aforesaid method subject to the initial condition, we have

$$y(x, s) = \frac{e^{-x} + x}{s} - \frac{1}{s^2} L \left[ y - \frac{\partial^2 y}{\partial x^2} \right] \quad (4.11)$$

The inverse of the Laplace transform implies that

$$y(x, t) = e^{-x} + x - L^{-1} \left[ \frac{1}{s^2} L \left[ y - \frac{\partial^2 y}{\partial x^2} \right] \right]. \quad (4.12)$$

Now, we apply the homotopy perturbation method; we have

$$\sum_{n=0}^{\infty} p^n y_n(x, t) = e^{-x} + x - p \left( L^{-1} \left[ \frac{1}{s^2} L \left[ y - \frac{\partial^2 y}{\partial x^2} \right] \right] \right). \quad (4.13)$$

Comparing the coefficient of like powers of  $p$ , we have

$$\begin{aligned} p^0 : y_0(x, t) &= e^{-x} + x, \\ p^1 : y_1(x, t) &= -L^{-1} \left[ \frac{1}{s^2} L \left[ y_0 - \frac{\partial^2 y_0}{\partial x^2} \right] \right] = \frac{-xt^2}{2!}, \\ p^2 : y_2(x, t) &= -L^{-1} \left[ \frac{1}{s^2} L \left[ y_1 - \frac{\partial^2 y_1}{\partial x^2} \right] \right] = \frac{xt^4}{4!}. \end{aligned} \quad (4.14)$$

Proceeding in a similar manner, we have

$$\begin{aligned} p^3 : y_3(x, t) &= \frac{-xt^6}{6!}, \\ p^4 : y_4(x, t) &= \frac{xt^8}{8!}, \\ p^n : y_n(x, t) &= \frac{(-1)^n xt^{2n}}{2n!}, \end{aligned} \quad (4.15)$$

so that the solution  $y(x, t)$  is given by

$$\begin{aligned} y(x, t) &= y_0 + y_1 + y_2 + y_3 + \dots \\ &= e^{-x} + x - \frac{xt^2}{2!} + \frac{xt^4}{4!} - \frac{xt^6}{6!} + \dots + \frac{(-1)^n xt^{2n}}{2n!} \\ &= e^{-x} + x \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots + \frac{(-1)^n t^{2n}}{2n!} + \dots \right) \\ &= e^{-x} + x \text{Cos}(t), \end{aligned} \quad (4.16)$$

which is the exact solution of the problem.

*Example 4.3.* Consider the following homogeneous nonlinear PDE (Burger equation) [33]:

$$\frac{\partial y}{\partial t} - y \frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2} = 0, \quad (4.17)$$

with the following conditions:

$$y(x, 0) = 1 - x, \quad y(0, t) = \frac{1}{(1+t)}, \quad y(1, t) = 0. \quad (4.18)$$

By applying the aforesaid method subject to the initial condition, we have

$$y(x, s) = \frac{1-x}{s} + \frac{1}{s} L \left[ \frac{\partial^2 y}{\partial x^2} + y \frac{\partial y}{\partial x} \right]. \quad (4.19)$$

The inverse of the Laplace transform implies that

$$y(x, t) = 1 - x + L^{-1} \left[ \frac{1}{s} L \left[ \frac{\partial^2 y}{\partial x^2} + y \frac{\partial y}{\partial x} \right] \right]. \quad (4.20)$$

Now, we apply the homotopy perturbation method; we have

$$\sum_{n=0}^{\infty} p^n y_n(x, t) = 1 - x + p \left( L^{-1} \left[ \frac{1}{s} \left\{ L \left[ \frac{\partial^2 y}{\partial x^2} \right] + L \left[ \sum_{n=0}^{\infty} p^n H_n(y) \right] \right\} \right] \right), \quad (4.21)$$

where  $H_n(y)$  are He's polynomials. The first few components of He's polynomials are given by

$$\begin{aligned} H_0(y) &= y_0 \frac{\partial y_0}{\partial x} = -(1-x), \\ H_1(y) &= y_0 \frac{\partial y_1}{\partial x} + y_1 \frac{\partial y_0}{\partial x} = 2(1-x)t, \\ H_2(y) &= y_0 \frac{\partial y_2}{\partial x} + y_1 \frac{\partial y_1}{\partial x} + y_2 \frac{\partial y_0}{\partial x} = -3(1-x)t^2, \\ &\vdots \end{aligned} \quad (4.22)$$

Comparing the coefficient of like powers of  $p$ , we have

$$\begin{aligned} p^0: \quad y_0(x, t) &= 1 - x, \\ p^1: \quad y_1(x, t) &= L^{-1} \left[ \frac{1}{s} \left\{ L \left[ \frac{\partial^2 y_0}{\partial x^2} \right] + L[H_0(y)] \right\} \right] = -(1-x)t, \\ p^2: \quad y_2(x, t) &= L^{-1} \left[ \frac{1}{s} \left\{ L \left[ \frac{\partial^2 y_1}{\partial x^2} \right] + L[H_1(y)] \right\} \right] = (1-x)t^2, \end{aligned} \quad (4.23)$$



Proceeding in a similar manner, we have

$$\begin{aligned} p^3 : y_3(x, t) &= -(1-x)t^3, \\ p^4 : y_4(x, t) &= (1-x)t^4, \\ &\vdots \end{aligned} \tag{4.24}$$

so that the solution  $y(x, t)$  is given by

$$\begin{aligned} y(x, t) &= y_0 + y_1 + y_2 + y_3 + \dots \\ &= (1-x) - (1-x)t + (1-x)t^2 - (1-x)t^3 + \dots \\ &= (1-x) \left[ 1 - t + t^2 - t^3 + t^4 - \dots \right] \\ &= (1-x)(1+t)^{-1} = \frac{(1-x)}{(1+t)}, \end{aligned} \tag{4.25}$$

which is the exact solution of the problem.

*Example 4.4.* Consider the following homogeneous nonlinear PDE [33]:

$$\frac{\partial y}{\partial t} - y - y \frac{\partial^2 y}{\partial x^2} - \left( \frac{\partial y}{\partial x} \right)^2 = 0, \tag{4.26}$$

with the following conditions:

$$y(x, 0) = \sqrt{x}, \quad y(0, t) = 0, \quad y(1, t) = e^t. \tag{4.27}$$

By applying the aforesaid method subject to the initial condition, we have

$$y(x, s) = \frac{\sqrt{x}}{s} + \frac{1}{s} L \left[ y + y \frac{\partial^2 y}{\partial x^2} + \left( \frac{\partial y}{\partial x} \right)^2 \right]. \tag{4.28}$$

The inverse of the Laplace transform implies that

$$y(x, t) = \sqrt{x} + L^{-1} \left[ \frac{1}{s} L \left[ y + y \frac{\partial^2 y}{\partial x^2} + \left( \frac{\partial y}{\partial x} \right)^2 \right] \right]. \tag{4.29}$$

Now, we apply the homotopy perturbation method; we have

$$\sum_{n=0}^{\infty} p^n y_n(x, t) = \sqrt{x} + p \left( L^{-1} \left[ \frac{1}{s} \left\{ L[y] + L \left[ \sum_{n=0}^{\infty} p^n H_n(y) \right] \right\} \right] \right), \tag{4.30}$$

where  $H_n(y)$  are He's polynomials. The first few components of He's polynomials are given by

$$\begin{aligned} H_0(y) &= y_0 \frac{\partial^2 y_0}{\partial x^2} + \left( \frac{\partial y_0}{\partial x} \right)^2 = 0, \\ H_1(y) &= y_0 \frac{\partial^2 y_1}{\partial x^2} + y_1 \frac{\partial^2 y_0}{\partial x^2} + 2 \frac{\partial y_0}{\partial x} \frac{\partial y_1}{\partial x} = 0, \\ H_2(y) &= y_0 \frac{\partial^2 y_2}{\partial x^2} + y_1 \frac{\partial^2 y_1}{\partial x^2} + y_2 \frac{\partial^2 y_0}{\partial x^2} + \left( \frac{\partial y_1}{\partial x} \right)^2 + 2 \frac{\partial y_0}{\partial x} \frac{\partial y_2}{\partial x} = 0, \\ &\vdots \end{aligned} \tag{4.31}$$

Comparing the coefficient of like powers of  $p$ , we have

$$\begin{aligned} p^0 : y_0(x, t) &= \sqrt{x}, \\ p^1 : y_1(x, t) &= L^{-1} \left[ \frac{1}{s} \{ L[y_0] + L[H_0(y)] \} \right] = \sqrt{x} t, \\ p^2 : y_2(x, t) &= L^{-1} \left[ \frac{1}{s} \{ L[y_1] + L[H_1(y)] \} \right] = \frac{\sqrt{x} t^2}{2!}. \end{aligned} \tag{4.32}$$

Proceeding in a similar manner, we have

$$\begin{aligned} p^3 : y_3(x, t) &= \frac{\sqrt{x} t^3}{3!}, \\ p^4 : y_4(x, t) &= \frac{\sqrt{x} t^4}{4!}, \\ &\vdots \end{aligned} \tag{4.33}$$

so the solution  $y(x, t)$  is given by

$$\begin{aligned} y(x, t) &= y_0 + y_1 + y_2 + y_3 + \dots \\ &= \sqrt{x} + \frac{\sqrt{x} t}{1!} + \frac{\sqrt{x} t^2}{2!} + \frac{\sqrt{x} t^3}{3!} + \dots \\ &= \sqrt{x} \left( 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots + \frac{t^n}{n!} + \dots \right) \\ &= \sqrt{x} e^t, \end{aligned} \tag{4.34}$$

which is the exact solution of the problem.

## 5. Comparison of Rate of Convergence of HPM and He-Laplace Method

*Example 5.1.* Consider the following nonhomogeneous nonlinear PDE:

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2} + \left( \frac{\partial y}{\partial x} \right)^2 = 2x + t^4, \quad (5.1)$$

with the following conditions:

$$y(x, 0) = 0, \quad \frac{\partial y}{\partial t}(x, 0) = a, \quad y(0, t) = at, \quad \frac{\partial y}{\partial x}(0, t) = t^2. \quad (5.2)$$

According to the homotopy perturbation method, we have

$$\begin{aligned} H(v, p) = & \frac{\partial^2}{\partial t^2} (v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots) - (1-p) \frac{\partial^2 y_0}{\partial t^2} \\ & + p \left[ \frac{\partial^2}{\partial x^2} (v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots) + \left( \frac{\partial}{\partial x} (v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots) \right)^2 - 2x - t^4 \right] = 0. \end{aligned} \quad (5.3)$$

The initial approximation is chosen  $y_0 = at$ . By equating the coefficients of  $p$  to zero, we obtain

$$\text{Coefficient of } p^0 : \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial^2 y_0}{\partial t^2} = 0, \implies v_0 = y_0 = at,$$

$$\text{Coefficient of } p^1 : \frac{\partial^2 v_1}{\partial t^2} + \frac{\partial^2 v_0}{\partial t^2} + \frac{\partial^2 v_0}{\partial x^2} + \left( \frac{\partial v_0}{\partial x} \right)^2 - 2x - t^4 = 0, \implies v_1 = xt^2 + \frac{1}{30}t^6, \quad (5.4)$$

$$\text{Coefficient of } p^2 : \frac{\partial^2 v_2}{\partial t^2} + \frac{\partial^2 v_1}{\partial x^2} + \left( \frac{\partial}{\partial x} 2v_0v_1 \right)^2 = 0, \implies v_2 = 0;$$

similarly

$$\begin{aligned} v_3 &= -\frac{1}{30}t^6, \\ v_4 &= 0, \\ v_5 &= 0, \\ &\vdots \\ v_n &= 0. \end{aligned} \quad (5.5)$$

Therefore, we obtain

$$\begin{aligned} y(x, t) &= v_0 + v_1 + v_2 + v_3 + \dots \\ &= at + xt^2. \end{aligned} \quad (5.6)$$

*Note.* Now we solve the same problem using the He-Laplace method.

*Example 5.2.* Consider the following non-homogeneous nonlinear PDE:

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2} + \left( \frac{\partial y}{\partial x} \right)^2 = 2x + t^4, \quad (5.7)$$

with the following conditions:

$$y(x, 0) = 0, \quad \frac{\partial y}{\partial t}(x, 0) = a, \quad y(0, t) = at, \quad \frac{\partial y}{\partial x}(0, t) = t^2. \quad (5.8)$$

By applying the HE-Laplace method subject to the initial condition, we have

$$\begin{aligned} y(x, s) &= \frac{a}{s^2} - \frac{1}{s^2} L \left[ \frac{\partial^2 y}{\partial x^2} + \left( \frac{\partial y}{\partial x} \right)^2 \right] + \frac{1}{s^2} L [2x + t^4] \\ &= \frac{a}{s^2} + \frac{2x}{s^3} + \frac{4!}{s^7} - \frac{1}{s^2} L \left[ \frac{\partial^2 y}{\partial x^2} + \left( \frac{\partial y}{\partial x} \right)^2 \right]. \end{aligned} \quad (5.9)$$

The inverse of the Laplace transform implies that

$$y(x, t) = at + xt^2 + \frac{t^6}{30} - L^{-1} \left[ \frac{1}{s^2} L \left[ \frac{\partial^2 y}{\partial x^2} + \left( \frac{\partial y}{\partial x} \right)^2 \right] \right]. \quad (5.10)$$

Now, we apply the homotopy perturbation method, we have

$$\sum_{n=0}^{\infty} p^n y_n(x, t) = at + xt^2 + \frac{t^6}{30} - p \left( L^{-1} \left[ \frac{1}{s^2} \left\{ L \left[ \frac{\partial^2 y}{\partial x^2} \right] + L \left[ \sum_{n=0}^{\infty} p^n H_n(y) \right] \right\} \right] \right), \quad (5.11)$$

where  $H_n(\mathbf{y})$  are He's polynomials. The first few components of He's polynomials are given by

$$\begin{aligned} H_0(\mathbf{y}) &= \left(\frac{\partial y_0}{\partial x}\right)^2 = t^4, \\ H_1(\mathbf{y}) &= 2\left(\frac{\partial y_0}{\partial x}\right) \times \left(\frac{\partial y_1}{\partial x}\right) = 0, \\ H_2(\mathbf{y}) &= \left(\frac{\partial y_1}{\partial x}\right)^2 + 2\frac{\partial y_0}{\partial x} \frac{\partial y_2}{\partial x} = 0, \\ &\vdots \end{aligned} \tag{5.12}$$

Comparing the coefficient of like powers of  $p$ , we have

$$\begin{aligned} p^0 : y_0(x, t) &= at + xt^2 + \frac{t^6}{30}, \\ p^1 : y_1(x, t) &= -L^{-1} \left[ \frac{1}{s^2} \left\{ L \left[ \frac{\partial^2 y_0}{\partial x^2} \right] + L[H_0(\mathbf{y})] \right\} \right] = -\frac{t^6}{30}, \\ p^2 : y_2(x, t) &= -L^{-1} \left[ \frac{1}{s^2} \left\{ L \left[ \frac{\partial^2 y_1}{\partial x^2} \right] + L[H_1(\mathbf{y})] \right\} \right] = 0. \end{aligned} \tag{5.13}$$

Proceeding in a similar manner, we have

$$\begin{aligned} p^3 : y_3(x, t) &= 0, \\ p^4 : y_4(x, t) &= 0, \\ &\vdots \end{aligned} \tag{5.14}$$

so that the solution  $y(x, t)$  is given by

$$\begin{aligned} y(x, t) &= y_0 + y_1 + y_2 + y_3 + \dots \\ &= at + xt^2 + \frac{t^6}{30} - \frac{t^6}{30} + 0 + 0 + 0 + \dots \\ &= at + xt^2, \end{aligned} \tag{5.15}$$

which is the exact solution of the problem.

*Remark 5.3.* From comparison, it is clear that the rate of convergence of He-Laplace method is faster than homotopy perturbation method (HPM). Also it can be seen the following demerits in the HPM.

- (1) Choice of initial approximation is compulsory.
- (2) According to the steps of homotopy, perturbation procedure operator  $L$  should be "easy to handle." We mean that it must be chosen in such a way that one has no difficulty in subsequently solving systems of resulting equations. It should be noted that this condition does not restrict  $L$  to be linear. In some cases, as was done by He to solve the Lighthill equation, a nonlinear choice of  $L$  may be more suitable, but it is strongly recommended for beginners to take a linear operator as  $L$ .

## 6. Conclusions and Discussions

In this paper, the He-Laplace method is employed for solving linear and nonlinear partial differential equations, that is, heat and wave equations. In previous papers [6, 15, 34–38] many authors have already used Adomian polynomials to decompose the nonlinear terms in equations. The solution procedure is simple, but the calculation of Adomian polynomials is complex. To overcome this shortcoming, we proposed a He-Laplace method using He's polynomials [31, 32, 39]. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical results.

## Acknowledgment

The First author acknowledges the financial support provided by the Indian Academy of sciences SRF 2011, Bangalore, India.

## References

- [1] L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers*, Birkhäuser, 1997.
- [2] E. V. Krishnamurthy and S. K. Sen, *Numerical Algorithm Computations in Science and Engineering*, East-West Press, 2001.
- [3] G. Adomian, "Solution of physical problems by decomposition," *Computers & Mathematics with Applications*, vol. 27, no. 9-10, pp. 145–154, 1994.
- [4] Y. Cherruault and G. Adomian, "Decomposition methods: a new proof of convergence," *Mathematical and Computer Modelling*, vol. 18, no. 12, pp. 103–106, 1993.
- [5] M. Wadati, H. Segur, and M. J. Ablowitz, "A new Hamiltonian amplitude equation governing modulated wave instabilities," *Journal of the Physical Society of Japan*, vol. 61, no. 4, pp. 1187–1193, 1992.
- [6] A.-M. Wazwaz, "A comparison between the variational iteration method and Adomian decomposition method," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 129–136, 2007.
- [7] A.-M. Wazwaz, "A new algorithm for calculating Adomian polynomials for nonlinear operators," *Applied Mathematics and Computation*, vol. 111, no. 1, pp. 53–69, 2000.
- [8] A.-M. Wazwaz and A. Gorguis, "Exact solutions for heat-like and wave-like equations with variable coefficients," *Applied Mathematics and Computation*, vol. 149, no. 1, pp. 15–29, 2004.
- [9] A. Golbabai and M. Javidi, "A variational iteration method for solving parabolic partial differential equations," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 987–992, 2007.
- [10] J.-H. He, "Variational iteration method for autonomous ordinary differential systems," *Applied Mathematics and Computation*, vol. 114, no. 2-3, pp. 115–123, 2000.
- [11] M. Tatari and M. Dehghan, "On the convergence of He's variational iteration method," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 121–128, 2007.

- [12] A.-M. Wazwaz, "The variational iteration method: a powerful scheme for handling linear and nonlinear diffusion equations," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 933–939, 2007.
- [13] J. Biazar and H. Ghazvini, "He's homotopy perturbation method for solving systems of Volterra Integral equations," *Chaos, Solitons, Fractals*, vol. 39, pp. 370–377, 2009.
- [14] M. Dehghan, "Weighted finite difference techniques for the one-dimensional advection-diffusion equation," *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 307–319, 2004.
- [15] Y. Khan and F. Austin, "Application of the Laplace decomposition method to nonlinear homogeneous and non-homogeneous advection equations," *Zeitschrift fuer Naturforschung A*, vol. 65, pp. 1–5, 2010.
- [16] M. Madani and M. Fathizadeh, "Homotopy perturbation algorithm using Laplace transformation," *Nonlinear Science Letters A*, vol. 1, pp. 263–267, 2010.
- [17] S. T. Mohyud-Din and A. Yildirim, "Homotopy perturbation method for advection problems," *Nonlinear Science Letter A*, vol. 1, pp. 307–312, 2010.
- [18] J.-H. He, "Recent development of the homotopy perturbation method," *Topological Methods in Nonlinear Analysis*, vol. 31, no. 2, pp. 205–209, 2008.
- [19] J.-H. He, "New interpretation of homotopy perturbation method. Addendum," *International Journal of Modern Physics B*, vol. 20, no. 18, pp. 2561–2568, 2006.
- [20] J.-H. He, "A coupling method of a homotopy technique and a perturbation technique for non-linear problems," *International Journal of Non-Linear Mechanics*, vol. 35, no. 1, pp. 37–43, 2000.
- [21] J.-H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.
- [22] J.-H. He, "Homotopy perturbation method: a new nonlinear analytical technique," *Applied Mathematics and Computation*, vol. 135, no. 1, pp. 73–79, 2003.
- [23] J.-H. He, "A simple perturbation approach to Blasius equation," *Applied Mathematics and Computation*, vol. 140, no. 2-3, pp. 217–222, 2003.
- [24] J. H. He, "Application of homotopy perturbation method to nonlinear wave equation," *Chaos, Solitons, Fractals*, vol. 26, pp. 295–300, 2005.
- [25] J.-H. He, "Homotopy perturbation method for solving boundary value problems," *Physics Letters A*, vol. 350, no. 1-2, pp. 87–88, 2006.
- [26] P. R. Sharma and G. Methi, "Homotopy perturbation method approach for solution of equation to unsteady flow of a polytropic gas," *Journal of Applied Sciences Research*, vol. 6, no. 12, pp. 2057–2062, 2010.
- [27] D. D. Ganji and M. Rafei, "Solitary wave solutions for a generalized Hirota-Satsuma coupled KdV equation by homotopy perturbation method," *Physics Letters A*, vol. 356, no. 2, pp. 131–137, 2006.
- [28] S. Abbasbandy, "Application of He's homotopy perturbation method to functional integral equations," *Chaos, Solitons and Fractals*, vol. 31, no. 5, pp. 1243–1247, 2007.
- [29] D. D. Ganji and A. Sadighi, "Application of He's homotopy perturbation method to nonlinear coupled systems of reaction-diffusion equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, pp. 411–418, 2006.
- [30] A. Ghorbani and J. Saberi-Nadjafi, "He's homotopy perturbation method for calculating Adomian polynomials," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 8, pp. 229–232, 2007.
- [31] A. Ghorbani, "Beyond Adomian polynomials: he polynomials," *Chaos, Solitons and Fractals*, vol. 39, no. 3, pp. 1486–1492, 2009.
- [32] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, "Travelling wave solutions of seventh-order generalized KdV equation using He's polynomials," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, pp. 227–233, 2009.
- [33] M. A. Jafari and A. Aminataei, "Improved homotopy perturbation method," *International Mathematical Forum*, vol. 5, no. 29-32, pp. 1567–1579, 2010.
- [34] J. Biazar, M. Gholami Porshokuhi, and B. Ghanbari, "Extracting a general iterative method from an Adomian decomposition method and comparing it to the variational iteration method," *Computers & Mathematics with Applications*, vol. 59, no. 2, pp. 622–628, 2010.
- [35] S. Islam, Y. Khan, N. Faraz, and F. Austin, "Numerical solution of logistic differential equations by using the Laplace decomposition method," *World Applied Sciences Journal*, vol. 8, pp. 1100–1105, 2010.
- [36] S. A. Khuri, "A Laplace decomposition algorithm applied to a class of nonlinear differential equations," *Journal of Applied Mathematics*, vol. 1, no. 4, pp. 141–155, 2001.
- [37] Y. Khan, "An effective modification of the Laplace decomposition method for nonlinear equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, pp. 1373–1376, 2009.

- [38] E. Yusufoglu, "Numerical solution of Duffing equation by the Laplace decomposition algorithm," *Applied Mathematics and Computation*, vol. 177, no. 2, pp. 572–580, 2006.
- [39] Y. Khan and Q. Wu, "Homotopy perturbation transform method for nonlinear equations using He's polynomials," *Computers & Mathematics with Applications*, vol. 61, no. 8, pp. 1963–1967, 2011.





# Hindawi

Submit your manuscripts at  
<http://www.hindawi.com>

