ON SOME SUFFICIENT CONDITIONS FOR THE BLOW-UP SOLUTIONS OF THE NONLINEAR GINZBURG-LANDAU-SCHRÖDINGER EVOLUTION EQUATION

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Investigation of the blow-up solutions of the problem in finite time of the first mixed-value problem with a homogeneous boundary condition on a bounded domain of n-dimensional Euclidean space for a class of nonlinear Ginzburg-Landau-Schrödinger evolution equation is continued. New simple sufficient conditions have been obtained for a wide class of initial data under which collapse happens for the given new values of parameters.

1. Introduction

In the present paper, the investigation of the blow-up of solutions of the problem for the first mixed-value problem of the Ginzburg-Landau-Schrödinger equation is continued.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. We consider the following mixed-value problem:

$$u_t = (\alpha + i\beta)\Delta u + f(u) + (\eta + i\mu)u, \quad x \in \Omega,\ t > 0, \tag{1.1a}$$

$$u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \tag{1.1b}$$

$$u(x,t)|_{\partial\Omega} = 0, \quad t \geq 0. \tag{1.1c}$$

Here $f(u) = (\omega + i\gamma)|u|^{1+\rho}$, $\{\alpha,\beta,\omega,\gamma,\eta,\mu\} \in \mathbb{R}$, $\rho \in \mathbb{R}_+$, $\alpha^2 + \beta^2 \neq 0$, and $\omega^2 + \gamma^2 \neq 0$.

We meet (1.1a) in different fields of applied physics, in nonlinear quantum mechanics, and in the theory of propagation of light waves in a nonlinear media (see, e.g., [4, 14]). For $\alpha = 0$, $\eta = 0$, $f(u) = i\gamma|u|^\rho u$, $\gamma\beta > 0$, and $\rho n \geq 4$, the question on blow-up solutions of problem (1.1) is considered in the case $\mu = 0$ in [6] and in the case $\mu > 0$ in [7]; the Cauchy problem for (1.1a) in the case $\mu = 0$ is considered in [2, 6, 9, 11], and so forth. In the case $\alpha = 0$, $\beta = 1$, $\eta = 0$, $\mu = 0$, $f(u) = \gamma|u|^\rho u$, $\rho > 0$, $\gamma \neq 0$, papers by P. L. Lions, T. Cazenave, B. Weissler Fred, W. A. Strauss, J. Shatah, T. Kato, F. Merle, M. Tsutsumi, Y. Tsutsumi, H. Nawa, J. Ginibre, G. Velo, and so forth (see the references in [9, 10]) are devoted to the different properties of the solutions of the Cauchy problem for (1.1a).

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The global solvability of problem (1.1) for $\alpha = 0, \beta = 1, \eta = 0, \mu = 0, f(u) = |u|^{\rho}u, \rho > 0$ is investigated by Lions in [3]; for $\alpha = 0, \beta = -1, \eta = 0, \mu = 0, f(u) = -i\nu_1|u|^{\rho_1}u - i\nu_2|u|^{\rho_2}u$, $\nu_2 > 0$, $\nu_1 \in \mathbb{R}$, $\rho_1 > 0$, $\rho_2 > 0$ by Vladimirov in [12], the author in [7], and others; for $\alpha = 0$, $\beta = 1$, $\eta = 0$, $\mu = 0$, $f(u) = \gamma|u|^2u$, $\gamma \neq 0$, $n = 2$ by Brézis and Gallouet in [1]; for $f(u) = |u|^{\rho}u$, $\rho > 0$ in [5]; and so forth.

In [8], the problem on blow-up of solutions of problem (1.1) is considered and in the case $a_0 = \lambda_0\alpha - \eta \neq 0$, where λ_0 is the first eigenvalue of the spectral problem (2.1), sufficient conditions on u_0 are suggested under which collapse happens for the given values of the parameters of (1.1a). The conditions on u_0 suggested in [8] are cumbersome. In the present paper, the simpler sufficient conditions on u_0 are offered under which in any value a_0 for given values of the parameters of (1.1a), the solutions of the problem (1.1) end with singularity.

The obtained results are stated in Theorems 3.1, 3.2, and 3.3. The proofs of these theorems are based on the Lemma 4.1, which is deduced from the equality for the solutions of the problem (1.1) and nontrivial solutions of the spectral problem (2.1) (Statement 4.2).

2. Notations

Let λ_0 be the first eigenvalue and $v_0(x)$ the corresponding first eigenfunction of the following problem:

$$\begin{gathered} \Delta v + \lambda v = 0, \quad x \in \Omega, \\ v|_{\partial\Omega} = 0. \end{gathered} \tag{2.1}$$

It is known that $\lambda_0 > 0$, $v_0(x) \in C^2(\Omega) \cap C(\overline{\Omega})$, and $v_0(x) > 0$ for all $x \in \Omega$ (see, e.g., [13, page 434]). Without loss of the generality, we will consider that

$$\int_{\Omega} v_0(x)dx = 1. \tag{2.2}$$

Notations. $a_0 = \lambda_0\alpha - \eta$, $b_0 = \lambda_0\beta - \mu$, $k_1 = c_1\omega + c_2\gamma$, $k_2 = c_1\gamma - c_2\omega$, where $\{c_1, c_2\} \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$, for $c_2 = 0$ we are to set $c_1 = 1$, for $c_1 = 0 - c_2 = 1$; $\tilde{y}_0 = c_1(\operatorname{Re} u_0, v_0) + c_2(\operatorname{Im} u_0, v_0)$, $(\cdot,\cdot)$ is a scalar product in $L_2(\Omega)$; $\|\cdot\|$ is a norm in $L_2(\Omega)$, $\|\cdot\|_q$ is a norm in $L_q(\Omega)$, $q \geq 1$, $\overset{\circ}{W}{}_2^1(\Omega)$, $W_2^2(\Omega)$ are the Sobolev spaces, $B(\Omega) \equiv \overset{\circ}{W}{}_2^1(\Omega) \cap W_2^2(\Omega) \cap L_{\rho+1}(\Omega)$.

We pass to the statement of the obtained results.

3. The results

We formulate the results in the form of the following three theorems.

Theorem 3.1. *Let λ_0 be the first eigenvalue, let $v_0(x)$ be a corresponding first eigenfunction of problem (2.1), satisfying the norm condition (2.2), and let $b_0 \neq 0$, $k_1 \neq 0$, and $\varphi =$ $\operatorname{sign}(k_1k_2b_0)\arcsin(|k_2|/\sqrt{k_1^2 + k_2^2})$. Further, let the initial function $u_0 \in B(\Omega)$ be such that*

$$y_0 = \operatorname{sign}(k_1)\,\tilde{y}_0 \tag{3.1}$$

satisfies the condition

$$y_0 \geq \left[\frac{\chi}{(1-\sin\varphi)}\right]^{1/\rho}; \tag{3.2}$$

here,

$$\chi = \begin{cases} \dfrac{|b_0|}{\rho\chi_0} & \text{for } a_0 \leq 0, \\ |b_0| \dfrac{\exp(a_0\rho t_p)}{\rho\chi_0} & \text{for } a_0 > 0, \end{cases} \tag{3.3}$$

where

$$\chi_0 = \frac{\sqrt{\gamma^2+\omega^2}\sqrt{c_1^2+c_2^2}}{(|c_1|+|c_2|)^{\rho+1}}, \qquad t_\rho = \frac{\pi/2-\varphi}{|b_0|}. \tag{3.4}$$

Then the solution $u(x,t)$ of problem (1.1) from the class $C([0,T],B(\Omega))\cap C^1([0,T],L_2(\Omega))$ blows up in a finite time $t_{\max}$, that is, for $t\to t_{\max}^-$,

$$\begin{aligned} \|u(\cdot,t)\| &\longrightarrow \infty, & \|u(\cdot,t)\|_{\rho+1} &\longrightarrow \infty, \\ \|\nabla u(\cdot,t)\| &\longrightarrow \infty, & \|u(\cdot,t)\|_{W_2^2(\Omega)} &\longrightarrow \infty. \end{aligned} \tag{3.5}$$

Moreover, $t_{\max} \leq t_k \leq t_\rho$, where

$$t_k = \frac{\arcsin(\sin\varphi+\chi/y_0^\rho)-\varphi}{|b_0|}. \tag{3.6}$$

Theorem 3.2. *Let λ_0 be the first eigenvalue, let $v_0(x)$ be a corresponding eigenfunction of problem (2.1), satisfying the norm condition (2.2), and let $b_0\neq 0$, $k_2\neq 0$, $\varphi=\arccos(|k_2|/\sqrt{k_1^2+k_2^2})$, and $\operatorname{sign}(k_1k_2b_0)=-1$ for $k_1\neq 0$. Further, let the initial function $u_0\in B(\Omega)$ be such that*

$$y_0 = -\operatorname{sign}(k_2b_0)\,\tilde{y}_0 \tag{3.7}$$

satisfies the condition

$$y_0 \geq \left[\frac{\chi}{(1+\cos\varphi)}\right]^{1/\rho}, \tag{3.8}$$

where χ is determined by formula (3.3) in which $t_\rho=(\pi-\varphi)/|b_0|$, and χ_0 is given by relation (3.4).

Then the statement of Theorem 3.1 is valid, where

$$t_k = \frac{\arccos(\cos\varphi-\chi/y_0^\rho)-\varphi}{|b_0|}. \tag{3.9}$$

Theorem 3.3. *Let λ_0 be the first eigenvalue and let $v_0(x)$ be a corresponding first eigenfunction of problem (2.1), satisfying the norm condition (2.2). Let $b_0 = 0$, $k_1 \neq 0$, $\omega \neq 0$, and $\gamma \neq 0$ (for $\gamma = 0$, $c_2 = 0$ for $\omega = 0 - c_1 = 0$ has to be taken). Let the initial function $u_0(x) \in B(\Omega)$ be such that*

$$y_0 = \text{sign}\,(k_1)\,\tilde{y}_0 \tag{3.10}$$

in the case $a_0 > 0$ satisfies the condition

$$y_0 > \left[\frac{a_0}{\chi_0}\right]^{1/\rho}, \tag{3.11}$$

where

$$\chi_0 = \frac{|k_1|}{(|c_1| + |c_2|)^{\rho+1}}; \tag{3.12}$$

in the case $a_0 \leq 0$ satisfies the condition $y_0 > 0$.

Then the statement of Theorem 3.1 is valid, where

$$t_k = -\frac{1}{a_0\rho}\ln\left(1 - \frac{a_0}{\chi_0 y_0^{\rho}}\right) \tag{3.13}$$

in the case $a_0 \neq 0$; in the case $a_0 = 0$, $t_k = 1/\rho\chi_0 y_0^{\rho}$.

4. Outline of the proof

4.1. Let $u(x,t) \in C([0,t_{\max}),B(\Omega)) \cap C^1([0,t_{\max}),L_2(\Omega))$ be the maximal solution of problem (1.1) in the sense that the interval $[0,t_{\max})$ is a maximal interval of the existence of the solution for problem (1.1) from the indicated class. Clearly, $t_{\max}$ is either finite or infinite. By proving the above stated theorems, we use the following lemma.

Lemma 4.1. *Let $u_0(x) \in B(\Omega)$, $u(x,t)$ be a maximal solution of problem (1.1) from the class $C([0,t_{\max}),B(\Omega)) \cap C^1([0,t_{\max}),L_2(\Omega))$, let λ_0 be the first eigenvalue, and let $v_0(x)$ be a corresponding first eigenfunction of problem (2.1), satisfying the norm condition (2.2). On the interval $[0,t_{\max})$, the following functions are defined:*

$$\begin{aligned} y_1(t) &= \text{Re}\,[(u,v_0)\exp(zt)], \\ y_2(t) &= \text{Im}\,[(u,v_0)\exp(zt)], \end{aligned} \tag{4.1}$$

where $z = a_0 + ib_0$.

Then

(1) *in the case $b_0 \neq 0$ and $k_1 \neq 0$ for the function $y(t) = \text{sign}(k_1)(c_1y_1(t) + c_2y_2(t))$ with the condition $y_0 = \text{sign}(k_1)\tilde{y}_0 > 0$ on the interval $[0,t^*)$, where $t^* = \min(t_{\max},t_\rho)$, $t_\rho = (\pi/2 - \varphi)/|b_0|$, $\varphi = \text{sign}(k_1k_2b_0)\arcsin(|k_2|/\sqrt{k_1^2 + k_2^2})$, the following differential*

inequality is valid:

$$\frac{dy}{dt} \geq \chi^* \cos(|b_0|t + \varphi) y^{1+\rho}; \tag{4.2}$$

here,

$$\chi^* = \begin{cases} \chi_0 & \text{for } a_0 \leq 0, \\ \chi_0 \exp(-a_0 \rho t_\rho) & \text{for } a_0 > 0, \end{cases} \tag{4.3}$$

and χ_0 has been determined in Theorem 3.1 by formula (3.4);

(2) *in the case $b_0 \neq 0$, $k_2 \neq 0$, and $\operatorname{sign}(k_1 k_2 b_0) = -1$ in the case $k_1 \neq 0$ for the function $y(t) = -\operatorname{sign}(k_2 b_0)(c_1 y_1(t) + c_2 y_2(t))$ with the condition $y_0 = -\operatorname{sign}(k_2 b_0)\tilde{y}_0 > 0$ on the interval $[0, t^*)$, where $t^* = \min(t_{\max}, t_\rho)$, $t_\rho = (\pi - \varphi)/|b_0|$, $\varphi = -\operatorname{sign}(k_1 k_2 b_0)$ $\arccos(|k_2|/\sqrt{k_1^2 + k_2^2})$, the following differential inequality is valid:*

$$\frac{dy}{dt} \geq \chi^* \sin(|b_0|t + \varphi) y^{1+\rho}; \tag{4.4}$$

here χ^* is determined by formula (4.3) in which $t_\rho = (\pi - \varphi)/|b_0|$;

(3) *in the case $b_0 = 0$, $k_1 \neq 0$, $\omega \neq 0$, and $\gamma \neq 0$ for the function $y(t) = \operatorname{sign}(k_1)(c_1 y_1(t) + c_2 y_2(t))$ with the condition $y_0 = \operatorname{sign}(k_1)\tilde{y}_0 > 0$ on the interval $[0, t_{\max})$, the following differential inequality is valid:*

$$\frac{dy}{dt} \geq \chi_0 e^{-a_0 \rho t} y^{1+\rho}, \tag{4.5}$$

where $\chi_0 = |k_1|/(|c_1| + |c_2|)^{1+\rho}$.

The above-mentioned lemma is proved on the ground of one suggestion. Now, we pass to the statement.

4.2. An auxiliary affirmation (on an integrodifferential identity for the solution $u(x,t)$ of problem (1.1) and solution $(\lambda, v(x))$ of problem (2.1)). Let $u(x,t) \in C([0, t_{\max}), B(\Omega)) \cap C^1([0, t_{\max}), L_2(\Omega))$ be the maximal solution of problem (1.1) and let $(\lambda, v(x))$ be any nontrivial solution of problem (2.1). On the interval $[0, t_{\max})$, we introduce the following functions:

$$\begin{aligned} y_1(t) &= \frac{1}{2} \int_\Omega \left[e^{zt} u(x,t) + e^{\bar{z}t} \bar{u}(x,t) \right] v(x) dx, \\ y_2(t) &= -\frac{i}{2} \int_\Omega \left[e^{zt} u(x,t) - e^{\bar{z}t} \bar{u}(x,t) \right] v(x) dx, \end{aligned} \tag{4.6}$$

where $z = a + ib$, $\bar{z} = a - ib$, $a = \lambda\alpha - \eta$, $b = \lambda\beta - \eta$, and $\bar{u}(x,t)$ is a complexly adjoint function to the $u(x,t)$.

The following statement is valid.

Statement 4.2. *Let $u(x,t)$ be a maximal solution of problem (1.1) from the class $C([0,t_{\max}), B(\Omega)) \cap C^1([0,t_{\max}), L_2(\Omega))$ and let $(\lambda, v(x))$ be any nontrivial solution of problem (2.1). Let $y_1(t)$, $y_2(t)$ be functions determined on $[0,t_{\max})$ by relations (4.6), respectively.*

Then for the functions $y_1(t)$, $y_2(t)$ on the interval $[0,t_{\max})$, the following relations are valid:

$$\frac{dy_1}{dt} = \exp(at)[\omega\cos(bt) - \gamma\sin(bt)]I_\rho(t), \tag{4.7}$$

$$\frac{dy_2}{dt} = \exp(at)[\omega\sin(bt) - \gamma\cos(bt)]I_\rho(t), \tag{4.8}$$

where $I_\rho(t) = \int_\Omega |u(x,t)|^{1+\rho} v(x)dx$.

Proof. For dy_1/dt, we have

$$\frac{dy_1}{dt} = R_1(t) + R_2(t), \tag{4.9}$$

where

$$\begin{aligned} R_1(t) &= \frac{1}{2}\int_\Omega [ze^{zt}u(x,t) + \bar{z}e^{\bar{z}t}\overline{u}(x,t)]v(x)dx, \\ R_2(t) &= \frac{1}{2}\int_\Omega [ze^{zt}u_t(x,t) + \bar{z}e^{\bar{z}t}\overline{u_t}(x,t)]v(x)dx. \end{aligned} \tag{4.10}$$

Taking into account (1.1a) in the right-hand side of the R_2, we get

$$\begin{aligned} R_2(t) &= \frac{1}{2}\int_\Omega \Big\{e^{zt}[(\alpha+i\beta)\Delta u + f(u) + (\eta+i\mu)u] \\ &\qquad + e^{\bar{z}t}[(\alpha-i\beta)\Delta\overline{u} + \overline{f(u)} + (\eta+i\mu)\overline{u}]\Big\}v(x)dx \\ &= \frac{1}{2}\Big\{(\alpha+i\beta)e^{zt}\int_\Omega \Delta uv\,dx + (\alpha-i\beta)e^{\bar{z}t}\int_\Omega \overline{\Delta u}v\,dx \\ &\qquad + \int_\Omega [e^{zt}f(u) + e^{\bar{z}t}\overline{f(u)}]v(x)dx + (\eta+i\mu)e^{zt}\int_\Omega uv\,dx \\ &\qquad + (\eta-i\mu)e^{\bar{z}t}\int_\Omega \overline{u}v\,dx\Big\}. \end{aligned} \tag{4.11}$$

Due to the second Green formula, we have

$$\int_\Omega \Delta uv\,dx = \int_\Omega u\Delta v\,dx = -\lambda\int_\Omega uv\,dx. \tag{4.12}$$

Hence,

$$\begin{aligned}
R_2(t) &= \frac{1}{2}\Big\{-\lambda(\alpha+i\beta)e^{zt}\int_\Omega uv\,dx-\lambda(\alpha-i\beta)e^{\bar{z}t}\int_\Omega \bar{u}v\,dx \\
&\quad+\int_\Omega\big[e^{zt}f(u)+e^{\bar{z}t}\overline{f(u)}\big]v(x)dx \\
&\quad+(\eta+i\mu)e^{zt}\int_\Omega uv\,dx+(\eta-i\mu)e^{\bar{z}t}\int_\Omega \bar{u}v\,dx\Big\} \\
&= \frac{1}{2}\Big\{-(\lambda\alpha-\eta)e^{zt}\int_\Omega uv\,dx-(\lambda\alpha-\eta)e^{\bar{z}t}\int_\Omega \bar{u}v\,dx \\
&\quad-i(\lambda\beta-\mu)e^{zt}\int_\Omega uv\,dx+i(\lambda\beta-\mu)e^{\bar{z}t}\int_\Omega \bar{u}v\,dx \\
&\quad+\int_\Omega\big[e^{zt}f(u)+e^{\bar{z}t}\overline{f(u)}\big]v\,dx\Big\} \\
&= \frac{1}{2}\Big\{-ae^{zt}\int_\Omega uv\,dx-ae^{\bar{z}t}\int_\Omega \bar{u}v\,dx-ibe^{zt}\int_\Omega uv\,dx+ibe^{\bar{z}t}\int_\Omega \bar{u}v\,dx \\
&\quad+\int_\Omega\big[e^{zt}f(u)+e^{\bar{z}t}\overline{f}(u)\big]v\,dx\Big\} \\
&= \frac{1}{2}\Big\{-(a+ib)e^{zt}\int_\Omega uv\,dx-(a-ib)e^{\bar{z}t}\int_\Omega \bar{u}v\,dx \\
&\quad+\int_\Omega\big[e^{zt}f(u)+e^{\bar{z}t}\overline{f}(u)\big]v\,dx\Big\} \\
&= -\frac{1}{2}\Big[ze^{zt}\int_\Omega uv\,dx+\bar{z}e^{\bar{z}t}\int_\Omega \bar{u}v\,dx\Big]+\int_\Omega\frac{e^{zt}f(u)+e^{\bar{z}t}\overline{f}(u)}{2}v\,dx \\
&= -R_1(t)+\int_\Omega\frac{e^{zt}f(u)+e^{\bar{z}t}\overline{f}(u)}{2}v\,dx.
\end{aligned} \tag{4.13}$$

Finally, for dy_1/dt, we get the following relation:

$$\begin{aligned}
\frac{dy_1}{dt} &= R_1(t)+R_2(t)=\int_\Omega\frac{e^{zt}f(u)+e^{\bar{z}t}\overline{f}(u)}{2}v\,dx \\
&= e^{at}\Big\{\frac{e^{ibt}(\omega+i\gamma)+e^{-ibt}(\omega-i\gamma)}{2}\Big\}\int_\Omega |u|^{1+\rho}v\,dx \\
&= e^{at}\Big(\omega\frac{e^{ibt}+e^{-ibt}}{2}-\gamma\frac{e^{ibt}-e^{-ibt}}{2i}\Big)I_\rho(t) \\
&= e^{at}\big[\omega\cos(bt)-\gamma\sin(bt)\big]I_\rho(t),
\end{aligned} \tag{4.14}$$

where $I_\rho(t)=\int_\Omega |u(x,t)|^{1+\rho}v\,dx$. The proof of relation (4.8) is similar to that of relation (4.7) for $y_1(t)$. Hence, we omit it here. The proof of the statement is over. □

4.3. Proof of Lemma 4.1. Let $u(x,t)$ be the maximal solution of problem (1.1) from the class $C([0,t_{\max}),B(\Omega))\cap C^1([0,t_{\max}),L_2(\Omega))$. Further, let c_1, c_2 be arbitrary real numbers such that $c_1^2+c_2^2\neq 0$. Multiplying (4.7) by c_1, (4.8) by c_2, and then adding the results, we get the following equation:

$$\frac{d}{dt}(c_1y_1 + c_2y_2) = e^{at}(k_1 \cos bt - k_2 \sin bt) I_\rho(t). \tag{4.15}$$

4.3.1. Proof of Lemma 4.1(1). Let $b \neq 0$ and $k_1 \neq 0$. We represent the function $\Phi(t) = k_1 \cos bt - k_2 \sin bt$ in the following form:

$$\begin{aligned}\Phi(t) &= |k_1| \operatorname{sign}(k_1) \cos(|b|t) - |k_2| \operatorname{sign}(k_2 b) \sin(|b|t) \\ &= \operatorname{sign}(k_1)\left[|k_1| \cos(|b|t) - |k_2| \operatorname{sign}(k_1k_2b) \sin(|b|t)\right] \\ &= \operatorname{sign}(k_1)\sqrt{k_1^2 + k_2^2}\left[\frac{|k_1|}{\sqrt{k_1^2+k_2^2}} \cos(|b|t) - \frac{|k_2|}{\sqrt{k_1^2+k_2^2}} \operatorname{sign}(k_1k_2b) \sin(|b|t)\right].\end{aligned} \tag{4.16}$$

Introducing the notations

$$\cos\varphi_0 = \frac{|k_1|}{\sqrt{k_1^2+k_2^2}}, \qquad \sin\varphi_0 = \frac{|k_2|}{\sqrt{k_1^2+k_2^2}} \tag{4.17}$$

for $\Phi(t)$, we have the following expression:

$$\Phi(t) = \operatorname{sign}(k_1)\sqrt{k_1^2+k_2^2}\cos(|b|t+\varphi), \tag{4.18}$$

where

$$\varphi = \operatorname{sign}(k_1k_2b)\varphi_0, \qquad \varphi_0 = \arcsin\frac{|k_2|}{\sqrt{k_1^2+k_2^2}}. \tag{4.19}$$

Substituting it into (4.15), we get for all $t \in [0, t_{\max})$ the following equation:

$$\frac{dy}{dt} = e^{at}\sqrt{k_1^2+k_2^2}\cos(|b|t+\varphi) I_\rho(t), \tag{4.20}$$

where $y = \operatorname{sign}(k_1)(c_1y_1 + c_2y_2)$. The function $\cos(|b|t+\varphi)$ in the segment $[0, t_\rho]$, where $t_\rho = (\pi/2 - \varphi)/|b|$, is nonnegative. The function

$$I_\rho(t) = \int_\Omega |u(x,t)|^{1+\rho} v(x)dx \tag{4.21}$$

will be positive for all $t \in [0, t_{\max})$ if $v(x) = v_0(x)$. Therefore, choosing $\lambda = \lambda_0$ and $v(x) = v_0(x)$, we see obviously that the right-hand side of (4.20) has the positive sign in $t \in [0, t^*)$, where $t^* = \min(t_{\max}, t_\rho)$. Suppose that $y_0 = \operatorname{sign}(k_1)\tilde{y}_0 > 0$. Then from (4.20), we deduce that $y(t)$ in $[0, t^*)$ strictly increases, and hence is strictly positive. For $y(t)$ in

$[0,t^*)$, we have the following estimate:

$$\begin{aligned} y(t) &\le (|c_1||y_1| + |c_2||y_2|) \\ &\le (|c_1| + |c_2|)e^{a_0 t}\int_\Omega |u(x,t)|v_0(x)dx \\ &= (|c_1| + |c_2|)e^{a_0 t}\int_\Omega |u| v_0^{1/(\rho+1)} v_0^{\rho/(\rho+1)} dx \quad \text{(by Hölder's inequality for integrals)} \\ &\le (|c_1| + |c_2|)e^{a_0 t}\left(\int_\Omega |u(x,t)|^{\rho+1} v_0(x)dx\right)^{1/(\rho+1)} \\ &\quad \times \left(\int_\Omega v_0(x)dx\right)^{\rho/(\rho+1)} \quad \text{(by the norm condition (2.2))} \\ &= (|c_1| + |c_2|)e^{a_0 t} I_\rho^{1/(\rho+1)}(t). \end{aligned} \tag{4.22}$$

From this estimate, we deduce for all $t \in [0,t^*)$ the inequality

$$I_\rho(t) \ge \frac{e^{-a_0(1+\rho)t}}{(|c_1| + |c_2|)^{1+\rho}} y^{1+\rho}(t), \tag{4.23}$$

due to which, by (4.20) for $y(t)$, we finally obtain the nonlinear differential inequality

$$\frac{dy}{dt} \ge \frac{e^{-a_0\rho t}\sqrt{k_1^2 + k_2^2}}{(|c_1| + |c_2|)^{1+\rho}} \cos(|b_0|t + \varphi) y^{1+\rho} \tag{4.24}$$

in which, further taking into account that $e^{-a_0\rho t} \ge 1$ in the case $a_0 \le 0$, $e^{-a_0\rho t} > e^{-a_0\rho t_\rho}$ for all $t \in [0,t^*)$ in the case $a_0 > 0$, we get the nonlinear differential inequality

$$\frac{dy}{dt} \ge \chi^* \cos(|b_0|t + \varphi) y^{1+\rho} \tag{4.25}$$

with the initial condition $y_0 = y(0) = \operatorname{sign}(k_1)\tilde{y}_0 > 0$. Here, χ^* is determined by formula (4.3), and χ_0 by (3.4).

The proof of the first part of lemma is over.

4.3.2. Proof of Lemma 4.1(2). Let $b \neq 0$ and $k_2 \neq 0$. For $\Phi(t) = k_1 \cos bt - k_2 \sin bt$, we have

$$\begin{aligned} \Phi(t) &= -\operatorname{sign}(k_2 b)[|k_2|\sin(|b|t) - |k_2|\operatorname{sign}(k_1 k_2 b)\cos(|b|t)] \\ &= -\operatorname{sign}(k_2 b)\sqrt{k_1^2 + k_2^2}\left[\frac{|k_2|}{\sqrt{k_1^2 + k_2^2}}\sin(|b|t) - \frac{|k_1|}{\sqrt{k_1^2 + k_2^2}}\operatorname{sign}(k_1 k_2 b)\cos(|b|t)\right]. \end{aligned} \tag{4.26}$$

Introducing the notations

$$\cos\varphi_0 = \frac{|k_2|}{\sqrt{k_1^2 + k_2^2}}, \qquad \sin\varphi_0 = \frac{|k_1|}{\sqrt{k_1^2 + k_2^2}}, \tag{4.27}$$

finally for $\Phi(t)$, we have the following expression:

$$\begin{aligned}\Phi(t) &= -\operatorname{sign}(k_2 b)\sqrt{k_1^2+k_2^2}\left[\cos\varphi_0 \sin(|b|t) - \operatorname{sign}(k_1 k_2 b)\sin\varphi_0 \cos(|b|t)\right]\\ &= -\operatorname{sign}(k_2 b)\sqrt{k_1^2+k_2^2}\sin(|b|t+\varphi),\end{aligned} \tag{4.28}$$

where

$$\varphi = \operatorname{sign}(k_1 k_2 b)\varphi_0, \qquad \varphi_0 = \arccos\frac{|k_2|}{\sqrt{k_1^2+k_2^2}}. \tag{4.29}$$

Substituting this expression for $\Phi(t)$ into (4.15) for all $t \in [0, t_{\max})$, we get the following equation:

$$\frac{dy}{dt} = e^{at}\sqrt{k_1^2+k_2^2}\sin(|b|t+\varphi)I_\rho(t), \tag{4.30}$$

where $y = -\operatorname{sign}(k_2 b)(c_1 y_1 + c_2 y_2)$. Let $k_1 \neq 0$ and $\operatorname{sign}(k_1 k_2 b) = -1$. Then $\sin(|b|t + \varphi_0)$ in the segment $[0, t_\rho]$, where $t_\rho = (\pi - \varphi_0)/|b|$, is nonnegative. In addition, $I_\rho(t) = \int_\Omega |u(x,t)|^{1+\rho} v(x)dx$ will be positive for all $t \in [0, t_{\max})$ if we have to take $v(x) = v_0(x)$. Hence, under choosing $\lambda = \lambda_0$ and $v(x) = v_0(x)$ for all $t \in [0, t^*)$, where $t^* = \min(t_{\max}, t_\rho)$ by virtue of (4.30), we conclude that $y(t)$ strictly increases; therefore, $y(t) \geq y_0$. Further, by analogical considerations, which have been done in the proof of the first part of the lemma, we establish the following inequality:

$$I_\rho(t) \geq \chi_0 e^{-a_0(1+\rho)t} y^{1+\rho}(t). \tag{4.31}$$

Taking into account the last estimate for $I_\rho(t)$ in (4.30), we get the following nonlinear differential inequality:

$$\frac{dy}{dt} \geq \chi_0 e^{-a_0 \rho t}\sin(|b_0|t+\varphi_0)y^{1+\rho} \tag{4.32}$$

with the initial data $y_0 = -\operatorname{sign}(k_2 b_0)\tilde{y}_0 > 0$ from which, obviously as in the proof of the first part of the lemma, one has

$$\frac{dy}{dt} \geq \chi^* \sin(|b_0|t+\varphi_0)y^{1+\rho}, \tag{4.33}$$

where χ^* is determined by formula (4.3) with its t_ρ.

The proof of the second part of the lemma is over.

4.3.3. Proof of Lemma 4.1(3). Let $\omega \neq 0, \gamma \neq 0, \{c_1, c_2\} \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$, $\lambda = \lambda_0$, $v(x) = v_0(x)$, and $b_0 = \lambda_0 \beta - \nu = 0$. Then from (4.15), we deduce that the following equation is true:

$$\frac{dy}{dt} = e^{a_0 t}|k_1| I_\rho(t), \tag{4.34}$$

where $y = \operatorname{sign}(k_1)(c_1 y_1 + c_2 y_2)$ from which similarly to the proof of the first and second parts of the lemma, obviously one has

$$\frac{dy}{dt} \geq \chi_0 e^{-a_0 \rho t} y^{1+\rho}, \tag{4.35}$$

where $\chi_0 = |k_1|/(|c_1| + |c_2|)^{1+\rho}$ with the initial condition $y_0 = \operatorname{sign}(k_1)\tilde{y}_0 > 0$. The proof of the third part of the lemma is over.

4.4. Proof of the theorems. We will prove in detail only Theorem 3.1 because the proofs of Theorems 3.2 and 3.3 are similar to that of Theorem 3.1, hence we will omit them here. Let all conditions of Theorem 3.1 be fulfilled; $u(x,t)$ is the maximal solution of problem (1.1). Let $t_{\max}$ be finite. In this case, we show that $t_{\max} \leq t_k$. We will prove this claim by contradiction, that is, we assume that $t_{\max} > t_k$. By virtue of the first part of the lemma for the function $y = \operatorname{sign}(k_1)(c_1 y_1 + c_2 y_2)$ under conditions of Theorem 3.1, the following nonlinear differential inequality is fulfilled for $t \in [0, t^*)$, $t^* = \min(t_{\max}, t_\rho)$, where t_ρ is determined in Theorem 3.1:

$$\frac{dy}{dt} \geq \chi^* \cos\left(|b_0| t + \varphi\right) y^{1+\rho} \tag{4.36}$$

with initial data $y_0 = \operatorname{sign}(k_1)\tilde{y}_0 > 0$ from which, after separation of the variables by the well-known procedure, we conclude that for $y(t)$, the following lower bound estimate is valid:

$$y(t) \geq \frac{y_0}{F(t)^{1/\rho}}, \tag{4.37}$$

where $F(t) = 1 - y_0 \rho \chi^* [\sin(|b_0| t + \varphi) - \sin\varphi]/|b_0|$.

To finish the proof of Theorem 3.1, one has to estimate the norms $\|u(\cdot,t)\|$, $\|u(\cdot,t)\|_{1+\rho}$, $\|\nabla u(\cdot,t)\|$, and $\|u(\cdot,t)\|_{W_2^2(\Omega)}$ from below by $y(t)$ in $t \in [0, t^*)$. For $y(t)$ on the base of the definitions of $y_1(t)$ and $y_2(t)$ by the Hölder inequality for integrals, we have the following estimate:

$$\begin{aligned} y(t) &\leq e^{a_0 t}\left(|c_1| + |c_2|\right) \int_\Omega |u(x,t)| v_0(x)\,dx \\ &\leq e^{a_0 t}\left(|c_1| + |c_2|\right) \|u(\cdot,t)\|_{1+\delta} \|v_0\|_{(1+\delta)/\delta} \end{aligned} \tag{4.38}$$

for any admissible positive δ. From this inequality for the norm $\|u(\cdot,t)\|_{1+\delta}$, we obtain the lower estimate

$$\|u(\cdot,t)\|_{1+\delta} \geq \frac{e^{-a_0 t}}{\left(|c_1| + |c_2|\right)\|v_0\|_{(1+\delta)/\delta}} y(t), \tag{4.39}$$

from which, by virtue of the Poincaré inequality $\|\nabla u(\cdot,t)\| \geq \text{const}\,\|u(\cdot,t)\|$, we have

$$\|\nabla u(\cdot,t)\| \geq c e^{-a_0 t} y(t) \tag{4.40}$$

(here and below, by c we will denote different constants which are independent of t and different norms of $u(x,t)$), and by virtue of Sobolev's inequality

$$||u(\cdot,t)||_{W_2^2(\Omega)} \geq \text{const}\,||u(\cdot,t)||, \tag{4.41}$$

we have

$$||u(\cdot,t)||_{W_2^2(\Omega)} \geq ce^{-a_0 t}y(t). \tag{4.42}$$

From these estimates and (4.37) for $y(t)$ in $t \in [0,t^*)$ for the norms $\|u(\cdot,t)\|$, $\|u(\cdot,t)\|_{1+\rho}$, $\|\nabla u(\cdot,t)\|$, and $\|u(\cdot,t)\|_{W_2^2(\Omega)}$, we get the following lower estimates:

$$\begin{aligned} ||u(\cdot,t)|| &\geq \frac{ce^{-a_0\rho t}}{R(t)}, \qquad & ||u(\cdot,t)||_{1+\rho} &\geq \frac{ce^{-a_0\rho t}}{R(t)}, \\ ||\nabla u(\cdot,t)|| &\geq \frac{ce^{-a_0\rho t}}{R(t)}, \qquad & ||u(\cdot,t)||_{W_2^2(\Omega)} &\geq \frac{ce^{-a_0\rho t}}{R(t)}, \end{aligned} \tag{4.43}$$

where $R(t) = F^{1/\rho}(t)$.

We pay attention to these estimates. Function $F(t)$ is defined and continues for all $t \geq 0$. At the point $t = 0$, it has the value $F(0) = 1$. We calculate its value at the point $t = t_\rho$. We have $F(t_\rho) = 1 - y_0^\rho(1 - \sin\varphi)/\chi$, where χ has been determined in Theorem 3.1 by formula (3.3). By virtue of the condition put on y_0, $y_0 \geq [\chi/(1 - \sin\varphi)]^{1/\rho}$, it follows that $F(t_\rho) \leq 0$. Hence, the function $F(t)$ in the segment $[0,t_\rho]$ decreasingly intersects it at the unique point $t_k \in (0,t_\rho]$, which is the unique root in $(0,t_\rho]$ of the trigonometric equation

$$\sin\left(|b_0|\,t + \varphi\right) = \sin\varphi + \frac{\chi}{y_0^\rho} \tag{4.44}$$

and is expressed by the formula

$$t_k = \frac{\arcsin\left(\sin\varphi + \chi/y_0^\rho\right) - \varphi}{|b_0|}. \tag{4.45}$$

It is clear that $F(t) < 0$ for $t > t_k$, so $R(t)$ has been determined only in the segment $[0,t_k]$. By our assumption, the solution $u(x,t)$ of problem (1.1) from the class $C([0,t_{\max}),B(\Omega)) \cap C^1([0,t_{\max}),L_2(\Omega))$ exists in $[0,t_k] \cup [t_k,t^*)$. Therefore, owing to our assumption in $[0,t_k]$, $y(t)$, $\|u(\cdot,t)\|$, $\|u(\cdot,t)\|_{1+\rho}$, $\|\nabla u(\cdot,t)\|$, and $\|u(\cdot,t)\|_{W_2^2(\Omega)}$ are determined. But from estimates (4.43), it follows that $\|u(\cdot,t)\| \to \infty$, $\|u(\cdot,t)\|_{1+\rho} \to \infty$, $\|\nabla u(\cdot,t)\| \to \infty$, and $\|u(\cdot,t)\|_{W_2^2(\Omega)} \to \infty$ as $t \to t_k$. We obtained the contradiction as consequences of it. One has to state that $t_{\max} \leq t_k$, and therefore, the claim of Theorem 3.1 is true. The proof of Theorem 3.1 is over.

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