Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 762762, 18 pages
doi:10.1155/2011/762762

Research Article

Sufficient and Necessary Center Conditions for the Poincaré Systems $P(2,2n)$ $(n \leq 5)$

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Received 22 August 2011; Revised 27 October 2011; Accepted 28 October 2011

Academic Editor: Mohamad Alwash

We obtain sufficient and necessary center conditions for the Poincaré system $P(2,2n)$ $(n \leq 5)$. The necessity of the condition is derived from the first $2n$ focal values by symbolic computation with Maple, and the sufficiency is proved by Volokitin's method.

1. Introduction

Research on Hilbert's sixteenth problem [1, 2] in general usually proceeds by the investigation on specific classes of polynomial systems. Much effort has been devoted in recent years to the investigation of various systems with cubic or quintic polynomials for the center problem [3–9]. We are interested in a certain family of polynomial systems of the form

$$\dot{x} = -y + xR(x,y), \qquad \dot{y} = x + yR(x,y), \tag{1.1}$$

with $\dot{x} = dx/dt$, $\dot{y} = dy/dt$, and $R(x,y) = \sum_{i=1}^{n} R_i(x,y)$, where R_i is a homogeneous polynomial of degree i. The centers of these systems are called uniformly isochronous centers [10]. The case in which $R(x,y)$ is of degree i has been investigated in [10, 11].

In the nonhomogeneous case [12], the pioneering studies mainly focus on the systems with $R(x,y) = R_1 + R_2$ [13], $R(x,y) = R_1 + R_3$ [14], $R(x,y) = R_1 + R_2 + R_3 (R_1^2 + R_2^2 + R_3^2 \neq 0)$ [15], $R(x,y) = R_1 + R_2 + R_3 + R_4$ ($R_4 \neq 0$ and only one R_i not equal to zero, for $i = 1,2,3$) [16].

In this paper, we consider the system (1.1) with $R(x,y) = R_2 + R_{2n}(n > 1)$, where

$$R_2 = a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2, \qquad R_{2n} = \sum_{i=0}^{2n} b_{2n-i,i}x^{2n-i}y^i, \tag{1.2}$$

and $b_{2,0}, b_{1,1}, b_{0,2}, b_{2n-i,i}$ $(i = 0,\ldots,2n)$ are real constants. We call the systems above as $P(2,2n)$. In [17, 18], the authors prove that in the system $P(2,2n)$ the uniformly isochronous centers are time reversible, which was done by imposing the existence of a transversal commutator. However, for any concrete n, it is difficult to obtain explicit form of the conditions for the origin to be a center for the system $P(2,2n)$ due to increasing expansion of computation during the management of large expressions. Sufficient and necessary conditions for the origin to be a center are obtained in Volokitin [19, 20] for $n = 2$ and 3 and in Xu and Lu [21] for $n = 4$.

In this paper, we shall consider the system $P(2,10)$ and obtain sufficient and necessary conditions for the origin to be a center. In Section 2, the main result and the basic method are stated, and the necessary and sufficient center conditions are proved in Sections 3 and 4, respectively. In Section 5, sufficient and necessary center conditions for systems $P(2,2n)$ $(n = 2, 3$ and $4)$ are given in tight form of polynomial systems.

2. Center Condition

In this section, the technique in [22] is adopted, and a recursive formula for focal values is obtained.

For $n = 5$, system $P(2,2n)$ takes the form

$$\dot{x} = -y + x\left(R_2(x,y) + R_{10}(x,y)\right), \qquad \dot{y} = x + y\left(R_2(x,y) + R_{10}(x,y)\right). \tag{2.1}$$

In polar coordinate system, system (2.1) can be written as

$$\dot{r} = pr^3 + qr^{11}, \qquad \dot{\theta} = 1, \tag{2.2}$$

where $\dot{r} = dr/d\theta$, p and q are the functions of θ:

$$p = a_{2,0}\cos^2(\theta) + a_{1,1}\cos(\theta)\sin(\theta) + a_{0,2}\sin^2(\theta), \qquad q = \sum_{i=0}^{10} b_{10-i,i}\cos^{10-i}(\theta)\sin^i(\theta). \tag{2.3}$$

Let $r(\theta,c)$ be the solution of (2.2) with $r(0,c) = c$. For $c > 0$ small enough, we write

$$r(\theta,c) = \sum_{m=1}^{\infty} a_m(\theta)c^m, \tag{2.4}$$

where $a_1(0) = 1, a_m(0) = 0$ $(m > 1)$. Substituting (2.4) into (2.2) and equating the coefficients of c, we obtain

$$\dot{a}_m = p \sum_{\substack{h+j+l=m \\ h,j,l\geq 1}} a_h a_j a_l + q \sum_{\substack{s_1+s_2+\cdots+s_{11}=m \\ s_1,s_2,\ldots,s_{11}\geq 1}} a_{s_1} a_{s_2} \cdots a_{s_{11}}. \tag{2.5}$$

For (2.5), we can easily obtain $\dot{a}_1 = 0, \dot{a}_{2k} = 0$ $(k = 1, 2, \ldots)$. Thus we can deduce

$$a_m = \begin{cases} 1, & m = 1, \\ 0, & m = 2k \ (k = 1, 2, \ldots), \\ \displaystyle\int_0^\theta \Phi_m d\theta, & m = 2k+1 \ (k = 1, 2, \ldots), \end{cases} \tag{2.6}$$

where

$$\Phi_m = p \sum_{\substack{h+j+l=m \\ h,j,l\geq 1 \\ h,j,l \text{ is odd}}} a_h a_j a_l + q \sum_{\substack{s_1+s_2+\cdots+s_{11}=m \\ s_1,s_2,\ldots,s_{11}\geq 1 \\ s_1,s_2,\ldots,s_{11} \text{ is odd}}} a_{s_1} a_{s_2} \cdots a_{s_{11}}. \tag{2.7}$$

The value of $a_{2k+1}(2\pi)$ $(k = 1, 2, \ldots)$ is called the kth focal value. Let $L_k = a_{2k+1}$ $(k = 1, 2, \ldots)$; then the origin is a center for system (2.1) if and only if

$$D_k \triangleq L_k(2\pi) = a_{2k+1}(2\pi) = 0, \quad (k = 1, 2, \ldots). \tag{2.8}$$

In fact, we have

$$L_k = \int_0^\theta \Psi_k d\theta \quad (k = 1, 2, \ldots), \tag{2.9}$$

where

$$\Psi_k = p \sum_{\substack{h+j+l=k-1 \\ h,j,l\geq 0}} L_h L_j L_l + q \sum_{\substack{s_1+s_2+\ldots+s_{11}=k-5 \\ s_1,s_2,\ldots,s_{11}\geq 0}} L_{s_1} L_{s_2} \cdots L_{s_{11}}, \tag{2.10}$$

and $L_0 = a_1 = 1$.

The main result of the present paper is presented in the following theorem.

Theorem 2.1. *The origin is a center for system* (2.1) *if and only if the following conditions are satisfied:*

$$\begin{aligned}
0 &= a_{2,0} + a_{0,2}, \\
0 &= 63b_{0,10} + 3b_{6,4} + 3b_{4,6} + 7b_{8,2} + 7b_{2,8} + 63b_{10,0}, \\
0 &= 126a_{1,1}b_{0,10} + 21a_{2,0}b_{3,7} + 15a_{2,0}b_{5,5} + 63a_{2,0}b_{1,9} - 189b_{10,0}a_{1,1} + 7a_{1,1}b_{2,8} \\
&\quad - 3a_{1,1}b_{6,4} - 14a_{1,1}b_{8,2} + 21a_{2,0}b_{7,3} + 63a_{2,0}b_{9,1},
\end{aligned}$$

$$
\begin{aligned}
0 ={}& -4a_{2,0}^2b_{8,2}+a_{1,1}^2b_{2,8}-108a_{2,0}^2b_{10,0}-4a_{2,0}^2b_{2,8}+27a_{1,1}^2b_{10,0}+27a_{1,1}^2b_{0,10}\\
&+a_{1,1}^2b_{8,2}^2-108a_{2,0}b_{0,10}+3a_{1,1}a_{2,0}b_{3,7}-3a_{1,1}a_{2,0}b_{7,3}-18a_{2,0}a_{1,1}b_{9,1}\\
&+18a_{2,0}a_{1,1}b_{1,9},\\
0 ={}& 72a_{2,0}^2a_{1,1}^3b_{10,0}+2a_{1,1}^2a_{2,0}^3b_{7,3}+16a_{2,0}^4a_{1,1}b_{6,4}-18a_{2,0}^3a_{1,1}^2b_{3,7}+3a_{1,1}^4a_{2,0}b_{7,3}\\
&+8a_{2,0}^5b_{7,3}+96a_{2,0}^5b_{1,9}+9a_{1,1}^5b_{0,10}-a_{1,1}^5b_{8,2}-36a_{1,1}^5b_{10,0}+96a_{2,0}^5b_{9,1}\\
&+8a_{2,0}^5b_{3,7}-156a_{2,0}^3a_{1,1}^2b_{1,9}-36a_{2,0}^3a_{1,1}^2b_{9,1}-228a_{2,0}^2a_{1,1}^3b_{0,10}\\
&-12a_{2,0}^2a_{1,1}^3b_{8,2}-4a_{1,1}^3a_{2,0}^2b_{6,4}+9a_{2,0}a_{1,1}^4b_{1,9}+64a_{2,0}^4a_{1,1}b_{8,2}\\
&+768a_{2,0}^4a_{1,1}b_{0,10}+288a_{2,0}^4a_{1,1}b_{10,0}+27a_{1,1}^4a_{2,0}b_{9,1},\\
0 ={}& a_{1,1}^6b_{8,2}+45a_{1,1}^6b_{10,0}-3a_{1,1}^5a_{2,0}b_{7,3}-36a_{1,1}^5a_{2,0}b_{9,1}-28a_{2,0}^2a_{1,1}^4b_{0,10}\\
&+16a_{2,0}^2a_{1,1}^4b_{8,2}+4a_{1,1}^4a_{2,0}^2b_{6,4}-148a_{2,0}^2a_{1,1}^4b_{10,0}+4a_{2,0}^3a_{1,1}^3b_{7,3}\\
&+76a_{2,0}^3a_{1,1}^3b_{9,1}-28a_{2,0}^3a_{1,1}^3b_{1,9}+240a_{2,0}^4a_{1,1}^2b_{10,0}+240a_{2,0}^4a_{1,1}^2b_{0,10}\\
&-16a_{2,0}^5a_{1,1}b_{9,1}+16a_{2,0}^5a_{1,1}b_{1,9}-64a_{2,0}^6b_{0,10}-64a_{2,0}^6b_{10,0},\\
0 ={}& -35b_{10,0}a_{1,1}^9+35a_{2,0}b_{9,1}a_{1,1}^8-30a_{1,1}^7a_{2,0}^2b_{8,2}+350a_{1,1}^7a_{2,0}^2b_{10,0}\\
&-340a_{1,1}^6a_{2,0}^3b_{9,1}+20a_{1,1}^6a_{2,0}^3b_{7,3}+280a_{1,1}^5a_{2,0}^4b_{8,2}-8a_{1,1}^5a_{2,0}^4b_{6,4}\\
&-600a_{1,1}^5a_{2,0}^4b_{10,0}+480a_{1,1}^4a_{2,0}^5b_{9,1}-176a_{1,1}^4a_{2,0}^5b_{7,3}+64a_{1,1}^3a_{2,0}^6b_{6,4}\\
&-288a_{1,1}^3a_{2,0}^6b_{8,2}+800a_{1,1}^3a_{2,0}^6b_{10,0}+96a_{1,1}^2a_{2,0}^7b_{7,3}-448a_{1,1}^2a_{2,0}^7b_{9,1}\\
&+128a_{1,1}a_{2,0}^8b_{8,2}-640a_{1,1}a_{2,0}^8b_{10,0}+128a_{2,0}^9b_{9,1}+128a_{2,0}^9b_{1,9}.
\end{aligned}
\tag{2.11}
$$

3. The Proof of the Necessity

By simple computation in terms of (2.10) with Maple, we obtain

$$
\begin{aligned}
\Psi_1 &= pL_0^3,\\
\Psi_2 &= 3pL_0^2L_1,\\
\Psi_3 &= p\left(3L_0^2L_2+3L_0L_1^2\right),\\
\Psi_4 &= p\left(3L_0^2L_3+6L_0L_1L_2+L_1^3\right),\\
\Psi_5 &= p\left(3L_0^2L_4+6L_0L_1L_3+3L_0L_2^2+3L_1^2L_2\right)+qL_0^{11},\\
\Psi_6 &= p\left(3L_0^2L_5+6L_0L_1L_4+6L_0L_2L_3+3L_1^2L_3+3L_1L_2^2\right)+11qL_0^{10}L_1,
\end{aligned}
$$

$$\Psi_7 = p\left(3L_1^2L_4 + 6L_1L_2L_3 + 3L_0L_3^2 + L_2^3 + 3L_0^2L_6 + 6L_0L_1L_5 + 6L_0L_2L_4\right) + q\left(11L_0^{10}L_2 + 55L_0^9L_1^2\right),$$

$$\Psi_8 = p\left(3L_2^2L_3 + 3L_1^2L_5 + 6L_1L_2L_4 + 6L_0L_3L_4 + 6L_0L_1L_6 + 6L_0L_2L_5 + 3L_1L_3^2 + 3L_0^2L_7\right) + q\left(11L_0^{10}L_3 + 110L_0^9L_1L_2 + 165L_0^8L_1^3\right),$$

$$\Psi_9 = p\left(6L_0L_2L_6 + 6L_0L_1L_7 + 3L_0^2L_8 + 6L_1L_3L_4 + 6L_1L_2L_5 + 3L_2^2L_4 + 3L_2L_3^2 + 3L_1^2L_6 + 3L_0L_4^2 + 6L_0L_3L_5\right) + q\left(110L_0^9L_1L_3 + 55L_0^9L_2^2 + 495L_0^8L_1^2L_2 + 330L_0^7L_1^4 + 11L_0^{10}L_4\right),$$

$$\Psi_{10} = p\left(L_0^2L_9 + L_2^2L_5 + 2L_1L_2L_6 + L_1^2L_7 + 2L_0L_2L_7 + 4L_0L_3L_6 + 4L_2L_3L_4 + 2L_1L_4^2 + 4L_1L_3L_5 + 2L_0L_1L_8 + 4L_0L_4L_5 + L_3^3\right) + q\left(110L_0^9L_2L_3 + 110L_0^9L_1L_4 + 11L_0^{10}L_5 + 495L_0^8L_1L_2^2 + 495L_0^8L_1^2L_3 + 1320L_0^7L_1^3L_2 + 462L_0^6L_1^5\right),$$

$$\Psi_{11} = p\left(6L_0L_3L_7 + 6L_0L_2L_8 + 3L_0^2L_{10} + 3L_0L_5^2 + 6L_0L_4L_6 + 3L_3^2L_4 + 3L_2L_4^2 + 6L_2L_3L_5 + 3L_2^2L_6 + 3L_1^2L_8 + 6L_1L_4L_5 + 6L_1L_2L_7 + 6L_1L_3L_6 + 6L_0L_1L_9\right) + q\left(11L_0^{10}L_6 + 110L_0^9L_1L_5 + 55L_0^9L_3^2 + 110L_0^9L_2L_4 + 495L_0^8L_1^2L_4 + 990L_0^8L_1L_2L_3 + 165L_0^8L_2^3 + 1320L_0^7L_1^3L_3 + 1980L_0^7L_1^2L_2^2 + 2310L_0^6L_1^4L_2 + 462L_0^5L_1^6\right). \quad (3.1)$$

Substituting (3.1) into (2.9), after a simple calculation, we obtain

$$L_1 = \frac{1}{2}a_{2,0}\cos(\theta)\sin(\theta) - \frac{1}{2}a_{0,2}\cos(\theta)\sin(\theta) + \frac{1}{2}a_{1,1} + \frac{1}{2}a_{2,0}\theta - \frac{1}{2}a_{1,1}\cos^2(\theta) + \frac{1}{2}a_{0,2}\theta, \quad (3.2)$$

and the first focal value $D_1 = L_1(2\pi) = a_{2,0}\pi + a_{0,2}\pi$. From (2.8), we can see that

$$0 = a_{2,0} + a_{0,2} \quad (3.3)$$

must be satisfied if the origin is a center for system (2.1).

Following the discussion above, we can compute the first kth focal values recursively by the following algorithm, which consists of three steps.

Step 1 (Initialization). Let

$$
\begin{gathered}
L_0 = 1, \qquad k = 1, \\
p(\theta) = a_{2,0}\cos^2(\theta) + a_{1,1}\cos(\theta)\sin(\theta) - a_{2,0}\sin^2(\theta), \\
q(\theta) = \sum_{i=0}^{10} b_{10-i,i}\cos^{10-i}(\theta)\sin^i(\theta).
\end{gathered}
\tag{3.4}
$$

Step 2 (Computation of kth focal value). L_k can be obtained in terms of (2.10) and (3.1) with Maple. Then the kth focal value D_k can be obtained by simple calculation.

Step 3 (Let $k = k + 1$, and go to Step 2). Using computer algebra and writing a Maple code applying the algorithm above, we obtain

$$
\begin{aligned}
D_k &= 0 \quad (k = 1, 2, \ldots, 4), \\
D_5 &= \frac{1}{128}\pi(63b_{0,10} + 3b_{6,4} + 63b_{10,0} + 3b_{4,6} + 7b_{8,2} + 7b_{2,8}), \\
D_6 &= \frac{1}{128}\pi(7a_{1,1}b_{4,6} + 14a_{2,0}b_{7,3} + 10a_{2,0}b_{5,5} + 21b_{10,0}a_{1,1} + 21a_{1,1}b_{2,8} \\
&\qquad + 7a_{1,1}b_{8,2} + 42a_{2,0}b_{9,1} + 5a_{1,1}b_{6,4} + 231a_{1,1}b_{0,10} + 14a_{2,0}b_{3,7} + 42a_{2,0}b_{1,9}), \\
D_7 &= \frac{5}{512}\pi\Big(132a_{2,0}^2b_{10,0} + 5a_{1,1}^2b_{6,4} + 5a_{1,1}^2b_{8,2} + 20a_{2,0}^2b_{6,4} + 33a_{1,1}^2b_{2,8} + 36a_{2,0}^2b_{8,2} \\
&\qquad + 20a_{2,0}^2b_{4,6} + 132a_{1,1}a_{2,0}b_{1,9} + 9a_{1,1}^2b_{4,6} + 36a_{2,0}b_{9,1}a_{1,1} + 20a_{2,0}a_{1,1}b_{7,3} \\
&\qquad + 20a_{2,0}a_{1,1}b_{5,5} + 132a_{2,0}^2b_{0,10} + 36a_{2,0}^2b_{2,8} + 429a_{1,1}^2b_{0,10} + 9a_{1,1}^2b_{10,0} + 36a_{1,1}a_{2,0}b_{3,7}\Big), \\
D_8 &= \frac{5}{4096}\pi\Big(792a_{2,0}^3b_{9,1} + 360a_{2,0}^3b_{7,3} + 6435a_{1,1}^3b_{0,10} + 280a_{2,0}^3b_{5,5} + 45a_{1,1}^3b_{10,0} \\
&\qquad + 99a_{1,1}^3b_{4,6} + 35a_{1,1}^3b_{8,2} + 429a_{1,1}^3b_{2,8} + 360a_{2,0}^3b_{3,7} + 792a_{2,0}^3b_{1,9} \\
&\qquad + 540a_{1,1}a_{2,0}^2b_{8,2} + 210a_{2,0}a_{1,1}^2b_{7,3} + 270a_{1,1}^2a_{2,0}b_{9,1} + 540a_{2,0}^2a_{1,1}b_{4,6} \\
&\qquad + 5148a_{2,0}^2a_{1,1}b_{0,10} + 2574a_{2,0}a_{1,1}^2b_{1,9} + 420a_{2,0}^2a_{1,1}b_{6,4} + 1188\,{}^2a_{2,0}\,b_{10,0}a_{1,1} \\
&\qquad + 1188a_{2,0}^2a_{1,1}b_{2,8} + 270a_{2,0}a_{1,1}^2b_{5,5} + 594a_{1,1}^2a_{2,0}b_{3,7} + 45a_{1,1}^3b_{6,4}\Big), \\
D_9 &= \frac{35}{32768}\pi\Big(1120a_{2,0}^3a_{1,1}b_{7,3} + 1120a_{2,0}^3a_{1,1}b_{5,5} + 17160a_{2,0}^2a_{1,1}^2b_{0,10} \\
&\qquad + 1760a_{2,0}^3a_{1,1}b_{3,7} + 1144a_{2,0}a_{1,1}^3b_{3,7} + 5720a_{2,0}a_{1,1}^3b_{1,9} + 1760a_{2,0}^3a_{1,1}b_{9,1} \\
&\qquad + 440a_{2,0}a_{1,1}^3b_{5,5} + 3432a_{2,0}^2a_{1,1}^2b_{2,8} + 840a_{2,0}^2a_{1,1}^2b_{8,2} + 280a_{1,1}^3a_{2,0}b_{9,1} \\
&\qquad + 880a_{2,0}^4b_{2,8} + 143a_{1,1}^4b_{4,6} + 2288a_{2,0}^4b_{0,10} + 2288a_{2,0}^4b_{10,0} + 35a_{1,1}^4b_{10,0} \\
&\qquad + 55a_{1,1}^4b_{6,4} + 1320a_{1,1}^2a_{2,0}^2b_{10,0} + 4576a_{1,1}a_{2,0}^3b_{1,9} + 1320a_{2,0}^2a_{1,1}^2b_{4,6} \\
&\qquad + 280a_{1,1}^3a_{2,0}b_{7,3} + 12155a_{1,1}^4b_{0,10} + 840a_{2,0}^2a_{1,1}^2b_{6,4} + 560a_{2,0}^4b_{6,4} + 35a_{1,1}^4b_{8,2} \\
&\qquad + 560a_{2,0}^4b_{4,6} + 880a_{2,0}^4b_{8,2} + 715a_{1,1}^4b_{2,8}\Big),
\end{aligned}
$$

$$
\begin{aligned}
D_{10} = {} & \frac{11}{32768}\pi^2(7b_{2,8} + 63b_{0,10} + 63b_{10,0} + 3b_{6,4} + 7b_{8,2} + 3b_{4,6}) \\
& + \frac{7}{16384}\pi\Big(3080a_{1,1}^3a_{2,0}^2b_{10,0} + 34320a_{1,1}a_{2,0}^4b_{0,10} + 17160b_{2,8}a_{1,1}^3a_{2,0}^2 + 11440a_{1,1}a_{2,0}^4b_{2,8} \\
& \qquad + 2520a_{1,1}^3a_{2,0}^2b_{8,2} + 5040a_{1,1}^2a_{2,0}^3b_{7,3} + 24310a_{1,1}^4a_{2,0}b_{1,9} + 6160a_{1,1}a_{2,0}^4b_{8,2} \\
& \qquad + 2016a_{2,0}^5b_{5,5} + 2464a_{2,0}^5b_{3,7} + 6160a_{2,0}^4a_{1,1}b_{4,6} + 4290a_{2,0}a_{1,1}^4b_{3,7} \\
& \qquad + 6160a_{1,1}^2a_{2,0}^3b_{9,1} + 2431b_{2,8}a_{1,1}^5 + 6160a_{2,0}^3a_{1,1}^2b_{5,5} + 11440a_{1,1}a_{2,0}^4b_{10,0} \\
& \qquad + 143a_{1,1}^5b_{6,4} + 429a_{1,1}^5b_{4,6} + 630a_{1,1}^4a_{2,0}b_{9,1} + 770a_{1,1}^4a_{2,0}b_{7,3} \\
& \qquad + 5040a_{1,1}a_{2,0}^4b_{6,4} + 97240a_{1,1}^3a_{2,0}^2b_{0,10} + 46189a_{1,1}^5b_{0,10} + 77a_{1,1}^5b_{8,2} \\
& \qquad + 4576a_{2,0}^5b_{9,1} + 1430a_{1,1}^4a_{2,0}b_{5,5} + 2464a_{2,0}^5b_{7,3} + 11440a_{2,0}^3a_{1,1}^2b_{3,7} \\
& \qquad + 63a_{1,1}^5b_{10,0} + 34320a_{1,1}^2a_{2,0}^3b_{1,9} + 3080a_{1,1}^3a_{2,0}^2b_{6,4} \\
& \qquad + 5720a_{1,1}^3a_{2,0}^2b_{4,6} + 4576a_{2,0}^5b_{1,9}\Big),
\end{aligned}
$$

$$
\begin{aligned}
D_{11} = {} & \frac{3}{4096}\pi^2(7b_{2,8} + 63b_{10,0} + 3b_{6,4} + 7b_{8,2} + 63b_{0,10} + 3b_{4,6}) \\
& \times (7a_{1,1}b_{8,2} + 10b_{5,5}a_{2,0} + 5a_{1,1}b_{6,4} + 14b_{7,3}a_{2,0} + 7a_{1,1}b_{4,6} + 42b_{9,1}a_{2,0} \\
& \quad + 14b_{3,7}a_{2,0} + 231a_{1,1}b_{0,10} + 21b_{2,8}a_{1,1} + 21a_{1,1}b_{10,0} + 42a_{2,0}b_{1,9}) + \frac{1}{1966080}\pi \\
& \times \Big(10752a_{2,0}b_{7,3}b_{3,7} + 126a_{1,1}b_{10,0}b_{3,7} + 580671a_{1,1}b_{0,10}b_{9,1} + 113022a_{1,1}b_{7,3}b_{0,10} \\
& \quad + 23667a_{1,1}b_{9,1}b_{10,0} - 61296a_{2,0}b_{10,0}b_{4,6} + 663a_{1,1}b_{1,9}b_{4,6} + 23587200a_{2,0}^5a_{1,1}b_{1,9} \\
& \quad + 19656000a_{2,0}^3a_{1,1}^3b_{3,7} + 158722200a_{2,0}^2a_{1,1}^4b_{0,10} + 7046a_{2,0}b_{5,5}b_{3,7} \\
& \quad + 7620480a_{2,0}^5a_{1,1}b_{7,3} + 9525600a_{2,0}^4a_{1,1}^2b_{8,2} - 143332a_{2,0}b_{2,8}b_{10,0} - 129948a_{2,0}b_{4,6}b_{0,10} \\
& \quad + 9100a_{1,1}b_{2,8}b_{7,3} + 9879a_{1,1}b_{9,1}b_{6,4} + 50388b_{0,10}a_{1,1}b_{3,7} - 268736a_{2,0}b_{2,8}b_{0,10} \\
& \quad + 140a_{1,1}b_{8,2}b_{3,7} + 234a_{1,1}b_{6,4}b_{3,7} + 9525600a_{2,0}^4a_{1,1}^2b_{6,4} + 5012280a_{2,0}a_{1,1}^5b_{3,7} \\
& \quad + 437a_{1,1}b_{6,4}b_{5,5} + 54145a_{1,1}b_{5,5}b_{0,10} + 11007360a_{2,0}^5a_{1,1}b_{3,7} + 100245600a_{1,1}^2a_{2,0}^4b_{0,10} \\
& \quad - 129168a_{2,0}b_{6,4}b_{0,10} + 13759200a_{2,0}^4a_{1,1}^2b_{4,6} + 57330a_{1,1}^6b_{8,2} + 122850a_{1,1}^6b_{6,4} \\
& \quad - 12318a_{2,0}b_{8,2}^2 + 7862400a_{2,0}^6b_{0,10} + 4914a_{2,0}b_{3,7}^2 - 3198a_{2,0}b_{4,6}^2 + 2645370a_{1,1}^6b_{2,8} \\
& \quad - 16354a_{2,0}b_{2,8}^2 - 11856a_{2,0}b_{6,4}b_{8,2} + 15555a_{1,1}b_{9,1}b_{4,6} + 7802a_{2,0}b_{7,3}b_{5,5} \\
& \quad + 3439800a_{2,0}^2a_{1,1}^4b_{6,4} + 29484000a_{1,1}^2a_{2,0}^4b_{2,8} + 78636b_{9,1}^2a_{2,0} + 39690a_{1,1}^6b_{10,0} \\
& \quad + 2540160a_{2,0}^6b_{6,4} - 881790a_{2,0}b_{0,10}^2 + 417690a_{1,1}^6b_{4,6} + 4199a_{1,1}b_{1,9}b_{2,8} \\
& \quad + 329b_{8,2}a_{1,1}b_{5,5} + 55552770a_{1,1}^6b_{0,10} - 12220a_{2,0}b_{4,6}b_{8,2} + 3669120a_{2,0}^6b_{2,8} \\
& \quad + 63a_{1,1}b_{1,9}b_{10,0} + 6350400a_{2,0}^3a_{1,1}^3b_{9,1} + 91a_{1,1}b_{8,2}b_{1,9} + 476280a_{2,0}a_{1,1}^5b_{9,1} \\
& \quad - 2946a_{2,0}b_{6,4}^2 + 195b_{6,4}a_{1,1}b_{1,9} + 25061400a_{2,0}^2a_{1,1}^4b_{2,8} - 132672a_{2,0}b_{10,0}b_{8,2} \\
& \quad + 11367a_{1,1}b_{9,1}b_{8,2} + 385a_{1,1}b_{10,0}b_{5,5} + 43050a_{2,0}b_{9,1}b_{3,7} + 1474200a_{2,0}a_{1,1}^5b_{5,5} \\
& \quad - 14768a_{2,0}b_{4,6}b_{2,8} + 2381400a_{2,0}^2a_{1,1}^4b_{8,2} + 21268b_{1,9}a_{2,0}b_{5,5} + 7620480a_{2,0}^5a_{1,1}b_{5,5} \\
& \quad - 258076a_{2,0}b_{0,10}b_{8,2} + 30498b_{1,9}a_{2,0}b_{3,7} + 129024a_{2,0}b_{1,9}b_{9,1} + 1302b_{7,3}a_{1,1}b_{8,2} \\
& \quad + 687960a_{2,0}a_{1,1}^5b_{7,3} + 44766a_{2,0}b_{9,1}b_{7,3} + 30956b_{9,1}a_{2,0}b_{5,5} + 949a_{1,1}b_{4,6}b_{5,5} \\
& \quad + 3094a_{1,1}b_{3,7}b_{2,8} + 66830400a_{2,0}^3a_{1,1}^3b_{1,9} + 7371000a_{2,0}^2a_{1,1}^4b_{4,6}
\end{aligned}
$$

$$+ 3913a_{1,1}b_{5,5}b_{2,8} + 32214b_{1,9}a_{2,0}b_{7,3} - 28672a_{2,0}b_{8,2}b_{2,8} - 60516a_{2,0}b_{10,0}b_{6,4}$$
$$+ 5838a_{2,0}b_{7,3}^2 + 2381400a_{2,0}^2a_{1,1}^4b_{10,0} - 1290240a_{2,0}b_{0,10}b_{10,0} + 2004a_{1,1}b_{7,3}b_{10,0}$$
$$+ 624a_{1,1}b_{3,7}b_{4,6} + 88179a_{1,1}b_{1,9}b_{0,10} + 3669120a_{2,0}^6b_{8,2} + 2538a_{1,1}b_{7,3}b_{4,6}$$
$$- 408450a_{2,0}b_{10,0}^2 + 7862400a_{2,0}^6b_{10,0} + 13759200a_{2,0}^4a_{1,1}^2b_{10,0}$$
$$+9172800a_{2,0}^3a_{1,1}^3b_{5,5} - 14404a_{2,0}b_{2,8}b_{6,4} + 2560a_{2,0}b_{5,5}^2 + 50267b_{2,8}b_{9,1}a_{1,1}\Big). \tag{3.5}$$

Usually Wu's method [23] is used to do the triangular sets reduction for the focal values, that is, $D_5, \ldots, D_{11}$. The method provides a standard algorithm [24] to handle the reduction of polynomials; however, we here just take use the idea of Wu's method and do the reduction by substitution method due to the speciality of our case. Thus we can obtain the relations (2.11). Therefore, the necessary part of Theorem 2.1 is proved.

4. The Proof of the Sufficiency

Case 1 ($a_{1,1} = 0$). From (2.11), we have that either

(i) $a_{0,2} = a_{2,0} = 0$, $63b_{0,10} + 3b_{6,4} + 3b_{4,6} + 7b_{8,2} + 7b_{2,8} + 63b_{10,0} = 0$, or

(ii) $a_{0,2} + a_{2,0} = 0$, $b_{0,10} + b_{10,0} = 0$, $b_{1,9} + b_{9,1} = 0$, $b_{2,8} + b_{8,2} = 0$, $b_{3,7} + b_{7,3} = 0$, $b_{4,6} + b_{6,4} = 0$, $b_{5,5} = 0$.

For Case 1, system (2.1) takes the form

$$\dot{x} = -y + xR_{10}(x, y), \qquad \dot{y} = x + yR_{10}(x, y). \tag{4.1}$$

The center condition of system (4.1) is $\int_0^{2\pi} R_{10}(\sin(\theta), \cos(\theta))d\theta = 0$ (see ([10]). That is,

$$63b_{0,10} + 3b_{6,4} + 3b_{4,6} + 7b_{8,2} + 7b_{2,8} + 63b_{10,0} = 0. \tag{4.2}$$

For Case 2, system (2.1) takes the form

$$\dot{x} = -y + xT, \qquad \dot{y} = x + yT, \tag{4.3}$$

where $T = a_{0,2}(y^2 - x^2) + b_{0,10} + (y^{10} - x^{10}) + b_{1,9}(xy^9 - x^9y) + b_{2,8}(x^2y^8 - x^8y^2) + b_{3,7}(x^3y^7 - x^7y^3) + b_{4,6}(x^4y^6 - x^6y^4))$. For system (4.3), the following linear transformation:

$$(x, y) = (u, v)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{4.4}$$

leads to

$$\dot{u} = -v + u^2vJ(u, v), \qquad \dot{v} = u + uv^2J(u, v), \tag{4.5}$$

where $J(u,v) = J_{0,0} + J_{8,0}u^8 + J_{6,2}u^6v^2 + J_{4,4}u^4v^4 + J_{2,6}u^2v^6 + J_{0,8}v^8$,

$$\begin{aligned}
J_{0,0} &= a_{0,2},\\
J_{8,0} &= \frac{1}{256}(3b_{2,8} + b_{4,6} + 5b_{0,10} + 4b_{1,9} + 2b_{3,7}),\\
J_{6,2} &= \frac{1}{256}(24b_{1,9} - 4b_{3,7} + 60b_{0,10} + 4b_{2,8} - 4b_{4,6}),\\
J_{4,4} &= \frac{1}{256}(-14b_{2,8} + 126b_{0,10} + 6b_{4,6}),\\
J_{2,6} &= \frac{1}{256}(4b_{3,7} - 24b_{1,9} + 60b_{0,10} - 4b_{4,6} + 4b_{2,8}),\\
J_{0,8} &= \frac{1}{256}(-4b_{1,9} + 3b_{2,8} + b_{4,6} + 5b_{0,10} - 2b_{3,7}).
\end{aligned} \tag{4.6}$$

Obviously, in this case, system (4.5) satisfies the condition of symmetry principle [25, page 135]; thus the origin is a center for the system.

Case 2 ($a_{1,1} \neq 0, a_{2,0} = 0$). From (2.11), we have

$$b_{10,0} = b_{8,2} = b_{6,4} = b_{4,6} = b_{2,8} = b_{0,10} = 0. \tag{4.7}$$

Thus system (2.1) takes the form

$$\dot{x} = -y + x^2y(a_{1,1} + S), \qquad \dot{y} = x + xy^2(a_{1,1} + S), \tag{4.8}$$

where $S = b_{9,1}x^8 + b_{7,3}x^6y^2 + b_{5,5}x^4y^4 + b_{3,7}x^2y^6 + b_{1,9}y^8$. Obviously, system (4.8) satisfies the condition of symmetry principle; thus the origin is a center for the system.

Case 3 ($a_{2,0} \neq 0,\ a_{1,1} \neq 0$). In this case, (2.11) takes the form

$$\begin{aligned}
a_{0,2} &= -a_{2,0},\\
b_{4,6} &= -\frac{7}{3}b_{2,8} - 21b_{0,10} - b_{6,4} - \frac{7}{3}b_{8,2} - 21b_{10,0},\\
b_{5,5} &= -\frac{1}{15a_{2,0}}(-14a_{1,1}b_{8,2} + 21a_{2,0}b_{7,3} + 63a_{2,0}b_{9,1} - 3a_{1,1}b_{6,4} - 189a_{1,1}b_{10,0}\\
&\qquad +21a_{2,0}b_{3,7} + 126a_{1,1}b_{0,10} + 63a_{2,0}b_{1,9} + 7a_{1,1}b_{2,8}),\\
b_{2,8} &= -\frac{1}{-4a_{2,0}^2 + a_{1,1}^2}\Big(3a_{1,1}a_{2,0}b_{3,7} - 18a_{1,1}a_{2,0}b_{9,1} + a_{1,1}^2b_{8,2} + 27a_{1,1}^2b_{10,0}\\
&\qquad - 4a_{2,0}^2b_{8,2} - 108a_{2,0}^2b_{10,0} - 3a_{1,1}a_{2,0}b_{7,3} - 108a_{2,0}^2b_{0,10} + 27a_{1,1}^2b_{0,10}\\
&\qquad +18a_{1,1}a_{2,0}b_{1,9}\Big),
\end{aligned}$$

$$
\begin{aligned}
b_{3,7} =\ & \frac{1}{2a_{2,0}^3\left(-4a_{2,0}^2+9a_{1,1}^2\right)} \\
& \times \Big(3a_{1,1}^4 a_{2,0} b_{7,3} + 9a_{1,1}^5 b_{0,10} + 16a_{1,1}a_{2,0}^4 b_{6,4} - a_{1,1}^5 b_{8,2} \\
& \quad + 96a_{2,0}^5 b_{1,9} + 768a_{1,1}a_{2,0}^4 b_{0,10} - 36a_{1,1}^2 a_{2,0}^3 b_{9,1} - 228a_{1,1}^3 a_{2,0}^2 b_{0,10} \\
& \quad + 288a_{1,1}a_{2,0}^4 b_{10,0} - 4a_{1,1}^3 a_{2,0}^2 b_{6,4} + 8a_{2,0}^5 b_{7,3} - 156a_{1,1}^2 a_{2,0}^3 b_{1,9} \\
& \quad - 12a_{1,1}^3 a_{2,0}^2 b_{8,2} + 2a_{1,1}^2 a_{2,0}^3 b_{7,3} + 9a_{1,1}^4 a_{2,0} b_{1,9} + 27a_{1,1}^4 a_{2,0} b_{9,1} \\
& \quad + 72a_{1,1}^3 a_{2,0}^2 b_{10,0} + 64a_{1,1}a_{2,0}^4 b_{8,2} + 96a_{2,0}^5 b_{9,1} - 36a_{1,1}^5 b_{10,0}\Big), \\
b_{0,10} =\ & \frac{1}{4a_{2,0}^2\left(7a_{1,1}^4 - 60a_{1,1}^2 a_{2,0}^2 + 16a_{2,0}^4\right)} \\
& \times \Big(-148a_{1,1}^4 a_{2,0}^2 b_{10,0} + 76a_{1,1}^3 a_{2,0}^3 b_{9,1} + a_{1,1}^6 b_{8,2} + 4a_{1,1}^4 a_{2,0}^2 b_{6,4} + 4a_{1,1}^3 a_{2,0}^3 b_{7,3} \\
& \quad - 28a_{1,1}^3 a_{2,0}^3 b_{1,9} + 16a_{1,1}a_{2,0}^5 b_{1,9} - 3a_{1,1}^5 a_{2,0} b_{7,3} + 16a_{1,1}^4 a_{2,0}^2 b_{8,2} - 16a_{1,1}a_{2,0}^5 b_{9,1} \\
& \quad - 36a_{1,1}^5 a_{2,0} b_{9,1} + 240a_{1,1}^2 a_{2,0}^4 b_{10,0} + 45a_{1,1}^6 b_{10,0} - 64a_{2,0}^6 b_{10,0}\Big), \\
b_{1,9} =\ & \frac{1}{128a_{2,0}^9}\Big(-280a_{1,1}^5 a_{2,0}^4 b_{8,2} - 20a_{1,1}^6 a_{2,0}^3 b_{7,3} - 480a_{1,1}^4 a_{2,0}^5 b_{9,1} \\
& \quad - 64a_{1,1}^3 a_{2,0}^6 b_{6,4} - 800a_{1,1}^3 a_{2,0}^6 b_{10,0} - 128a_{1,1}a_{2,0}^8 b_{8,2} - 96a_{1,1}^2 a_{2,0}^7 b_{7,3} \\
& \quad + 176a_{1,1}^4 a_{2,0}^5 b_{7,3} - 35a_{1,1}^8 a_{2,0} b_{9,1} + 30a_{1,1}^7 a_{2,0}^2 b_{8,2} + 8a_{1,1}^5 a_{2,0}^4 b_{6,4} \\
& \quad - 128a_{2,0}^9 b_{9,1} + 448a_{1,1}^2 a_{2,0}^7 b_{9,1} + 288a_{1,1}^3 a_{2,0}^6 b_{8,2} + 340a_{1,1}^6 a_{2,0}^3 b_{9,1} \\
& \quad + 640a_{1,1}a_{2,0}^8 b_{10,0} + 600a_{1,1}^5 a_{2,0}^4 b_{10,0} + 35a_{1,1}^9 b_{10,0} - 350a_{1,1}^7 a_{2,0}^2 b_{10,0}\Big).
\end{aligned}
\tag{4.9}
$$

Substituting (4.9) into (2.1), we get

$$
\begin{aligned}
\dot{x} &= -y + x\left(a_{2,0}x^2 + a_{1,1}xy - a_{2,0}y^2\right)H(x,y), \\
\dot{y} &= x + y\left(a_{2,0}x^2 + a_{1,1}xy - a_{2,0}y^2\right)H(x,y),
\end{aligned}
\tag{4.10}
$$

where

$$
\begin{aligned}
H(x,y) =\ & \frac{1}{128a_{2,0}^9}\Big(128a_{2,0}^9 + 128b_{10,0}a_{2,0}^8 x^8 + \left(-128b_{10,0}a_{1,1}a_{2,0}^7 + 128b_{9,1}a_{2,0}^8\right)yx^7 \\
& + \left(128a_{1,1}^2 b_{10,0}a_{2,0}^6 - 128a_{1,1}b_{9,1}a_{2,0}^7 + 128b_{10,0}a_{2,0}^8 + 128a_{2,0}^8 b_{8,2}\right)y^2x^6 \\
& + \Big(128a_{1,1}^2 b_{9,1}a_{2,0}^6 + 128b_{9,1}a_{2,0}^8 - 256b_{10,0}a_{1,1}a_{2,0}^7 - 128{a_{1,1}}^3 b_{10,0}\mathrm{a}_{2,0}^5 \\
& \quad + 128a_{2,0}^8 b_{7,3} - 128a_{2,0}^7 a_{1,1}b_{8,2}\Big)y^3x^5 \\
& + \Big(-256a_{1,1}b_{9,1}a_{2,0}^7 - 128a_{1,1}^3 b_{9,1}a_{2,0}^5 + 384a_{1,1}^2 b_{10,0}a_{2,0}^6 + 128a_{2,0}^6 a_{1,1}^2 b_{8,2} \\
& \quad + 128a_{1,1}^4 b_{10,0}a_{2,0}^4 + 128a_{2,0}^8 b_{8,2} + 128b_{10,0}a_{2,0}^8 + 128a_{2,0}^8 b_{6,4} - 128a_{2,0}^7 a_{1,1}b_{7,3}\Big)y^4x^4
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(-880a_{1,1}^4 b_{9,1} a_{2,0}^4 - 512a_{1,1}^3 b_{10,0} a_{2,0}^5 + 128 b_{9,1} a_{2,0}^8 - 256 a_{2,0}^7 a_{1,1} b_{8,2} \\
&\quad + 880 a_{1,1}^5 b_{10,0} a_{2,0}^3 + 768 a_{2,0}^5 a_{1,1}^3 b_{8,2} + 384 a_{1,1}^2 b_{9,1} a_{2,0}^6 + 128 a_{2,0}^8 b_{7,3} \\
&\quad -544 a_{2,0}^6 a_{1,1}^2 b_{7,3} - 384 b_{10,0} a_{1,1} a_{2,0}^7 + 256 a_{2,0}^7 a_{1,1} b_{6,4}\Big) y^5 x^3 \\
&+ \Big(192 a_{2,0}^6 a_{1,1}^2 b_{6,4} - 800 a_{1,1}^5 b_{9,1} a_{2,0}^3 + 832 a_{1,1}^3 b_{9,1} a_{2,0}^5 + 688 a_{2,0}^4 a_{1,1}^4 b_{8,2} \\
&\quad - 384 a_{1,1} b_{9,1} a_{2,0}^7 + 192 a_{2,0}^7 a_{1,1} b_{7,3} + 800 a_{1,1}^6 b_{10,0} a_{2,0}^2 - 464 a_{2,0}^5 a_{1,1}^3 b_{7,3} + 128 a_{2,0}^8 b_{8,2} \\
&\quad +128 b_{10,0} a_{2,0}^8 - 1040 a_{1,1}^4 b_{10,0} a_{2,0}^4 + 768 a_{1,1}^2 b_{10,0} a_{2,0}^6 - 512 a_{2,0}^6 a_{1,1}^2 b_{8,2}\Big) y^6 x^2 \\
&+ \Big(-384 a_{1,1}^2 b_{9,1} a_{2,0}^6 - 160 a_{2,0}^4 a_{1,1}^4 b_{7,3} + 240 {a_{2,0}}^3 a_{1,1}^5 b_{8,2} + 96 a_{2,0}^6 a_{1,1}^2 b_{7,3} \\
&\quad + 64 a_{2,0}^5 a_{1,1}^3 b_{6,4} - 256 a_{2,0}^5 a_{1,1}^3 b_{8,2} - 480 a_{1,1}^5 b_{10,0} a_{2,0}^3 - 512 b_{10,0} a_{1,1} a_{2,0}^7 \\
&\quad + 128 a_{2,0}^7 a_{1,1} b_{8,2} + 280 a_{1,1}^7 b_{10,0} a_{2,0} + 400 a_{1,1}^4 b_{9,1} a_{2,0}^4 \\
&\quad -280 a_{1,1}^6 b_{9,1} a_{2,0}^2 + 640 a_{1,1}^3 b_{10,0} a_{2,0}^5 + 128 b_{9,1} a_{2,0}^8\Big) y^7 x \\
&+ \Big(16 a_{2,0}^5 a_{1,1}^3 b_{7,3} - 80 a_{1,1}^3 b_{9,1} a_{2,0}^5 + 60 a_{1,1}^5 b_{9,1} a_{2,0}^3 + 120 a_{1,1}^4 b_{10,0} a_{2,0}^4 \\
&\quad - 20 a_{2,0}^3 a_{1,1}^5 b_{7,3} - 70 a_{1,1}^6 b_{10,0} a_{2,0}^2 + 30 a_{2,0}^2 a_{1,1}^6 b_{8,2} + 32 a_{2,0}^6 a_{1,1}^2 b_{8,2} \\
&\quad - 160 a_{1,1}^2 b_{10,0} a_{2,0}^6 - 35 a_{1,1}^7 b_{9,1} a_{2,0} + 8 a_{2,0}^4 a_{1,1}^4 b_{6,4} + 128 b_{10,0} a_{2,0}^8 \\
&\quad +64 a_{1,1} b_{9,1} a_{2,0}^7 - 40 a_{2,0}^4 a_{1,1}^4 b_{8,2} + 35 a_{1,1}^8 b_{10,0}\Big) y^8\Big).
\end{aligned}
\tag{4.11}
$$

For system (4.10), we consider the following linear transformation:

$$
(x, y) = (u, v)\begin{pmatrix} t & -1 \\ 1 & t \end{pmatrix}, \tag{4.12}
$$

where the diagonal elements of the transformation matrix satisfy the equation $a_{2,0}t^2 - a_{1,1}t - a_{2.0} = 0$. Applying the transformation (4.12) to (4.10), we obtain

$$
\dot{u} = -v + u^2 v M \cdot I(u, v), \qquad \dot{v} = u + uv^2 M \cdot I(u, v), \tag{4.13}
$$

where $M = (4a_{2,0}t - a_{1,1} + a_{1,1}t^2)/128 a_{2,0}^9$, $I(u, v) = I_{0,0} + I_{8,0}u^8 + I_{6,2}u^6v^2 + I_{4,4}u^4v^4 + I_{2,6}u^2v^6 + I_{0,8}v^8$, and $I_{0,0}, I_{8,0}, I_{6,2}, I_{4,4}, I_{2,6}, I_{0,8}$ are given in the appendix.

Obviously, system (4.13) satisfies the conditions of symmetry principle, thus the origin is a center for the system. Therefore, the sufficient part of Theorem 2.1 is proved.

5. Center Conditions of $P(2, 2n)$ for $n = 2, 3$ and 4

By similar steps discussed above, we can obtain the following theorem.

Theorem 5.1. *The origin is a center for system $P(2, 8)$ if and only if the following conditions are satisfied:*

$$
\begin{aligned}
0 &= a_{2,0} + a_{0,2}, \\
0 &= 5b_{6,2} + 35b_{0,8} + 5b_{2,6} + 3b_{4,4} + 35b_{8,0},
\end{aligned}
$$

$$
\begin{aligned}
0 &= a_{1,1}b_{2,6} + 3a_{2,0}b_{5,3} + 14a_{1,1}b_{0,8} + 7a_{2,0}b_{1,7} + 3a_{2,0}b_{3,5} - a_{1,1}b_{6,2} + 7a_{2,0}b_{7,1} - 14a_{1,1}b_{8,0},\\
0 &= 7a_{2,0}b_{1,7}a_{1,1} - 14a_{2,0}a_{1,1}b_{7,1} - 3a_{2,0}a_{1,1}b_{5,3} + a_{1,1}^2 b_{6,2} + 21b_{8,0}a_{1,1}^2\\
&\quad + 7a_{1,1}^2 b_{0,8} - 2a_{2,0}^2 b_{6,2} - 56a_{2,0}^2 b_{0,8} - 2a_{2,0}^2 b_{2,6} - 56a_{2,0}^2 b_{8,0},\\
0 &= -3a_{1,1}^2 a_{2,0}b_{5,3} - 48a_{1,1}a_{2,0}^2 b_{8,0} + 18a_{1,1}^3 b_{8,0} - 11a_{1,1}^2 a_{2,0}b_{7,1} + a_{1,1}^3 b_{6,2}\\
&\quad - 64a_{2,0}^2 a_{1,1}b_{0,8} - 4a_{1,1}a_{2,0}^2 b_{6,2} + 10a_{1,1}^3 b_{0,8} + 10a_{1,1}^2 a_{2,0}b_{1,7} - 4a_{2,0}^3 b_{1,7} - 4a_{2,0}^3 b_{7,1},\\
0 &= 8a_{2,0}^3 a_{1,1}^3 b_{7,1} + 16a_{2,0}^4 a_{1,1}^2 b_{8,0} - 5a_{1,1}^5 a_{2,0}b_{7,1} + 5a_{1,1}^6 b_{8,0} - 16a_{2,0}^6 b_{8,0}\\
&\quad - 16a_{2,0}^6 b_{0,8} + 4a_{2,0}^2 a_{1,1}^4 b_{6,2} - 2a_{2,0}^3 a_{1,1}^3 b_{5,3} - 8a_{2,0}^5 a_{1,1}b_{7,1} - 4a_{2,0}^4 a_{1,1}^2 b_{6,2} - 10a_{2,0}^2 a_{1,1}^4 b_{8,0}.
\end{aligned}
\tag{5.1}
$$

The relations above give a complete set of relations in the sense given in [21], that is, this set of relations cover all the cases of the conditions which are sufficient and necessary for the system to be of center type.

Theorem 5.2. *The origin is a center for system $P(2,6)$ if and only if the following conditions are satisfied:*

$$
\begin{aligned}
0 &= a_{2,0} + a_{0,2},\\
0 &= 5b_{0,6} + b_{2,4} + b_{4,2} + 5b_{6,0},\\
0 &= 3a_{2,0}b_{3,3} - a_{1,1}b_{4,2} + 5a_{1,1}b_{0,6} + 5a_{2,0}b_{1,5} + 5a_{2,0}b_{5,1} - 10a_{1,1}b_{6,0},\\
0 &= a_{1,1}^2 b_{0,6} + b_{6,0}a_{1,1}^2 - 4a_{2,0}^2 b_{0,6} - 4a_{2,0}^2 b_{6,0} + a_{2,0}a_{1,1}b_{1,5} - a_{2,0}a_{1,1}b_{5,1},\\
0 &= 18a_{2,0}^2 a_{1,1}^3 b_{6,0} - 16a_{2,0}^3 a_{1,1}^2 b_{5,1} - 24a_{2,0}^4 a_{1,1}b_{6,0} - 3a_{1,1}^5 b_{6,0}\\
&\quad + 3a_{1,1}^4 a_{2,0}b_{5,1} + 8a_{2,0}^5 b_{1,5} + 8a_{2,0}^5 b_{5,1} - 2a_{1,1}^3 a_{2,0}^2 b_{4,2} + 8a_{1,1}a_{2,0}^4 b_{4,2}.
\end{aligned}
\tag{5.2}
$$

The relations above is another form as the result of paper [20].

Theorem 5.3. *The origin is a center for system $P(2,4)$ if and only if the following conditions are satisfied:*

$$
\begin{aligned}
0 &= a_{2,0} + a_{0,2},\\
0 &= 3b_{0,4} + b_{2,2} + 3b_{4,0},\\
0 &= a_{1,1}b_{0,4} + a_{2,0}b_{1,3} + a_{2,0}b_{3,1} - a_{1,1}b_{4,0},\\
0 &= -2a_{2,0}^2 b_{0,4} - a_{1,1}a_{2,0}b_{3,1} + a_{1,1}^2 b_{4,0} - 2a_{2,0}^2 b_{4,0}.
\end{aligned}
\tag{5.3}
$$

The relations above is another form of center condition for the system $P(2,4)$, which has been discussed by Volokitin in paper [19].

Appendix

$I_{0,0}, I_{8,0}, I_{6,2}, I_{4,4}, I_{2,6}, I_{0,8}$

$$
\begin{aligned}
I_{0,0} &= 128a_{2,0}^{9}, \\
I_{8,0} &= 128b_{10,0}a_{2,0}^{8}t^{8} + \left(128a_{2,0}^{7}b_{10,0}a_{1,1} - 128a_{2,0}^{8}b_{9,1}\right)t^{7} \\
&\quad + \left(128a_{2,0}^{6}a_{1,1}^{2}b_{10,0} + 128b_{10,0}a_{2,0}^{8} - 128a_{1,1}b_{9,1}a_{2,0}^{7} + 128a_{2,0}^{8}b_{8,2}\right)t^{6} \\
&\quad + \left(256a_{2,0}^{7}b_{10,0}a_{1,1} - 128a_{2,0}^{8}b_{9,1} + 128a_{2,0}^{5}a_{1,1}^{3}b_{10,0} - 128a_{2,0}^{6}a_{1,1}^{2}b_{9,1}\right. \\
&\qquad \left. -128a_{2,0}^{8}b_{7,3} + 128a_{2,0}^{7}a_{1,1}b_{8,2}\right)t^{5} \\
&\quad + \left(384a_{2,0}^{6}a_{1,1}^{2}b_{10,0} + 128b_{10,0}a_{2,0}^{8} - 256a_{1,1}b_{9,1}a_{2,0}^{7} + 128a_{2,0}^{4}a_{1,1}^{4}b_{10,0} - 128a_{2,0}^{7}a_{1,1}b_{7,3}\right. \\
&\qquad \left. +128a_{2,0}^{8}b_{8,2} + 128a_{2,0}^{6}a_{1,1}^{2}b_{8,2} + 128a_{2,0}^{8}b_{6,4} - 128a_{2,0}^{5}a_{1,1}^{3}b_{9,1}\right)t^{4} \\
&\quad + \left(-256a_{2,0}^{7}a_{1,1}b_{6,4} - 880a_{1,1}^{5}b_{10,0}a_{2,0}^{3} - 128a_{2,0}^{8}b_{7,3} + 512a_{2,0}^{5}a_{1,1}^{3}b_{10,0}\right. \\
&\qquad - 384a_{2,0}^{6}a_{1,1}^{2}b_{9,1} - 768a_{2,0}^{5}a_{1,1}^{3}b_{8,2} - 128a_{2,0}^{8}b_{9,1} + 256a_{2,0}^{7}a_{1,1}b_{8,2} \\
&\qquad \left. +384a_{2,0}^{7}b_{10,0}a_{1,1} + 880a_{1,1}^{4}b_{9,1}a_{2,0}^{4} + 544a_{2,0}^{6}a_{1,1}^{2}b_{7,3}\right)t^{3} \\
&\quad + \left(800a_{2,0}^{2}a_{1,1}^{6}b_{10,0} + 128a_{2,0}^{8}b_{8,2} - 800a_{2,0}^{3}a_{1,1}^{5}b_{9,1} - 512a_{2,0}^{6}a_{1,1}^{2}b_{8,2}\right. \\
&\qquad + 192a_{2,0}^{7}a_{1,1}b_{7,3} + 832a_{2,0}^{5}a_{1,1}^{3}b_{9,1} + 768a_{2,0}^{6}a_{1,1}^{2}b_{10,0} + 128b_{10,0}a_{2,0}^{8} - 464a_{2,0}^{5}a_{1,1}^{3}b_{7,3} \\
&\qquad \left. -384a_{1,1}b_{9,1}a_{2,0}^{7} + 688a_{2,0}^{4}a_{1,1}^{4}b_{8,2} + 192a_{2,0}^{6}a_{1,1}^{2}b_{6,4} - 1040a_{2,0}^{4}a_{1,1}^{4}b_{10,0}\right)t^{2} \\
&\quad + \left(-128a_{2,0}^{7}a_{1,1}b_{8,2} - 64a_{2,0}^{5}a_{1,1}^{3}b_{6,4} - 640a_{2,0}^{5}a_{1,1}^{3}b_{10,0} - 280a_{1,1}^{7}b_{10,0}a_{2,0}\right. \\
&\qquad - 128a_{2,0}^{8}b_{9,1} + 512a_{2,0}^{7}b_{10,0}a_{1,1} - 240a_{2,0}^{3}a_{1,1}^{5}b_{8,2} + 160a_{2,0}^{4}a_{1,1}^{4}b_{7,3} \\
&\qquad - 400a_{1,1}^{4}b_{9,1}a_{2,0}^{4} + 480a_{1,1}^{5}b_{10,0}a_{2,0}^{3} - 96a_{2,0}^{6}a_{1,1}^{2}b_{7,3} \\
&\qquad \left. +384a_{2,0}^{6}a_{1,1}^{2}b_{9,1} + 256a_{2,0}^{5}a_{1,1}^{3}b_{8,2} + 280a_{1,1}^{6}b_{9,1}a_{2,0}^{2}\right)t \\
&\quad + 30a_{2,0}^{2}a_{1,1}^{6}b_{8,2} - 20a_{2,0}^{3}a_{1,1}^{5}b_{7,3} + 128b_{10,0}a_{2,0}^{8} \\
&\quad + 8a_{2,0}^{4}a_{1,1}^{4}b_{6,4} - 70a_{2,0}^{2}a_{1,1}^{6}b_{10,0} - 35a_{1,1}^{7}b_{9,1}a_{2,0} - 40a_{2,0}^{4}a_{1,1}^{4}b_{8,2} \\
&\quad + 60a_{2,0}^{3}a_{1,1}^{5}b_{9,1} - 80a_{2,0}^{5}a_{1,1}^{3}b_{9,1} + 120a_{2,0}^{4}a_{1,1}^{4}b_{10,0} + 16a_{2,0}^{5}a_{1,1}^{3}b_{7,3} \\
&\quad + 64a_{1,1}b_{9,1}a_{2,0}^{7} - 160a_{2,0}^{6}a_{1,1}^{2}b_{10,0} + 35a_{1,1}^{8}b_{10,0} + 32a_{2,0}^{6}a_{1,1}^{2}b_{8,2},
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
I_{6,2} &= \left(128a_{2,0}^{6}a_{1,1}^{2}b_{10,0} + 128b_{10,0}a_{2,0}^{8} - 128a_{1,1}b_{9,1}a_{2,0}^{7} + 128{a_{2,0}}^{8}b_{8,2}\right)t^{8} \\
&\quad + \left(-384a_{2,0}^{8}b_{7,3} + 384a_{2,0}^{5}a_{1,1}^{3}b_{10,0} + 384a_{2,0}^{7}a_{1,1}b_{8,2} - 128a_{2,0}^{7}b_{10,0}a_{1,1}\right. \\
&\qquad \left. -384a_{2,0}^{6}a_{1,1}^{2}b_{9,1} + 512a_{2,0}^{8}b_{9,1}\right)t^{7} \\
&\quad + \left(768a_{2,0}^{6}a_{1,1}^{2}b_{10,0} + 768a_{2,0}^{4}a_{1,1}^{4}b_{10,0} - 768a_{2,0}^{5}a_{1,1}^{3}b_{9,1} + 768a_{2,0}^{8}b_{6,4}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad +2816b_{10,0}a_{2,0}^{8}+768a_{2,0}^{6}a_{1,1}^{2}b_{8,2}-768a_{2,0}^{8}b_{8,2}-768a_{2,0}^{7}a_{1,1}b_{7,3}\Big)t^{6}\\
&+\Big(-1920a_{2,0}^{6}a_{1,1}^{2}b_{9,1}-2048a_{2,0}^{8}b_{9,1}+5440a_{2,0}^{6}a_{1,1}^{2}b_{7,3}\\
&\quad +640a_{2,0}^{7}a_{1,1}b_{8,2}-8800a_{1,1}^{5}b_{10,0}a_{2,0}^{3}+2688a_{2,0}^{7}b_{10,0}a_{1,1}-7680a_{2,0}^{5}a_{1,1}^{3}b_{8,2}\\
&\quad +8800a_{1,1}^{4}b_{9,1}a_{2,0}^{4}-2560a_{2,0}^{7}a_{1,1}b_{6,4}+640a_{2,0}^{8}b_{7,3}+3200a_{2,0}^{5}a_{1,1}^{3}b_{10,0}\Big)t^{5}\\
&+\Big(14528a_{2,0}^{5}a_{1,1}^{3}b_{9,1}-12000a_{2,0}^{3}a_{1,1}^{5}b_{9,1}-9728a_{2,0}^{6}a_{1,1}^{2}b_{8,2}-2048a_{2,0}^{8}b_{6,4}\\
&\quad +12000a_{2,0}^{2}a_{1,1}^{6}b_{10,0}+7296a_{2,0}^{6}a_{1,1}^{2}b_{10,0}+4928a_{2,0}^{7}a_{1,1}b_{7,3}+1792b_{10,0}a_{2,0}^{8}\\
&\quad +10320a_{2,0}^{4}a_{1,1}^{4}b_{8,2}-17648a_{2,0}^{4}a_{1,1}^{4}b_{10,0}-3584a_{1,1}b_{9,1}a_{2,0}^{7}+1792a_{2,0}^{8}b_{8,2}\\
&\quad -6960a_{2,0}^{5}a_{1,1}^{3}b_{7,3}+2880a_{2,0}^{6}a_{1,1}^{2}b_{6,4}\Big)t^{4}\\
&+\Big(-1344a_{2,0}^{5}a_{1,1}^{3}b_{6,4}+640a_{2,0}^{8}b_{7,3}-19840a_{2,0}^{5}a_{1,1}^{3}b_{10,0}+3840a_{2,0}^{7}a_{1,1}b_{6,4}\\
&\quad -21600a_{1,1}^{4}b_{9,1}a_{2,0}^{4}+12544a_{2,0}^{6}a_{1,1}^{2}b_{9,1}-5248a_{2,0}^{7}a_{1,1}b_{8,2}\\
&\quad +3360a_{2,0}^{4}a_{1,1}^{4}b_{7,3}+16896a_{2,0}^{5}a_{1,1}^{3}b_{8,2}-2048a_{2,0}^{8}b_{9,1}\\
&\quad -5880a_{1,1}^{7}b_{10,0}a_{2,0}-10176a_{2,0}^{6}a_{1,1}^{2}b_{7,3}+23280a_{1,1}^{5}b_{10,0}a_{2,0}^{3}\\
&\quad +7552a_{2,0}^{7}b_{10,0}a_{1,1}+5880a_{1,1}^{6}b_{9,1}a_{2,0}^{2}-5040a_{2,0}^{3}a_{1,1}^{5}b_{8,2}\Big)t^{3}\\
&+\Big(-11560a_{2,0}^{2}a_{1,1}^{6}b_{10,0}+840a_{2,0}^{2}a_{1,1}^{6}b_{8,2}+980a_{1,1}^{8}b_{10,0}+2816b_{10,0}a_{2,0}^{8}\\
&\quad -560a_{2,0}^{3}a_{1,1}^{5}b_{7,3}+7808a_{2,0}^{6}a_{1,1}^{2}b_{8,2}+16608a_{2,0}^{4}a_{1,1}^{4}b_{10,0}+6016a_{2,0}^{5}a_{1,1}^{3}b_{7,3}\\
&\quad +4864a_{1,1}b_{9,1}a_{2,0}^{7}+224a_{2,0}^{4}a_{1,1}^{4}b_{6,4}-2304a_{2,0}^{6}a_{1,1}^{2}b_{6,4}-12992a_{2,0}^{5}a_{1,1}^{3}b_{9,1}\\
&\quad +768{a_{2,0}}^{8}b_{6,4}-11392a_{2,0}^{6}a_{1,1}^{2}b_{10,0}+11280a_{2,0}^{3}a_{1,1}^{5}b_{9,1}-9376a_{2,0}^{4}a_{1,1}^{4}b_{8,2}\\
&\quad -980a_{1,1}^{7}b_{9,1}a_{2,0}-768a_{2,0}^{8}b_{8,2}-3072a_{2,0}^{7}a_{1,1}b_{7,3}\Big)t^{2}\\
&+\Big(5440a_{1,1}^{4}b_{9,1}a_{2,0}^{4}+448a_{2,0}^{5}a_{1,1}^{3}b_{6,4}+1680a_{2,0}^{3}a_{1,1}^{5}b_{8,2}-1120a_{2,0}^{4}a_{1,1}^{4}b_{7,3}\\
&\quad +6016a_{2,0}^{5}a_{1,1}^{3}b_{10,0}-3840a_{2,0}^{6}a_{1,1}^{2}b_{9,1}-6000a_{1,1}^{5}b_{10,0}a_{2,0}^{3}+512a_{2,0}^{8}b_{9,1}\\
&\quad -384a_{2,0}^{8}b_{7,3}+1960a_{1,1}^{7}b_{10,0}a_{2,0}-4096a_{2,0}^{5}a_{1,1}^{3}b_{8,2}-1960a_{1,1}^{6}b_{9,1}a_{2,0}^{2}\\
&\quad -768a_{2,0}^{7}a_{1,1}b_{6,4}-2432a_{2,0}^{7}b_{10,0}a_{1,1}+2304a_{2,0}^{6}a_{1,1}^{2}b_{7,3}+1664a_{2,0}^{7}a_{1,1}b_{8,2}\Big)t\\
&+800a_{2,0}^{2}a_{1,1}^{6}b_{10,0}+128a_{2,0}^{8}b_{8,2}-800a_{2,0}^{3}a_{1,1}^{5}b_{9,1}-512a_{2,0}^{6}a_{1,1}^{2}b_{8,2}\\
&+192a_{2,0}^{7}a_{1,1}b_{7,3}+832a_{2,0}^{5}a_{1,1}^{3}b_{9,1}+768a_{2,0}^{6}a_{1,1}^{2}b_{10,0}+128b_{10,0}a_{2,0}^{8}\\
&-464a_{2,0}^{5}a_{1,1}^{3}b_{7,3}-384a_{1,1}b_{9,1}a_{2,0}^{7}+688a_{2,0}^{4}a_{1,1}^{4}b_{8,2}+192a_{2,0}^{6}a_{1,1}^{2}b_{6,4}\\
&-1040a_{2,0}^{4}a_{1,1}^{4}b_{10,0},
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
I_{4,4}=&\Big(128b_{10,0}a_{2,0}^{8}+128a_{2,0}^{8}b_{6,4}-128a_{2,0}^{7}a_{1,1}b_{7,3}+128a_{2,0}^{4}a_{1,1}^{4}b_{10,0}\\
&+128a_{2,0}^{6}a_{1,1}^{2}b_{8,2}+384a_{2,0}^{6}a_{1,1}^{2}b_{10,0}-256a_{1,1}b_{9,1}a_{2,0}^{7}+128a_{2,0}^{8}b_{8,2}\\
&-128a_{2,0}^{5}a_{1,1}^{3}b_{9,1}\Big)t^{8}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(640a_{2,0}^{7}a_{1,1}b_{8,2}-1280a_{2,0}^{6}a_{1,1}^{2}b_{9,1}-3840a_{2,0}^{5}a_{1,1}^{3}b_{8,2}-1280a_{2,0}^{7}a_{1,1}b_{6,4}\\
&\quad-4400a_{1,1}^{5}b_{10,0}a_{2,0}^{3}+1920a_{2,0}^{5}a_{1,1}^{3}b_{10,0}+640a_{2,0}^{7}b_{10,0}a_{1,1}+4400a_{1,1}^{4}b_{9,1}a_{2,0}^{4}\\
&\quad+2720a_{2,0}^{6}a_{1,1}^{2}b_{7,3}\Big)t^{7}\\
&+\Big(14528a_{2,0}^{5}a_{1,1}{}^{3}b_{9,1}-12000a_{2,0}^{3}a_{1,1}^{5}b_{9,1}-9728a_{2,0}^{6}a_{1,1}^{2}b_{8,2}+12000a_{2,0}^{2}a_{1,1}^{6}b_{10,0}\\
&\quad-2048a_{2,0}^{8}b_{6,4}+7296a_{2,0}^{6}a_{1,1}^{2}b_{10,0}+4928a_{2,0}^{7}a_{1,1}b_{7,3}+10320a_{2,0}^{4}a_{1,1}^{4}b_{8,2}\\
&\quad+1792b_{10,0}a_{2,0}^{8}-17648a_{2,0}^{4}a_{1,1}^{4}b_{10,0}-3584a_{1,1}b_{9,1}a_{2,0}^{7}-6960a_{2,0}^{5}a_{1,1}^{3}b_{7,3}\\
&\quad+1792a_{2,0}^{8}b_{8,2}+2880\ a_{1,1}^{2}b_{6,4}\Big)t^{6}\\
&+\Big(9800a_{1,1}^{6}b_{9,1}a_{2,0}^{2}-9800a_{1,1}^{7}b_{10,0}a_{2,0}-19680a_{2,0}^{6}a_{1,1}^{2}b_{7,3}+9600a_{2,0}^{7}b_{10,0}a_{1,1}\\
&\quad+21120a_{2,0}^{6}a_{1,1}^{2}b_{9,1}+43200a_{1,1}^{5}b_{10,0}a_{2,0}^{3}-33920a_{2,0}^{5}a_{1,1}^{3}b_{10,0}-8400a_{2,0}^{3}a_{1,1}^{5}b_{8,2}\\
&\quad+5600a_{2,0}^{4}a_{1,1}^{4}b_{7,3}-8320a_{2,0}^{7}a_{1,1}b_{8,2}-40400a_{1,1}^{4}b_{9,1}a_{2,0}^{4}+32000a_{2,0}^{5}a_{1,1}^{3}b_{8,2}\\
&\quad-2240a_{2,0}^{5}a_{1,1}^{3}b_{6,4}+7680a_{2,0}^{7}a_{1,1}b_{6,4}\Big)t^{5}\\
&+\Big(-43488a^{5_{2,0}}a_{1,1}^{3}b_{9,1}+19680a_{2,0}^{5}a_{1,1}^{3}b_{7,3}+4608a_{2,0}^{8}b_{6,4}+27328a_{2,0}^{6}a_{1,1}^{2}b_{8,2}\\
&\quad+560a_{2,0}^{4}a_{1,1}^{4}b_{6,4}-1400a_{2,0}^{3}a_{1,1}^{5}b_{7,3}+12288b_{10,0}a_{2,0}^{8}-5632a_{2,0}^{8}b_{8,2}\\
&\quad+2450a_{1,1}^{8}b_{10,0}-36900a_{2,0}^{2}a_{1,1}^{6}b_{10,0}-12288a_{2,0}^{7}a_{1,1}b_{7,3}+15744a_{1,1}b_{9,1}a_{2,0}^{7}\\
&\quad-2450a_{1,1}^{7}b_{9,1}a_{2,0}-7680a_{2,0}^{6}a_{1,1}^{2}b_{6,4}-33216a_{2,0}^{6}a_{1,1}^{2}b_{10,0}+54608a_{2,0}^{4}a_{1,1}^{4}b_{10,0}\\
&\quad+36200a_{2,0}^{3}a_{1,1}^{5}b_{9,1}-30320a_{2,0}^{4}a_{1,1}^{4}b_{8,2}+2100a_{2,0}^{2}a_{1,1}^{6}b_{8,2}\Big)t^{4}\\
&+\Big(19680a_{2,0}^{6}a_{1,1}^{2}b_{7,3}+2240a_{2,0}^{5}a_{1,1}^{3}b_{6,4}+9800a_{1,1}^{7}b_{10,0}a_{2,0}-5600a_{2,0}^{4}a_{1,1}^{4}b_{7,3}\\
&\quad-9600a_{2,0}^{7}b_{10,0}a_{1,1}+8320a_{2,0}^{7}a_{1,1}b_{8,2}-9800a_{1,1}^{6}b_{9,1}a_{2,0}^{2}+8400a_{2,0}^{3}a_{1,1}^{5}b_{8,2}\\
&\quad-43200a_{1,1}^{5}b_{10,0}a_{2,0}^{3}-7680a_{2,0}^{7}a_{1,1}b_{6,4}+33920a_{2,0}^{5}a_{1,1}^{3}b_{10,0}+40400a_{1,1}^{4}b_{9,1}a_{2,0}^{4}\\
&\quad-21120a_{2,0}^{6}a_{1,1}^{2}b_{9,1}-32000a_{2,0}^{5}a_{1,1}b_{8,2}\Big)t^{3}\\
&+\Big(14528a_{2,0}^{5}a_{1,1}^{3}b_{9,1}-12000a_{2,0}^{3}a_{1,1}^{5}b_{9,1}-9728a_{2,0}^{6}a_{1,1}^{2}b_{8,2}+12000a_{2,0}^{2}a_{1,1}^{6}b_{10,0}\\
&\quad-2048a_{2,0}^{8}b_{6,4}+7296a_{2,0}^{6}a_{1,1}^{2}b_{10,0}+4928a_{2,0}^{7}a_{1,1}b_{7,3}+10320a_{2,0}^{4}a_{1,1}^{4}b_{8,2}\\
&\quad+1792b_{10,0}a_{2,0}^{8}-17648a_{2,0}^{4}a_{1,1}^{4}b_{10,0}-3584a_{1,1}b_{9,1}a_{2,0}^{7}-6960a_{2,0}^{5}a_{1,1}^{3}b_{7,3}\\
&\quad+1792a_{2,0}^{8}b_{8,2}+2880a_{2,0}^{6}a_{1,1}^{2}b_{6,4}\Big)t^{2}\\
&+\Big(3840a_{2,0}^{5}a_{1,1}^{3}b_{8,2}+1280a_{2,0}^{7}a_{1,1}b_{6,4}+1280a_{2,0}^{6}a_{1,1}^{2}b_{9,1}-640a_{2,0}^{7}a_{1,1}b_{8,2}\\
&\quad+4400a_{1,1}^{5}b_{10,0}a_{2,0}^{3}-640a_{2,0}^{7}b_{10,0}a_{1,1}-2720a_{2,0}^{6}a_{1,1}^{2}b_{7,3}-4400a_{1,1}^{4}b_{9,1}a_{2,0}^{4}\\
&\quad-1920a_{2,0}^{5}a_{1,1}^{3}b_{10,0}\Big)t+128b_{10,0}a_{2,0}^{8}+128a_{2,0}^{8}b_{6,4}-128a_{2,0}^{7}a_{1,1}b_{7,3}\\
&+128a_{2,0}^{4}a_{1,1}^{4}b_{10,0}+128a_{2,0}^{6}a_{1,1}^{2}b_{8,2}+384a_{2,0}^{6}a_{1,1}^{2}b_{10,0}-256a_{1,1}b_{9,1}a_{2,0}^{7}\\
&+128a_{2,0}^{8}b_{8,2}-128a_{2,0}^{5}a_{1,1}^{3}b_{9,1},
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
I_{2,6} = {} & \Big(128a_{2,0}^{8}b_{8,2} + 128b_{10,0}a_{2,0}^{8} + 768a_{2,0}^{6}a_{1,1}^{2}b_{10,0} + 832a_{2,0}^{5}a_{1,1}^{3}b_{9,1} \\
& - 464a_{2,0}^{5}a_{1,1}^{3}b_{7,3} + 800a_{2,0}^{2}a_{1,1}^{6}b_{10,0} - 512a_{2,0}^{6}a_{1,1}^{2}b_{8,2} + 688a_{2,0}^{4}a_{1,1}^{4}b_{8,2} \\
& - 1040a_{2,0}^{4}a_{1,1}^{4}b_{10,0} - 800a_{2,0}^{3}a_{1,1}^{5}b_{9,1} + 192a_{2,0}^{6}a_{1,1}^{2}b_{6,4} + 192a_{2,0}^{7}a_{1,1}b_{7,3} \\
& -384a_{1,1}b_{9,1}a_{2,0}^{7}\Big)t^{8} \\
& + \Big(3840a_{2,0}^{6}a_{1,1}^{2}b_{9,1} - 6016a_{2,0}^{5}a_{1,1}^{3}b_{10,0} - 512a_{2,0}^{8}b_{9,1} \\
& + 4096a_{2,0}^{5}a_{1,1}^{3}b_{8,2} - 448a_{2,0}^{5}a_{1,1}^{3}b_{6,4} + 1120a_{2,0}^{4}a_{1,1}^{4}b_{7,3} - 5440a_{1,1}^{4}b_{9,1}a_{2,0}^{4} \\
& + 384a_{2,0}^{8}b_{7,3} + 2432a_{2,0}^{7}b_{10,0}a_{1,1} - 1664{a_{2,0}}^{7}a_{1,1}b_{8,2} - 1680a_{2,0}^{3}a_{1,1}^{5}b_{8,2} \\
& - 2304a_{2,0}^{6}a_{1,1}^{2}b_{7,3} + 1960a_{1,1}^{6}b_{9,1}a_{2,0}^{2} - 1960a_{1,1}^{7}b_{10,0}a_{2,0} + 768a_{2,0}^{7}a_{1,1}b_{6,4} \\
& +6000a_{1,1}^{5}b_{10,0}a_{2,0}^{3}\Big)t^{7} \\
& + \Big(768a_{2,0}^{8}b_{6,4} - 768a_{2,0}^{8}b_{8,2} + 11280a_{2,0}^{3}a_{1,1}^{5}b_{9,1} \\
& - 11392a_{2,0}^{6}a_{1,1}^{2}b_{10,0} - 980a_{1,1}^{7}b_{9,1}a_{2,0} + 16608a_{2,0}^{4}a_{1,1}^{4}b_{10,0} - 3072a_{2,0}^{7}a_{1,1}b_{7,3} \\
& - 2304a_{2,0}^{6}a_{1,1}^{2}b_{6,4} + 4864a_{1,1}b_{9,1}a_{2,0}^{7} - 12992{a_{2,0}}^{5}a_{1,1}^{3}b_{9,1} - 560a_{2,0}^{3}a_{1,1}^{5}b_{7,3} \\
& + 980a_{1,1}^{8}b_{10,0} + 224a_{2,0}^{4}a_{1,1}^{4}b_{6,4} + 6016a_{2,0}^{5}a_{1,1}^{3}b_{7,3} + 2816b_{10,0}a_{2,0}^{8} \\
& +7808a_{2,0}^{6}a_{1,1}^{2}b_{8,2} + 840a_{2,0}^{2}a_{1,1}^{6}b_{8,2} - 11560a_{2,0}^{2}a_{1,1}^{6}b_{10,0} - 9376a_{2,0}^{4}a_{1,1}^{4}b_{8,2}\Big)t^{6} \\
& + \Big(5248a_{2,0}^{7}a_{1,1}b_{8,2} + 21600a_{1,1}^{4}b_{9,1}a_{2,0}^{4} - 7552a_{2,0}^{7}b_{10,0}a_{1,1} + 1344a_{2,0}^{5}a_{1,1}^{3}b_{6,4} \\
& - 3840a_{2,0}^{7}a_{1,1}b_{6,4} - 23280a_{1,1}^{5}b_{10,0}a_{2,0}^{3} - 640a_{2,0}^{8}b_{7,3} + 10176a_{2,0}^{6}a_{1,1}^{2}b_{7,3} \\
& - 5880a_{1,1}^{6}b_{9,1}a_{2,0}^{2} - 12544a_{2,0}^{6}a_{1,1}^{2}b_{9,1} - 3360a_{2,0}^{4}a_{1,1}^{4}b_{7,3} - 16896a_{2,0}^{5}a_{1,1}^{3}b_{8,2} \\
& +5040a_{2,0}^{3}a_{1,1}^{5}b_{8,2} + 19840a_{2,0}^{5}a_{1,1}^{3}b_{10,0} + 5880a_{1,1}^{7}b_{10,0}a_{2,0} + 2048a_{2,0}^{8}b_{9,1}\Big)t^{5} \\
& + \Big(-2048a_{2,0}^{8}b_{6,4} - 6960a_{2,0}^{5}a_{1,1}^{3}b_{7,3} - 12000a_{2,0}^{3}a_{1,1}^{5}b_{9,1} \\
& + 7296a_{2,0}^{6}a_{1,1}^{2}b_{10,0} - 9728a_{2,0}^{6}a_{1,1}^{2}b_{8,2} + 1792a_{2,0}^{8}b_{8,2} + 12000a_{2,0}^{2}a_{1,1}^{6}b_{10,0} \\
& + 14528a_{2,0}^{5}a_{1,1}^{3}b_{9,1} + 2880a_{2,0}^{6}a_{1,1}^{2}b_{6,4} - 3584a_{1,1}b_{9,1}a_{2,0}^{7} + 1792b_{10,0}a_{2,0}^{8} \\
& +10320a_{2,0}^{4}a_{1,1}^{4}b_{8,2} - 17648a_{2,0}^{4}a_{1,1}^{4}b_{10,0} + 4928a_{2,0}^{7}a_{1,1}b_{7,3}\Big)t^{4} \\
& + \Big(-640a_{2,0}^{7}a_{1,1}b_{8,2} + 2048a_{2,0}^{8}b_{9,1} - 640a_{2,0}^{8}b_{7,3} - 5440a_{2,0}^{6}a_{1,1}^{2}b_{7,3} \\
& - 3200a_{2,0}^{5}{a_{1,1}}^{3}b_{10,0} - 8800a_{1,1}^{4}b_{9,1}a_{2,0}^{4} + 7680a_{2,0}^{5}a_{1,1}^{3}b_{8,2} - 2688a_{2,0}^{7}b_{10,0}a_{1,1} \\
& +1920a_{2,0}^{6}a_{1,1}^{2}b_{9,1} + 8800a_{1,1}^{5}b_{10,0}a_{2,0}^{3} + 2560a_{2,0}^{7}a_{1,1}b_{6,4}\Big)t^{3} \\
& + \Big(-768a_{2,0}^{5}a_{1,1}^{3}b_{9,1} - 768a_{2,0}^{8}b_{8,2} + 768a_{2,0}^{6}a_{1,1}^{2}b_{10,0} + 2816b_{10,0}a_{2,0}^{8} \\
& +768a_{2,0}^{8}b_{6,4} + 768a_{2,0}^{6}a_{1,1}^{2}b_{8,2} - 768a_{2,0}^{7}a_{1,1}b_{7,3} + 768a_{2,0}^{4}a_{1,1}^{4}b_{10,0}\Big)t^{2} \\
& + \Big(384a_{2,0}^{8}b_{7,3} - 384a_{2,0}^{7}a_{1,1}b_{8,2} - 384a_{2,0}^{5}a_{1,1}^{3}b_{10,0} \\
& +384a_{2,0}^{6}a_{1,1}^{2}b_{9,1} - 512a_{2,0}^{8}b_{9,1} + 128a_{2,0}^{7}b_{10,0}a_{1,1}\Big)t \\
& + 128a_{2,0}^{6}a_{1,1}^{2}b_{10,0} + 128b_{10,0}a_{2,0}^{8} - 128a_{1,1}b_{9,1}a_{2,0}^{7} + 128a_{2,0}^{8}b_{8,2},
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
I_{0,8} = \Big(&16a_{2,0}^5 a_{1,1}^3 b_{7,3} - 35a_{1,1}^7 b_{9,1} a_{2,0} + 120a_{2,0}^4 a_{1,1}^4 b_{10,0} + 35a_{1,1}^8 b_{10,0} - 40a_{2,0}^4 a_{1,1}^4 b_{8,2} \\
&+ 32a_{2,0}^6 a_{1,1}^2 b_{8,2} + 128b_{10,0} a_{2,0}{}^8 - 70a_{2,0}^2 a_{1,1}^6 b_{10,0} + 30a_{2,0}^2 a_{1,1}^6 b_{8,2} - 160a_{2,0}^6 a_{1,1}^2 b_{10,0} \\
&-80a_{2,0}^5 a_{1,1}^3 b_{9,1} - 20a^3{}_{2,0} a_{1,1}^5 b_{7,3} + 60a_{2,0}^3 a_{1,1}^5 b_{9,1} + 64a_{1,1} b_{9,1} a_{2,0}^7 + 8a_{2,0}^4 a_{1,1}^4 b_{6,4}\Big) t^8 \\
&+ \Big(240a_{2,0}^3 a_{1,1}^5 b_{8,2} + 128a_{2,0}^7 a_{1,1} b_{8,2} - 256a_{2,0}^5 a_{1,1}^3 b_{8,2} + 640a_{2,0}^5 a_{1,1}^3 b_{10,0} \\
&\quad + 128a_{2,0}^8 b_{9,1} - 160a_{2,0}^4 a_{1,1}^4 b_{7,3} - 480a_{1,1}^5 b_{10,0} a_{2,0}^3 + 96a_{2,0}^6 a_{1,1}^2 b_{7,3} \\
&\quad + 400a_{1,1}^4 b_{9,1} a_{2,0}^4 - 512a_{2,0}^7 b_{10,0} a_{1,1} - 280a_{1,1}^6 b_{9,1} a_{2,0}^2 + 280a_{1,1}^7 b_{10,0} a_{2,0} \\
&\quad +64a_{2,0}^5 a_{1,1}^3 b_{6,4} - 384a_{2,0}^6 a_{1,1}^2 b_{9,1}\Big) t^7 \\
&+ \Big(128a_{2,0}^8 b_{8,2} + 128b_{10,0} a_{2,0}^8 + 768a_{2,0}^6 a_{1,1}^2 b_{10,0} + 832a_{2,0}^5 a_{1,1}^3 b_{9,1} \\
&\quad - 464a_{2,0}^5 a_{1,1}^3 b_{7,3} + 800a_{2,0}^2 a_{1,1}^6 b_{10,0} - 512a_{2,0}^6 a_{1,1}^2 b_{8,2} + 688a_{2,0}^4 a_{1,1}^4 b_{8,2} \\
&\quad - 1040a_{2,0}^4 a_{1,1}^4 b_{10,0} - 800a_{2,0}{}^3 \mathrm{a}_{1,1}^5 b_{9,1} + 192a_{2,0}^6 a_{1,1}^2 b_{6,4} + 192a_{2,0}^7 a_{1,1} b_{7,3} \\
&\quad -384a_{1,1} b_{9,1} a_{2,0}^7\Big) t^6 \\
&+ \Big(-512a_{2,0}^5 a_{1,1}^3 b_{10,0} + 384a_{2,0}^6 a_{1,1}^2 b_{9,1} - 256a_{2,0}^7 a_{1,1} b_{8,2} \\
&\quad + 128a_{2,0}^8 b_{7,3} - 544a_{2,0}^6 a_{1,1}^2 b_{7,3} + 768a_{2,0}^5 a_{1,1}^3 b_{8,2} + 256a_{2,0}^7 a_{1,1} b_{6,4} + 128a_{2,0}^8 b_{9,1} \\
&\quad +880a_{1,1}^5 b_{10,0} a_{2,0}^3 - 384a_{2,0}^7 b_{10,0} a_{1,1} - 880a_{1,1}^4 b_{9,1} a_{2,0}^4\Big) t^5 \\
&+ \Big(128a_{2,0}^6 a_{1,1}^2 b_{8,2} + 128b_{10,0} a_{2,0}^8 - 256a_{1,1} b_{9,1} a_{2,0}^7 + 128a_{2,0}^8 b_{6,4} + 128a_{2,0}^4 a_{1,1}^4 b_{10,0} \\
&\quad +128a_{2,0}^8 b_{8,2} + 384a_{2,0}^6 a_{1,1}^2 b_{10,0} - 128a_{2,0}^7 a_{1,1} b_{7,3} - 128a_{2,0}^5 a_{1,1}^3 b_{9,1}\Big) t^4 \\
&+ \Big(-128a_{2,0}^5 a_{1,1}^3 b_{10,0} - 256a_{2,0}^7 b_{10,0} a_{1,1} + 128a_{2,0}^6 a_{1,1}^2 b_{9,1} \\
&\quad +128a_{2,0}^8 b_{9,1} + 128a_{2,0}^8 b_{7,3} - 128a_{2,0}^7 a_{1,1} b_{8,2}\Big) t^3 \\
&+ \Big(128a_{2,0}^6 a_{1,1}^2 b_{10,0} + 128b_{10,0} a_{2,0}^8 - 128a_{1,1} b_{9,1} a_{2,0}^7 + 128a_{2,0}^8 b_{8,2}\Big) t^2 \\
&+ \Big(128a_{2,0}^8 b_{9,1} - 128a_{2,0}^7 b_{10,0} a_{1,1}\Big) t + 128b_{10,0} a_{2,0}^8.
\end{aligned}
\tag{A.5}
$$

Acknowledgments

The authors are very grateful to the referees for the useful remarks and suggestions. This paper is partially supported by a National Key Basic Research of China (Grant no. 2004CB318000).

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