STRONG ASYMPTOTICS FOR L_p EXTREMAL POLYNOMIALS OFF A COMPLEX CURVE

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We study the asymptotic behavior of $L_p(\sigma)$ extremal polynomials with respect to a measure of the form $\sigma = \alpha + \gamma$, where α is a measure concentrated on a rectifiable Jordan curve in the complex plane and γ is a discrete measure concentrated on an infinite number of mass points.

1. Introduction

Let F be a compact subset of the complex plane $\mathbb{C}$ and let B be a metric space of functions defined on F. We suppose that B contains the set of monic polynomials. Then the extremal or general Chebyshev polynomial T_n of degree n is a monic polynomial that minimizes the distance between zero and the set of all monic polynomials of degree n, that is,

$$\operatorname{dist}(T_n, 0) = \min\{\operatorname{dist}(Q_n, 0) : Q_n(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0\} = m_n(B). \tag{1.1}$$

Recently, a series of results concerning the asymptotic of the extremal polynomials was established for the case of $B = L_p(F,\sigma)$, $1 \le p \le \infty$, where σ is a Borel measure on F; see, for example, [3, 7, 8, 12]. When $p = 2$, we have the special case of orthogonal polynomials with respect to the measure σ. A lot of research work has been done on this subject; see, for example, [1, 4, 5, 9, 11, 13]. The case of the spaces $L_p(F,\sigma)$, where $0 < p < \infty$ and F is a closed rectifiable Jordan curve with some smoothness conditions, was studied by Geronimus [2]. An extension of Geronimus's result has been given by Kaliaguine [3] who found asymptotics when $0 < p < \infty$ and the measure σ has a decomposition of the form

$$\sigma = \alpha + \gamma, \tag{1.2}$$

where α is a measure supported on a closed rectifiable Jordan curve E as defined in [2] and γ is a discrete measure with a finite number of mass points.

In this paper, we generalize Kaliaguine's work [3] in the case where $1 \le p < \infty$ and the support of the measure σ is a rectifiable Jordan curve E plus an infinite discrete set of

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mass points which accumulate on E. More precisely, $\sigma = \alpha + \gamma$, where the measure α and its support E are defined as in [3], that is,

$$d\alpha(\xi) = \rho(\xi)|d\xi|, \quad \rho \geq 0,\ \rho \in L^1(E, |d\xi|); \tag{1.3}$$

γ is a discrete measure concentrated on $\{z_k\}_{k=1}^{\infty} \subset \text{Ext}(E)$ ($\text{Ext}(E)$ is the exterior of E), that is,

$$\gamma = \sum_{k=1}^{+\infty} A_k \delta(z - z_k), \quad A_k > 0,\ \sum_{k=1}^{+\infty} A_k < \infty. \tag{1.4}$$

Note that the result of the special case $p = 2$ is also a generalization of [4]. More precisely, in the proof of Theorem 4.3, we show that condition [4, page 265, (17)] imposed on the points $\{z_k\}_{k=1}^{\infty}$ is redundant.

2. The $H^p(\Omega,\rho)$ spaces ($1 \leq p < \infty$)

Let E be a rectifiable Jordan curve in the complex plane, $\Omega = \text{Ext}(E)$, $G = \{z \in \mathbb{C},\ |z| > 1\}$ (∞ belongs to Ω and G).

We denote by Φ the conformal mapping of Ω into G with $\Phi(\infty) = \infty$ and $1/C(E) = \lim_{z\to\infty}(\Phi(z)/z) > 0$, where $C(E)$ is the logarithmic capacity of E. We denote $\Psi = \Phi^{-1}$.

Let ρ be an integrable nonnegative weight function on E satisfying the Szegö condition

$$\int_E (\log \rho(\xi))\,|\Phi'(\xi)|\,|d\xi| > -\infty. \tag{2.1}$$

Condition (2.1) allows us to construct the so-called Szegö function D associated with the curve E and the weight function ρ:

$$D(z) = \exp\left\{-\frac{1}{2p\pi}\int_{-\pi}^{+\pi} \frac{w + e^{it}}{w - e^{it}} \log\left(\frac{\rho(\xi)}{|\Phi'(\xi)|}\right) dt\right\} \quad (w = \Phi(z),\ \xi = \Psi(e^{it})) \tag{2.2}$$

such that

(i) D is analytic in Ω, $D(z) \neq 0$ in Ω, and $D(\infty) > 0$;
(ii) $|D(\xi)|^{-p}|\Phi'(\xi)| = \rho(\xi)$ a.e. on E, where $D(\xi) = \lim_{z\to\xi} D(z)$.

We say that $f \in H^p(\Omega,\rho)$ if and only if f is analytic in Ω and $f_0\Psi/D_0\Psi \in H^p(G)$.

For $1 \leq p < \infty$, $H^p(\Omega,\rho)$ is a Banach space. Each function $f \in H^p(\Omega,\rho)$ has limit values a.e. on E and

$$\|f\|_{H^p(\Omega,\rho)}^p = \int_E |f(\xi)|^p \rho(\xi)|d\xi| = \lim_{R\to 1^+} \frac{1}{R}\int_{E_R} \frac{|f(z)|^p}{|D(z)|^p}|\Phi'(z)dz|, \tag{2.3}$$

where $E_R = \{z \in \Omega : |\Phi(z)| = R\}$.

Lemma 2.1 [3]. *If $f \in H^p(\Omega,\rho)$, then for every compact set $K \subset \Omega$, there is a constant C_K such that*

$$\sup\{|f(z)| : z \in K\} \leq C_K \|f\|_{H^p(\Omega,\rho)}. \tag{2.4}$$

3. The extremal problems

Let $1 \leq p < \infty$; we denote $\sigma_l = \alpha + \sum_{k=1}^{l} A_k \delta(z - z_k)$ and by $\mu(\rho)$, $\mu(l)$, $\mu^{\infty}(\rho)$, $m_{n,p}(\rho)$, $m_{n,p}(l)$, and $m_{n,p}(\sigma)$ the extremal values of the following problems, respectively:

$$\mu(\rho) = \inf \{\|\varphi\|_{H^p(\Omega,\rho)}^p : \varphi \in H^p(\Omega,\rho),\ \varphi(\infty) = 1\}, \tag{3.1}$$

$$\mu(l) = \inf \{\|\varphi\|_{H^p(\Omega,\rho)}^p : \varphi \in H^p(\Omega,\rho),\ \varphi(\infty) = 1,\ \varphi(z_k) = 0,\ k = 1,2,\dots,l\}, \tag{3.2}$$

$$\mu^{\infty}(\rho) = \inf \{\|\varphi\|_{H^p(\Omega,\rho)}^p : \varphi \in H^p(\Omega,\rho),\ \varphi(\infty) = 1,\ \varphi(z_k) = 0,\ k = 1,2,\dots\}, \tag{3.3}$$

$$m_{n,p}(\rho) = \min \{||Q_n||_{L_p(\alpha)} : Q_n(z) = z^n + \cdots\}, \tag{3.4}$$

$$m_{n,p}(l) = \min \{||Q_n||_{L_p(\sigma_l)} : Q_n(z) = z^n + \cdots\}, \tag{3.5}$$

$$m_{n,p}(\sigma) = \min \{||Q_n||_{L_p(\sigma)} : Q_n(z) = z^n + \cdots\}. \tag{3.6}$$

As usual,

$$\|f\|_{L_p(\sigma)} := \left(\int_E |f(\xi)|^p d\sigma(\xi)\right)^{1/p}. \tag{3.7}$$

We denote by φ^* and ψ^{∞} the extremal functions of problems (3.1) and (3.3), respectively.

Let $T_{n,p}^l(z)$ and $T_{n,p}(z)$ be the extremal polynomials with respect to the measures σ_l and σ, respectively, that is,

$$||T_{n,p}^l||_{L_p(\sigma_l)} = m_{n,p}(l), \qquad ||T_{n,p}||_{L_p(\sigma)} = m_{n,p}(\sigma). \tag{3.8}$$

Lemma 3.1. *Let $\varphi \in H^p(\Omega,\rho)$ such that $\varphi(\infty) = 1$ and $\varphi(z_k) = 0$ for $k = 1,2,\dots,$ and let*

$$B_{\infty}(z) = \prod_{k=1}^{+\infty} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \frac{|\Phi(z_k)|^2}{\Phi(z_k)} \tag{3.9}$$

be the Blaschke product. Then

(i) $B_{\infty} \in H^p(\Omega,\rho)$, $B_{\infty}(\infty) = 1$, $|B_{\infty}(\xi)| = \prod_{k=1}^{+\infty} |\Phi(z_k)|$ $(\xi \in E)$;
(ii) $\varphi/B_{\infty} \in H^p(\Omega,\rho)$ *and* $(\varphi/B_{\infty})(\infty) = 1$.

Proof. This lemma is proved for $p = 2$ in [1]. The proof is based on the fact that if $f \in H^2(U)$, where $U = \{z \in \mathbb{C},\ |z| < 1\}$, and B is the Blaschke product formed by the zeros of f, then $f/B \in H^2(U)$. It remains true in $H^p(U)$ for $1 \leq p < \infty$; see [6, 10]. □

Lemma 3.2. *An extremal function ψ^∞ of problem (3.3) is given by $\psi^\infty = \varphi^* B_\infty$; in addition,*

$$\mu^\infty(\rho) = \prod_{k=1}^{+\infty} \left(|\Phi(z_k)| \right)^p \mu(\rho). \tag{3.10}$$

Proof. If $\varphi \in H^p(\Omega,\rho)$, $\varphi(\infty) = 1$ and $\varphi(z_k) = 0$ for $k = 1,2,\dots$. Then by Lemma 2.1, we have $f = \varphi/B_\infty \in H^p(\Omega,\rho)$, $f(\infty) = 1$, and $|B_\infty(\xi)| = \prod_{k=1}^{+\infty} |\Phi(z_k)|$ for $\xi \in E$. These lead to

$$\|f\|^p = \left(\prod_{k=1}^{+\infty} |\Phi(z_k)| \right)^{-p} \|\varphi\|^p. \tag{3.11}$$

Thus

$$\mu(\rho) \le \left(\prod_{k=1}^{+\infty} |\Phi(z_k)| \right)^{-p} \mu^\infty(\rho). \tag{3.12}$$

On the other hand, since the function $\psi^\infty = \varphi^* B_\infty \in H^p(\Omega,\rho)$, $\varphi(\infty) = 1$ and $\varphi(z_k) = 0$ for $k = 1,2,\dots$, we get

$$\mu^\infty(\rho) \le \|\psi^\infty\|^p = \left(\prod_{k=1}^{+\infty} |\Phi(z_k)| \right)^{p} \mu(\rho). \tag{3.13}$$

Finally, the lemma follows from (3.12) and (3.13). □

4. The main results

Definition 4.1. A measure $\sigma = \alpha + \gamma$ is said to belong to a class A if the absolutely continuous part α and the discrete part γ satisfy conditions (1.3), (1.4), and (2.1) and Blaschke's condition, that is,

$$\sum_{k=1}^{+\infty} \left(|\Phi(z_k)| - 1 \right) < \infty. \tag{4.1}$$

We denote $\lambda_n = \Phi^n - \Phi_n$, where Φ_n is the polynomial part of the Laurent expansion of Φ^n in the neighborhood of infinity.

Definition 4.2 [2]. A rectifiable curve E is said to be of class Γ if $\lambda_n(\xi) \to 0$ uniformly on E.

Theorem 4.3. *Let a measure $\sigma = \alpha + \gamma$ satisfy conditions (1.3), (1.4) and Blaschke's condition (4.1); then*

$$\lim_{l \to +\infty} m_{n,p}(l) = m_{n,p}(\sigma). \tag{4.2}$$

Proof. The extremal property of $T_{n,p}(z_k)$ gives

$$\begin{aligned}(m_{n,p}(\sigma))^p &\le \int_E |T^l_{n,p}(\xi)|^p \rho(\xi)|d\xi| + \sum_{k=1}^{l} A_k |T^l_{n,p}(z_k)|^p + \sum_{k=l+1}^{+\infty} A_k |T^l_{n,p}(z_k)|^p \\ &= (m_{n,p}(l))^p + \sum_{k=l+1}^{+\infty} A_k |T^l_{n,p}(z_k)|^p.\end{aligned} \tag{4.3}$$

On the other hand, from the extremal property of $T^l_{n,p}(z_k)$, we can write

$$\begin{aligned}m_{n,p}(l) &\le \left(\int_E |T_{n,p}(\xi)|^p \rho(\xi)|d\xi| + \sum_{k=1}^{l} A_k |T_{n,p}(z_k)|^p\right)^{1/p} \\ &\le m_{n,p}(\sigma) = C_n < \infty.\end{aligned} \tag{4.4}$$

Note that C_n does not depend on l; so for all $l = 1,2,3,\ldots,$

$$\left(\int_E |T^l_{n,p}(\xi)|^p \rho(\xi)|d\xi|\right)^{1/p} < C_n. \tag{4.5}$$

This implies that there is a constant C'_n independent of l such that for all $l = 1,2,3,\ldots,$

$$\max\{|T^l_{n,p}(z)|^p : |z| \le 2\} < C'_n. \tag{4.6}$$

Using (4.6) in (4.3) for large enough l with (4.4), we get

$$(m_{n,p}(l))^p \le (m_{n,p}(\sigma))^p \le (m_{n,p}(l))^p + C'_n \sum_{k=l+1}^{+\infty} A_k. \tag{4.7}$$

Letting $l \to \infty$, we obtain

$$\lim_{l\to\infty} m_{n,p}(l) = m_{n,p}(\sigma). \tag{4.8}$$

□

THEOREM 4.4. *Let $1 \le p < \infty$, $E \in \Gamma$, and let $\sigma = \alpha + \gamma$ be a measure which belongs to A. In addition, for all n and l,*

$$m_{n,p}(l) \le \left(\prod_{k=1}^{l} |\Phi(z_k)|\right) m_{n,p}(\rho). \tag{4.9}$$

Then the monic orthogonal polynomials $T_{n,p}(z)$ with respect to the measure σ have the following asymptotic behavior:

(i) $\lim_{n\to\infty}(m_{n,p}(\sigma)/(C(E))^n) = (\mu^\infty(\rho))^{1/p}$;
(ii) $\lim_{n\to\infty} \|T_{n,p}/[C(E)\Phi]^n - \psi^\infty\|_{H^p(\Omega,\rho)} = 0$;
(iii) $T_{n,p}(z) = [C(E)\Phi(z)]^n[\psi^\infty(z) + \varepsilon_n(z)]$,

where $\varepsilon_n(z) \to 0$ uniformly on compact subsets of Ω and ψ^∞ is an extremal function of problem (3.3).

Remark 4.5. For $p = 2$ and E the unit circle, condition (4.9) is proved (see [5, Theorem 5.2]). In this case, this condition can be written as $\gamma_n/\gamma_n^l \leq \prod_{k=1}^{l} |z_k|$, where $\gamma_n^l = 1/m_{n,2}(l)$ and $\gamma_n = 1/m_{n,2}(\rho)$ are, respectively, the leading coefficients of the orthonormal polynomials associated to the measures σ_l and α.

Proof of Theorem 4.4. Taking the limit when l tends to infinity in (4.9) and using Theorem 4.3, we get

$$\frac{m_{n,p}(\sigma)}{(C(E))^n} \leq \left(\prod_{k=1}^{+\infty} |\Phi(z_k)|\right) \frac{m_{n,p}(\rho)}{(C(E))^n}. \tag{4.10}$$

On the other hand, it is proved in [2] that

$$\lim_{n\to\infty} \frac{m_{n,p}(\rho)}{(C(E))^n} = (\mu(\rho))^{1/p}. \tag{4.11}$$

Using (4.10), (4.11), and Lemma 3.2, we obtain

$$\limsup_{n\to\infty} \frac{m_{n,p}(\sigma)}{(C(E))^n} \leq \left(\prod_{k=1}^{+\infty} |\Phi(z_k)|\right)(\mu(\rho))^{1/p} = (\mu^\infty(\rho))^{1/p}. \tag{4.12}$$

It is well known that (see [3, page 231])

$$\forall l > 0, \quad \mu(l) = \mu(\rho)\left(\prod_{k=1}^{l} |\Phi(z_k)|\right)^p. \tag{4.13}$$

We also have (see [3, Theorem 2.2])

$$\lim_{n\to\infty} \frac{m_{n,p}(l)}{(C(E))^n} = (\mu(l))^{1/p}. \tag{4.14}$$

From (4.4), we deduce that

$$\forall l > 0, \quad \frac{m_{n,p}(\sigma)}{(C(E))^n} \geq \frac{m_{n,p}(l)}{(C(E))^n}. \tag{4.15}$$

By passing to the limit when n tends to infinity in (4.15) and taking into account (4.13) and (4.14), we get

$$\forall l > 0, \quad \liminf_{n\to\infty} \frac{m_{n,p}(\sigma)}{(C(E))^n} \geq \left(\prod_{k=1}^{l} |\Phi(z_k)|\right)(\mu(\rho))^{1/p}. \tag{4.16}$$

Finally, by using Lemma 3.2, we obtain

$$\liminf_{n\to\infty}\frac{m_{n,p}(\sigma)}{(C(E))^n}\geq\left(\prod_{k=1}^{+\infty}|\Phi(z_k)|\right)(\mu(\rho))^{1/p}=(\mu^\infty(\sigma))^{1/p}. \tag{4.17}$$

Inequalities (4.12) and (4.17) prove Theorem 4.4(i).
We obtain (ii) by proceeding as in [3, pages 234, 235].
To prove (iii), we consider the function

$$\varepsilon_n=\frac{T_{n,p}}{[C(E)\Phi]^n}-\psi^\infty \tag{4.18}$$

which belongs to the space $H^p(\Omega,\rho)$. Then by applying Lemma 2.1, we obtain

$$\begin{aligned}\sup&\left\{\left|\frac{T_{n,p}(z)}{[C(E)\Phi(z)]^n}-\psi^\infty(z)\right|:z\in K\right\}\\&=\sup\{|\varepsilon_n(z)|:z\in K\}\leq C_K\|\varepsilon_n\|_{H^p(\Omega,\rho)}\longrightarrow 0\end{aligned} \tag{4.19}$$

for all compact subsets K of Ω. This achieves the proof of the theorem. □

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