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BASIC SINGULARITIES IN THE THEORY OF INTERNAL WAVES WITH SURFACE TENSION

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ABSTRACT. The study of linearized interface wave problems for two superposed fluids often involves the consideration of different types of singularities in one of the two fluids. In this paper the line and point singularities are investigated for the case when each fluid is of finite constant depth. The effect of surface tension at the surface of separation is included.

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1. INTRODUCTION.

The study of internal waves in two fluids problems has attracted many authors in recent years. It is found useful to permit singularities of one type or another to occur as an idealisation of, or an approximation to certain physical situations. Problems dealing with the generation of waves at the interface of two non-mixing fluids involve the consideration of singularities of different types in the fluids. In the case when bodies are present, waves may either be generated by the movement of the body or reflected from it. The two cases are essentially the same and the resulting motion can be described by the use of these singularities in a suitable way. For example, Gorgui [1] has investigated into these waves using a distribution of sources on the surface of the body.

The different types of singularities that can be used in two fluids problems have been presented by Gougui and Kassem [2] and Kassem [3] ; in both the effects of surface tension are neglected. In [2] , the authors considered the cases when the lower fluid is of finite constant depth and the upper fluid is of infinite height. While in [3] the author considered the cases when the two superposed fluids are both of finite thickness and obtained the potentials for motions resulting from multipoles submerged in one of the two fluids.

In this paper we give a complete survey for the basic line and point singularities when the superposed fluids are as in [3] of finite constant thickness confined between two rigid horizontal planes but we here take surface tension into consideration.

In the two dimensional motion, the line singularities considered are wave sources and multipoles singularities. Restriction is made to symmetric (or vertical) multipoles, but the corresponding antisymmetric (or horizontal) multipoles can be found similarly. For axisymmetric motion, the point singularities considered are multipole singularities. These time harmonic singularities are described by harmonic potential functions which satisfy two linearised conditions at the surface of separation, and uniqueness is ensured by requiring that there are only outgoing waves in the far field. The method used is an extension of that used in [2] or [3] . The results obtained by Rhodes-Robinson [4], Gorgui and Kassem [2] and Kassem [3] can be deduced as special cases.

2. STATEMENT OF THE PROBLEM.

We are concerned with the irrotational, incompressible and inviscid motion of the two superposed non-mixing fluids under the action of gravity and surface tension. Each fluid is of infinite horizontal extent. Taking the origin 0 at the mean level of the interface and the axis Oy pointing vertically downwards into the lower fluid, let the two fluids be confined between rigid horizontal planes $y = h$, $y = -h'$. The motion is simple harmonic with a small amplitude and angular frequency σ ; it is due to an oscillating singularity in one of the two fluids. In two dimensional motion we consider the singularity is either a line wave source or multipole and in axisymmetric motion it is a point multipole. In each case, the velocity potentials of the lower and upper fluids are simple harmonic with period $\frac{2\pi}{\sigma}$ and it is more convenient to use the complex valued potentials $\phi e^{-i\sigma t}$, $\phi' e^{-i\sigma t}$, of which the actual velocity potentials are the real parts.

These potentials satisfy a boundary value problem in which

$$\nabla^2 \phi = 0, \ \nabla^2 \phi' = 0 \tag{2.1}$$

in the regions occupied by the fluids, except at the singularity;

$$\frac{\partial\phi}{\partial y} = 0 \qquad \text{on} \qquad y = h, \tag{2.2}$$

$$\frac{\partial\phi'}{\partial y} = 0 \qquad \text{on} \qquad y = -h', \tag{2.3}$$

and the linearized boundary conditions

$$\left.\begin{array}{l} \dfrac{\partial\phi}{\partial y} = \dfrac{\partial\phi'}{\partial y}, \\ K\phi + \dfrac{\partial\phi}{\partial y} - s(K\phi' + \dfrac{\partial\phi}{\partial y}) + M\dfrac{\partial^3\phi}{\partial y^3} = 0 \end{array}\right\} \quad \text{on } y = 0 \tag{2.4}$$

where $K = \frac{\sigma^2}{g}$, $M = \frac{T}{\rho g}$, g is the gravity, T is the surface tension, ρ and $s\rho$ $(s < 1)$ are the densities of the lower and upper fluids respectively. These conditions are applied for each singularity considered. They are supplemented by the two general limiting conditions that ϕ or ϕ' behaves like a typical singular harmonic function near the singularity and the radiation condition that both functions represent outgoing waves in the far field.

3. SUBMERGED LINE SINGULARITIES.

Without loss of generality, the line singularity is place at the point $(o, \pm\ \eta)$ We consider only singularities symmetric in x - namely, a wave source and vertical multipoles. We define polar coordinates (R, θ) in the xy - plane based at the singularity position by the equations

$$x = R \sin\theta \ , \ y \pm \eta = R\cos\theta$$

according as the singularity is in the lower or upper fluid, so that R denotes the distance from the singularity.

(i) Wave source singularities

Here ϕ and ϕ' are solutions of the boundary - value problem stated above with ϕ having a logarithmic singularity at the source. If the source is at (o, η) then

$$\phi \sim \log R \quad \text{as} \quad R = [x^2 + (y-\eta)^2]^{\frac{1}{2}} \to o \tag{3.1}$$

Let

$$\phi = \log R + \alpha \log R' + \int_0^\infty [sf(k) + \{A\cosh k(h-y) + B\sinh ky\}\cos kx]dk,$$

$$\phi' = \alpha' \log R + \int_0^\infty [f(k) + \{A'\cosh k(h'+y) + B'\sinh ky\}\cos kx]dk,$$

where $R' = [x^2 + (y+\eta)^2]^{\frac{1}{2}}$ is the distance from the image point $(o, -\eta)$. It is obvious that ϕ, ϕ' as given above are harmonic. We choose α, α', $f(k)$, $A(k)$, $B(k)$, $A'(k)$, $B'(k)$ such that the two integrals converge and the boundary conditions on $y = h$, $y = -h'$ and $y = o$ are satisfied.

Under suitable conditions concerning differentiation under the integral sign and using the relations

$$\frac{\partial}{\partial y}(\log R) = \frac{\cos\theta}{R} = \begin{cases} \int_0^\infty e^{-k(y-\eta)}\cos kx\, dk \ , & y > \eta \ , \\ -\int_0^\infty e^{k(y-\eta)}\cos kx\, dk, & y < \eta \ , \end{cases}$$

the conditions (2.2) , (2.3) are satisfied if

$$B = -\frac{e^{-k(h-\eta)} + \alpha\, e^{-k(h+\eta)}}{k\cosh kh}, \quad B' = \frac{\alpha'\, e^{-k(h'+\eta)}}{k\cosh kh'} \ ,$$

and the interface conditions are satisfied if

$$1 + \alpha - s\,\alpha' = o,$$

$$A\sinh kh + A'\sinh kh' = \frac{\alpha + \alpha' - 1}{k}\, e^{-k\eta} + (B - B') \ ,$$

$$cA\cosh kh - A'[sc\cosh kh' - k(1+\beta k^2)\sinh kh'] = (1+\beta k^2)(\alpha' e^{-k\eta} - kB') \ ,$$

from which we have

$$\Delta A = \alpha'(1+\beta k^2)\, e^{-k\eta}\sinh^2 kh' \operatorname{sech} kh + [\frac{sc}{k}\cosh kh' - (1+\beta k^2)\sinh kh']\frac{F(k)}{\cosh kh},$$

$$\Delta A' = -\alpha'(1+\beta k^2)\, e^{-k\eta}\sinh kh \tanh kh' + \frac{c\, F(k)}{k} \ ,$$

where

$$F(k) = (\alpha + \alpha' - 1)\, e^{-k\eta}\cosh kh - e^{-k(h-\eta)} - \alpha\, e^{-k(h+\eta)}$$
$$- \alpha'\, e^{-k(h'+\eta)}\cosh kh \operatorname{sech} kh' \ ,$$

$$\Delta(k) = c(\cosh kh \sinh kh' + s\cosh kh' \sinh kh) - k(1+\beta k^2)\sinh kh \sinh kh' \ , \tag{3.2}$$

$$c = \frac{K}{1-s} \ , \quad \beta = \frac{M}{1-s} \ .$$

Since $F(o) = -2$, therefore, the integrals involved in the assumed expressions for A , A' converge if we choose $f(k)$ such that $k \Delta f(k) = 2c + O(k^2)$ in the neighbourhood of $k = o$ and if $\frac{dF(o)}{dk} = o$,

i.e., $$h(1 + \alpha) + h' \alpha' = o$$

Hence

$$\alpha' = o \quad , \alpha = -1 \ ,$$

$$F(k) = -2 \cosh k(h - \eta) \ ,$$

$$B = -2 \frac{e^{-kh}}{k} \frac{\sinh k\eta}{\cosh kh} \quad , \quad B' = o \ ,$$

$$A = 2[(1 + \beta k^2) \sinh kh' - \frac{sc}{k} \cosh kh'] \frac{\cosh k(h - \eta)}{\Delta \cosh kh} \ ,$$

$$A' = -2c \frac{\cosh k(h - \eta)}{k \Delta}$$

The condition imposed on $f(k)$ does not specify it completely. This introduces no difficulty since it is the velocities in which we are really interested. It is found convenient to take $f(k) = \frac{2c}{k\Delta}$.

Now, $\Delta(k)$ has a simple zero at $k = m$, say, on the real axis of k . This introduces simple poles for the integrals in $\emptyset$, $\emptyset'$. Below this pole we make an indentation of the contours of the integrations. Substituting in the above assumed forms for A , A' , B , B' we get

$$\emptyset = \log \frac{R}{R'} + 2 \int_0^\infty [\frac{cs}{k\Delta} + [\{k(1 + \beta k^2)\sinh kh' - sc \cosh kh'\} \times$$
$$\times \frac{\cosh k(h - \eta)}{k\Delta \cosh kh} \cosh k(h - y) - \frac{e^{-kh}}{k} \frac{\sinh k\eta}{\cosh kh} \sinh ky] \cos kx] dk,$$

$$\emptyset' = 2c \int_0^\infty \frac{1}{k \Delta}[1 - \cosh k(h - \eta) \cosh k(h' + y) \cos kx] \ dk.$$

Two other alternative forms which will be useful in the subsequent work are

$$\emptyset = \log R + (2s - 1) \log R' + 2 \int_0^\infty \frac{\delta}{\Delta}(1 + \beta k^2)\sinh^2 kh' \cosh k(k - \eta) \times$$
$$\times \cosh k(h - y) \cos kx \ dk + 2s \int_0^\infty \frac{1}{k} [\delta - e^{-kh'} +$$
$$[e^{-k(y + \eta)} - \delta\cosh k(h - \eta) \frac{\cosh kh'}{\cosh kh}\cosh k(h - y)] \cos kx \ ] \ dk$$
$$- 2 \int_0^\infty \frac{e^{-kh}}{k} \frac{\sinh k\eta \sinh ky}{\cosh kh} \cos kx \ dk \ , \qquad (3.4)$$

$$\emptyset' = 2 \log R - 2 \int_0^\infty \frac{\delta}{\Delta}(1 + \beta k^2) \sinh kh \sinh kh' \cosh k(h - \eta) \times$$
$$\times \cosh K(h' + y)\cos kx \ dk + 2 \int_0^\infty \frac{1}{k}[\delta - e^{-kh'} + [e^{-k(\eta - y)} -$$
$$- \delta \cosh k(h - \eta) \cosh k(h' + y)] \cos kx \ ] dk \ , \qquad (3.5)$$

where $$\delta^{-1} = \cosh kh \sinh kh' + s \cosh kh' \sinh kh \ . \qquad (3.6)$$

In the above expressions we neglected the constants

$$2s \log h' + 2s \int_0^\infty \frac{\delta}{\Delta}(1 + \beta k^2)\sinh kh' \sinh kh \ dk \quad \text{and}$$

$$2 \log h' + 2 \int_0^\infty \frac{\delta}{\Delta}(1 + \beta k^2)\sinh kh \sinh kh' \ dk$$ in $\emptyset$, $\emptyset'$ respectively.

By putting $2\cos kx = e^{ik|x|} + e^{-ik|x|}$, and rotating the contours in the indented integrals in $\emptyset$, $\emptyset'$ into contours in the first and fourth quadrants so that we must include the residue term at $k = m$, these integral tend to

$$- C(o;h,h') \frac{\cosh m(h - y)}{m \sinh mh} e^{im|x|} , \quad C(o;h,h') \frac{\cosh m(h' + y)}{m \sinh mh'} e^{im|x|}$$

as $|x| \to \infty$, where

$$C(n;h,h') = \frac{\pi i c\, m^n}{n!} \frac{[e^{-m(h - \eta)} + (-1)^n e^{m(h - \eta)}]\sinh^2 mh' \sinh mh}{(1 + 3\beta m^2)\sinh^2 mh \sinh^2 mh' + c(h \sinh^2 mh' + h' s \sinh^2 mh)} \tag{3.7}$$

When the source is at $(o, -\eta)$ in the upper field, we have

$$\emptyset' \sim \log R \quad \text{as} \quad R = [x^2 + (y + \eta)^2]^{\frac{1}{2}} \to o \tag{3.8}$$

Assume the forms

$$\emptyset = \alpha \log R + \int_0^{\infty} [s\, f(k) + \{A \cosh k(h - y) + B \sinh ky\} \cos kx]\, dk,$$

$$\emptyset' = \log R + \alpha' \log R' + \int_0^{\infty} [f(k) + \{A' \cosh k(h' + y) + B' \sinh ky\} \cdot$$

$$\cdot \cos kx]\, dk,$$

where $R' = [x^2 + (y - \eta)^2]^{\frac{1}{2}}$ is the distance from the image point (o, η) and proceed as above to get the potentials

$$\emptyset = 2cs \,⨍_0^{\infty} \frac{1}{k\Delta} [1 - \cosh k(h' - \eta) \cosh k(h - y) \cos kx]\, dk ,$$

$$\emptyset' = \log \frac{R}{R'} + 2 ⨍_0^{\infty} \{\frac{c}{k\Delta} + [(k(1 + \beta k^2)\sinh kh - c \cosh kh) \frac{\cosh k(h' - \eta)}{k\, \Delta \cosh kh'} \times$$

$$\times \cosh k(h' + y) + e^{-kh'} \frac{\sinh k\eta}{k \cosh kh'} \sinh ky] \cos kx \}\, dk ,$$

and the other two alternatives forms are

$$\emptyset = 2s \log R - 2s ⨍_0^{\infty} \frac{\delta}{\Delta} (1 + \beta k^2)\sinh kh \sinh kh' \cosh k(h' - \eta) \times$$

$$\times \cosh k(h - y)\cos kx dk + 2s \int_0^{\infty} \frac{1}{k} \quad (\delta - e^{-kh'}) +$$

$$+ [e^{-k(y + \eta)} - \delta \cosh k(h' - \eta)\cosh k(h - y)] \cos kx \quad dk , \tag{3.9}$$

$$\emptyset' = \log RR' + 2s ⨍_0^{\infty} \frac{\delta}{\Delta} (1 + \beta k^2)\sinh^2 kh \cosh k(h' - \eta)\cosh k(h' + y) \times$$

$$\times \cosh k(h' + y)\cos kx\, dk + 2 \int_0^{\infty} \frac{1}{k}[(\delta - e^{-kh'}) + [e^{-k(\eta - y)}$$

$$- \delta \cosh k(h' - \eta)\frac{\cosh kh}{\cosh kh'} \cosh k(h' + y)] \cos ky]\, dk +$$

$$+ 2 \int_0^{\infty} e^{-kh'} \frac{\sinh k\eta \sinh ky}{k \cosh kh'} \cos kx\, dk . \tag{3.10}$$

These potentials have the outgoing waves

$$-C(o;h',h) \frac{s \cosh m(h - y)}{m \sinh mh} e^{i\, m|x|} ,$$

$$C(o;h',h) \frac{s \cosh m(h' + y)}{m \sinh mh'} e^{i\, m|x|} ,$$

as $|x| \to \infty$ where $C(n;h,h')$ is given by (3.7).

(ii) Multipoles singularities

Here $\emptyset$, $\emptyset'$ are harmonic in the regions occupied by the two fluids except at the singularity. In the neighbourhood of this point

$$\phi \sim \frac{\cos(n+1)\theta}{R^{n+1}} \quad , \; n = o, 1, 2, \ldots \tag{3.11}$$

We consider first the case when the singularity is at (o, η) in the lower fluid. We try as solutions

$$\phi = \frac{\cos(n+1)}{R^{n+1}} + \int_0^\infty [A \cosh k(h-y) + B \sinh ky] \cos kx \, dk,$$

$$\phi' = \int_0^\infty A' \cosh k(h'+y) \cos kx \, dk,$$

and use the representations

$$\frac{\cos(n+1)\,\theta}{R^{n+1}} = \begin{cases} \frac{(-1)^{n+1}}{n!} \int_0^\infty k^n e^{k(y-\eta)} \cos kx \, dk, & y < \eta \,, \\ \frac{1}{n!} \int_0^\infty k^n e^{-k(y-\eta)} \cos kx \, dk, & y > \eta \,. \end{cases}$$

Conditions (2.2) - (2.4) are satisfied if

$$B = \frac{k^n e^{-k(h-\eta)}}{n! \cosh kh} \,,$$

$$A \sinh kh + A' \sinh kh' = B + \frac{(-1)^{n+1}}{n!} k^n e^{-k\eta},$$

$$-cA \cosh kh + [\, sc \cosh kh' - k(1+\beta k^2)\sinh kh' \,]A' = \frac{(-1)^{n+1}}{n!} ck^n e^{-k\eta} \,.$$

These determine A, B, A', which when substituted in the above assumed forms, give

$$\phi = \frac{\cos(n+1)\theta}{R^{n+1}} + \frac{1}{n!} \int_0^\infty k^n e^{-k(h-\eta)} \frac{\sinh ky}{\cosh kh} \cos kx \, dk$$

$$+ \frac{1}{n!} \int_0^\infty \frac{k^n}{\Delta} P(n) \cos kx \, dk \,, \tag{3.12}$$

$$\phi = \frac{c}{n!} \int_0^\infty \frac{k^n}{\Delta} [e^{-k(h-\eta)} + (-1)^{n+1} e^{k(h-\eta)}] \cosh k(h'+y) \cos kx \, dk, \tag{3.13}$$

where

$$P(n) = (-1)^{n+1}[c(s \cosh kh' - \sinh kh') - k(1+\beta k^2)\sinh kh'] e^{-k\eta}$$

$$+ [cs \cosh kh' - k(1+\beta k^2)\sinh kh'] \frac{e^{-k(h-\eta)}}{\cosh kh} \quad , \tag{3.14}$$

and Δ is given by equation (3.2).

As $|x| \to \infty$, we have the outgoing waves

$$\phi \sim (n+1)\, C(n+1\,;\, h, h') \,\frac{\cosh m(h-y)}{m \sinh mh}\, e^{im|x|} \,,$$

$$\phi \sim -(n+1)\, C(n+1;\, h, h')\, \frac{\cosh m(h'+y)}{m \sinh mh'}\, e^{im|x|} \quad ,$$

where $C(n; h, h')$ is given by equation (3.7).

If the singularity is at $(o, -\eta)$ in the upper fluid we try as solutions

$$\phi = \int_0^\infty A \cosh k(h-y) \cos kx \, dk \,,$$

$$\phi' = \frac{\cos(n+1)\theta}{R^{n-1}} + \int_0^\infty [A' \cosh k(h'+y) + B' \sinh ky] \cos kx \, dk;$$

where $R = [x^2 + (y+\eta)^2]^{\frac{1}{2}}$.

Proceeding as above leads to

$$\phi = \frac{sc}{n!} \int_0^\infty \frac{k^n}{\Delta} [e^{k(h'-\eta)} + (-1)^{n+1} e^{-k(h'-\eta)}] \cos k(h-y) \cos kx \, dk, \tag{3.15}$$

$$\phi' = \frac{\cos(n+1)\theta}{R^{n+1}} - \frac{(-1)^{n+1}}{n\,!} \int_0^\infty \frac{k^n e^{-k(h'-\eta)}}{\cosh kh'} \sinh ky \cos kx \, dk$$
$$+ \frac{1}{n!} \oint_0^\infty \frac{k^n}{\Delta} Q(n) \cosh k(h'+y) \cos kx \, dk, \tag{3.16}$$

where

$$Q(n) = [c(\cosh kh - s \sinh kh) - k(1+ \beta k^2)\sinh kh]\, e^{-k\eta}$$
$$+ (-1)^{n+1}[c \cosh kh - k(1+ \beta h^2)\sinh kh] \frac{e^{-k(h' - \eta)}}{\cosh kh'}. \tag{3.17}$$

The contours of the integrals in (3.15), (3.16) are indented below the simple pole at $k = m$ to give the outgoing waves

$$\phi \sim s(n+1)\; C(n+1),\; h',\; h) \frac{\cosh m(h-y)}{n \sinh mh} e^{im|x|} ,$$
$$\phi' \sim -s(n+1)\; C(n+1;\; h',\; h) \frac{\cosh m(h'+y)}{n \sinh mh'} e^{im|x|} ,$$

as $|x| \to \infty$.

4. SUBMERGED POINT SINGULARITIES.

We now define cylindrical polar coordinates (r, ψ, y) with the origin 0 at the surface separating the two fluids and the y-axis pointing vertically downwards. We also define spherical polar coordinates (R, θ, ψ) based at the singularity taken at $(o, \pm \eta)$ by the equations

$$r = R \sin \theta \quad , \; y \mp \eta = R \cos \theta$$

We consider only point singularities for which Oy is an axis of symmetry so that the velocity potentials ϕ, ϕ' are independent of ψ.

When the singularity is at (o, η) in the lower fluid the boundary value problem for ϕ, ϕ' is given by (2.1) - (2.4) supplemented by the limiting condition

$$\phi \sim \frac{P_n(\cos\theta)}{R^{n+1}} \text{ as } R = [r^2 + (y-\eta)^2]^{\frac{1}{2}} \to o,\; n=o,1,2,... \tag{4.1}$$

If we try as solutions

$$\phi = \frac{P_n(\cos\theta)}{R^{n+1}} + \int_0^\infty [A \cosh k(h-y) + B \sinh ky]\, J_o(kx)dk,$$

$$\phi' = \int_0^\infty A' \cosh k(h'+y)\, J_o\,(kr)\, dk,$$

and using the representations

$$\frac{P_n\,(\cos\theta)}{R^{n+1}} = \begin{cases} \dfrac{(-1)^n}{n!} \displaystyle\int_0^\infty k^n e^{k(y-\eta)} J_o(kr)dk, & y < \eta, \\ \dfrac{1}{n!} \displaystyle\int_0^\infty k^n\, e^{-k(y-\eta)} J_o(kr)dk, & y > \eta, \end{cases}$$

conditions (2.2), (2.4) are satisfied if

$$B = \frac{k^n e^{-k(y-\eta)}}{n!\cosh kh} ,$$

$$A \sinh kh + A' \sinh kh' = B + \frac{(-1)^n}{n!}\; k^n\, e^{-k\eta},$$

$$-c\, A \cosh kh + [sc \cosh kh' - k(1 + \beta k^2)\sinh kh']A' = \frac{(-1)^n}{n!} ck^n\, e^{-k\eta}.$$

Solving these equations and substituting, we obtain the expressions

$$\phi = \frac{P_n(\cos\theta)}{R^{n+1}} - - + \frac{1}{n!} \int_0^\infty k^n e^{-k(h-\eta)} \frac{\sinh ky}{\cosh kh} J_o(kr)dk,$$

$$\frac{1}{n!} \fint_0^\infty \frac{k^n}{\Delta} P(n-1)\cosh k(h-y)\, J_o(kr)dk, \tag{4.2}$$

$$\phi' = \frac{c}{n!} \fint_0^\infty \frac{k^n}{\Delta} [e^{-h(h-\eta)} + (-1)^n e^{k(h-\eta)}] \cosh k(h'+y) J_o(kr)dk, \tag{4.3}$$

where Δ is given by (3.2), $P(n)$ is given by (3.14) and as before the contour of integration is indented below the simple root $k = m$ of $\Delta = o$ on the positive real k-axis, which ensures that the radiation conditions are satisfied. For, by putting

$$2 J_o(kr) = H_o^{(1)}(kr) + H_o^{(2)}(kr),$$

rotating the contours in each integral into contours in the first and fourth quadrants (where $H_o^{(1)}(kr)$, $H_o^{(2)}(kr)$ are respectively small) and including the residue term at $k = m$, we obtain the diverging waves

$$\phi \sim C(n;\, h,\, h') \frac{\cosh m(h-y)}{\sinh mh} H_o^{(1)}(mr),$$

$$\phi' \sim -C(n;\, h,\, h') \frac{\cosh m(h'+y)}{\sinh mh'} H_o^{(1)}(mr),$$

as $r \to \infty$, where $C(n;\, h,\, h')$ is given by equation (3.7).

In a similar manner we calculate the velocity potential when the multipole singularity is at $(o, -\eta)$ in the upper liquid. These are

$$\phi = \frac{sc}{n!} \fint_0^\infty \frac{k^n}{\Delta} [e^{k(h'-\eta)} + (-1)^n e^{-k(h'-\eta)}] \cosh k(h-y) J_o(kr)dk \tag{4.4}$$

$$\phi' = \frac{P_n(\cos\theta)}{R^{n+1}} - - \frac{(-1)^n}{n!} \int_0^\infty \frac{k^n e^{-k(h'-\eta)}}{\cosh kh'} \sinh ky\, J_o(kr)dk$$

$$+ \frac{1}{n!} \fint_0^\infty \frac{k^n}{\Delta} Q(n-1) \cosh k(h'+y) \cosh k(h'+y) J_o(kr)dk, \tag{4.5}$$

where $R = [r^2 + (y+\eta)^2]^{\frac{1}{2}}$ and $Q(n)$ is given by (3.17). This motion has the diverging cylindrical waves

$$\phi \sim C(n;\, h',\, h) \frac{s \cosh m(h-y)}{\sinh mh} H_o^{(1)}(mr),$$

$$\phi' \sim C(n;\, h',\, h) \frac{s \cosh m(h'+y)}{\sinh mh} H_o^{(1)}(mr),$$

as $r \to \infty$.

5. SUBMERGED SINGULARITIES; THE LOWER FLUID IS OF FINITE CONSTANT DEPTH AND THE UPPER IS UNBOUNDED.

A statement of the boundary-value problems for the velocity potentials ϕ, ϕ' for the different types of singularities can be easily written down. These of the present case are similar to the corresponding ones treated in sections 3 and 4 with condition (2.3) replaced by

$$\nabla\phi' \to o \text{ as } y \to -\infty$$

and the radiation condition taking the simplet forms

$$\phi \sim C \cosh m(h-y)\, e^{im|x|},$$

$$\phi' \sim -C\, e^{my}\, e^{im|x|},$$

as $|x| \to \infty$ for line singularities, and the forms

$$\phi \sim C \cosh m(h - y)\, H^{(1)} (mr),$$

$$\phi' \sim -C e^{my} H^{(1)} (mr),$$

as $r \to \infty$ for point singularities, where C is a constant multiplier and m is a simple root of the equation $\Delta = 0$ where now

$$\Delta = c (\cosh kh + s \sinh kh) - k(1 + \beta k^2) \sinh kh. \tag{5.1}$$

The determination of ϕ, ϕ' for each singularity can be carried out independently. This was done by the second author [5], where he assumed ϕ, ϕ' to have the appropriate forms. They may also be determined by letting $h' \to \infty$ in the formulae obtained in the above sections. The velocity potentials for the different cases are as follows:

(a) Line singularities

(i) For a wave source at (o, η) in the lower fluid,

$$\phi = \log R + (2s - 1) \log R' + 2 \int_0^\infty \frac{\delta}{\Delta} (1 + \beta k^2) \cosh k(h - \eta) \cosh k(h - y) \times$$

$$\times \cos kx\, dk + \int_0^\infty \frac{2}{k}[s\, e^{-k(y + \eta)} - e^{-kh} \frac{\sinh k\eta \sinh ky}{\cosh kh}$$

$$- \frac{s\delta \cosh k(h - \eta)\cosh k(h - y)}{\cosh kh}] \cos kx\, dk, \tag{5.2}$$

$$\phi' = 2 \log R - 2 \int_0^\infty \frac{\delta}{\Delta} (1 + \beta k^2) \sinh kh \cosh k(h - \eta)\, e^{ky} \cos kx\, dk$$

$$+ \int_0^\infty \frac{2}{k} [e^{-k\eta} - \delta\cosh k(h - \eta)]\, e^{ky} \cos kx\, dk, \tag{5.3}$$

where now $\delta^{-1} = \cosh kh + s \sinh kh$. The path of integration is along $Im(k) = o$ o to ∞, indented below the simple pole at $k = m$.

These potentials have the outgoing waves

$$-C(o) \frac{\cosh m(h - y)}{m \sinh mh} e^{im|x|}, \quad C(o) \frac{e^{my}}{m} e^{im|x|},$$

as $|x| \to \infty$, where

$$C(n) = \frac{\pi ic}{n!} m^n \frac{[e^{-m(h - \eta)} + (-1)^n e^{m(h - \eta)}] \sinh mh}{hc + (1 + 3\beta m^2) \sinh^2 mh}, \tag{5.4}$$

The above velocity potentials can be written in slightly different forms suitable for use in the next section. These are

$$\phi = \log R - \frac{1-s}{1+s} \log R' + 2 \int_0^\infty \frac{\delta}{\Delta} (1 + \beta k^2)\cosh k(h - \eta)\cosh k(h - y)\cos kx\, dk$$

$$+ \frac{2}{1+s} \int_0^\infty \frac{e^{-kh}}{k} [s^2 - \delta(\sinh k\eta + s \cosh k\eta)(\sinh ky + s \cosh ky) \cos kx]dk \tag{5.5}$$

$$\phi' = \frac{2}{1+s} \log R - 2 \int_0^\infty \frac{\delta}{\Delta}(1+\beta k^2) \sinh kh \cosh k(h-\eta)e^{ky}\cos kx\, dk + \frac{2}{1+s} \int_0^\infty \frac{e^{-kh}}{k} \times$$

$$\times [s - \delta(\sinh k\eta + s \cosh k\eta)e^{ky} \cos kx]dk. \tag{5.6}$$

When the wave source is at $(0, -\eta)$ in the upper fluid,

$$\phi = 2s \log R - 2s \int_0^\infty \frac{\delta}{\Delta} e^{-k\eta}(1+\beta k^2)\sinh kh \cosh k(h-y)\cos kx\, dk + 2s \int_0^\infty \frac{e^{-kh}}{k} \times$$

$$\times [e^{-ky} - \delta\cosh k(h-y)]\cos kx\, dk, \tag{5.7}$$

$$\phi' = \log R R' + 2s \int_0^\infty \frac{\delta}{\Delta}(1+\beta k^2) \sinh^2 kh\, e^{-k(\eta - y)} \cos kx\, dk + 2s \int_0^\infty \frac{\delta}{k} \sinh kh \times$$

$$\times e^{-k(\eta - y)} \cos kx\, dk. \qquad (5.8)$$

or as in the previous case,

$$\phi = \frac{2s}{1+s} \log R - 2s \int_0^\infty \frac{\delta}{\Delta} e^{-k\eta} (1+\beta k^2) \sinh kh \cosh k(h-y) \cos kx\, dk + \frac{2s}{1+s} \int_0^\infty \frac{e^{-kh}}{k} \times$$

$$\times [s - \delta e^{-k\eta}(\sinh ky + s \cosh ky) \cos kx]\, dk, \qquad (5.9)$$

$$\phi' = \log R + \frac{1-s}{1+s} \log R' + 2s \int_0^\infty \frac{\delta}{\Delta} (1+\beta k^2) e^{k(y-\eta)} \sinh^2 kh \cos kx\, dk + \frac{2s}{1+s} \int_0^\infty \frac{e^{-kh}}{k} \times$$

$$\times (1 - \delta e^{k(y-\eta)} \cos kx)\, dk \qquad (5.10)$$

These potentials have the outgoing waves

$$-C'(o) \frac{\cosh m(h-y)}{m \sinh mh} e^{im|x|}, \quad C'(o) \frac{e^{my}}{m} e^{im|x|},$$

as $|x| \to \infty$, where

$$C'(n) = \frac{-2\pi i\, sc\, m^n}{n!} \frac{e^{-m\eta} \sinh^2 mh}{hc + (1+3\beta m^2) \sinh^2 mh} \qquad (5.11)$$

(ii) Multipoles singularities

When the singularity is at $(0,\eta)$ in the lower fluid,

$$\phi = \frac{\cos(n+1)\theta}{R^{n+1}} + \frac{1}{n!} \int_0^\infty k^n e^{-k(h-\eta)} \frac{\sinh ky}{\sinh kh} \cos kx\, dk - \frac{1}{n!} \int_0^\infty \frac{k^n}{\Delta} [(-1)^{n+1} \times$$

$$[K + k(1 + \beta k^2)] e^{-k\eta} - [sc - k(1+\beta k^2)] \frac{e^{-k(h-\eta)}}{\cosh kh}] \cosh k(h-y) \cos kx\, dk, \qquad (5.12)$$

$$\phi' = \frac{c}{n!} \int_0^\infty \frac{k^n}{\Delta} [e^{-k(h-\eta)} + (-1)^{n+1} e^{k(h-\eta)}] e^{ky} \cos kx\, dk, \qquad (5.13)$$

which have the outgoing waves

$$(n+1)\, C(n+1) \frac{\cosh m(h-y)}{m \sinh mh} e^{im|x|}, \quad -(n+1)\, C(n+1) \frac{e^{my}}{m} e^{im|x|},$$

as $|x| \to \infty$, where $C(n)$ is given by (5.4), and when the singularity is at $(o, -\eta)$ in the upper fluid,

$$\phi = \frac{2\, sc}{n!} \int_0^\infty \frac{k^n}{\Delta} e^{-k\eta} \cosh k(h-y) \cos kx\, dk, \qquad (5.14)$$

$$\phi' = \frac{\cos(n+1)\theta}{R^{n+1}} + \frac{1}{n!} \int_0^\infty \frac{k^n}{\Delta} e^{-k\eta}[c(\cosh kh - s \sinh kh) - k(1+\beta k^2) \sinh kh] \times$$

$$\times e^{ky} \cos kx\, dk, \qquad (5.15)$$

and have the outgoing waves

$$(n+1)\, C'(n+1) \frac{\cosh m(h-y)}{m \sinh mh} e^{im|x|}, \quad -(n+1)\, C'(n+1) \frac{e^{my}}{m} e^{im|x|},$$

as $|x| \to \infty$, where $C'(n)$ is given by (5.11).

(b) Point singularities

For a singularity in the lower fluid,

$$\phi = \frac{P_n(\cos\theta)}{R^{n+1}} + \frac{1}{n!} \int_0^\infty k^n e^{-k(h - \eta)} \frac{\sinh ky}{\cosh kh} J_o(kr)\, dk - \frac{1}{n!} \int_0^\infty \frac{k^n}{\Delta} [(-1)^n[K + k(1+\beta k^2)]$$

$$- [sc - k(1+\beta k^2)] \frac{e^{-k(h-\eta)}}{\cosh kh}] \cosh k(h-y) J_o(kr)\, dk , \qquad (5.16)$$

$$\phi' = \frac{c}{n!} \int_0^\infty \frac{k^n}{\Delta} [e^{-k(h-\eta)} + (-1)^n e^{k(h-\eta)}] e^{ky} J_o(kr) dk, \qquad (5.17)$$

with the outgoing waves

$$C(n) \frac{\cosh m(h-y)}{m \sinh mh} H_o^{(1)} (mr), \quad -C(n) \frac{e^{my}}{m} H_o^{(1)} (mr)$$

as $r \to \infty$, $C(n)$ being given by (5.4).

When the singularity is in the upper fluid,

$$\phi = \frac{2\, sc}{n!} \int_0^\infty \frac{k^n}{\Delta} e^{-k\eta} \cosh k(h-y) J_o(kr)\, dk, \qquad (5.18)$$

$$\phi' = \frac{P_n(\cos\theta)}{R^{n+1}} + \frac{1}{n!} \int_0^\infty \frac{k^n}{\Delta} e^{-k\eta} [c(\cosh kh - s \sinh kh) - k(1+\beta k^2) \sinh kh] \times$$

$$\times\, e^{ky} J_o(kr)\, dk, \qquad (5.19)$$

and as $r \to \infty$, ϕ , ϕ' have the outgoing waves

$$C'(n) \frac{\cosh m(h-y)}{m \sinh mh} H_o^{(1)} (mr), \quad -C(n) \frac{e^{my}}{m} H_o^{(1)} (mr),$$

$C'(n)$ being given by (5.11).

6. SUBMERGED SINGULARITIES. BOTH FLUIDS INFINITE.

Here also the boundary value problem for the velocity potentials ϕ , ϕ' is similar to the corresponding ones in sections 3,4 except that conditions (2.2) and (2.3) are replaced by

$$\nabla\phi \to o \quad \text{as} \quad y \to \infty,$$
$$\nabla\phi' \to o \quad \text{as} \quad y \to -\infty$$

respectively, and the radiation condition takes the forms

$$\phi \sim Ce^{-my} e^{im|x|}, \quad \phi' \sim -Ce^{my} e^{im|x|}$$

as $|x| \to \infty$, for line singularities, and the forms

$$\phi \sim Ce^{-my} H_o^{(1)} (mr), \quad \phi' \sim -Ce^{my} H_o^{(1)} (mr)$$

as $r \to \infty$, for point singularities, where C is a constant multiplier and m is now the simple zero of the equation

$$k(1 + \beta k^2) - c(1 + s) = o$$

The evaluation of ϕ , ϕ' for each singularity can be carried out independently (see [5]). They may also be evaluated by letting h in the results of the previous section tend formally to infinity. The velocity potentials for the different singularities are as follows:

(a) Line singularities.

(i) Wave source.

The velocity potentials are

$$\phi = \log R - \frac{1-s}{1+s} \log R' - \frac{2}{1+s} \int_0^\infty \frac{(1 + \beta k^2) e^{-k(y+\eta)}}{k(1 + \beta k^2) - c(1+s)} \cos kx\, dk, \qquad (6.1)$$

$$\phi' = \frac{2}{1+s} \log R + \frac{2}{1+s} \int_0^\infty \frac{(1 + \beta k^2) e^{k(y-\eta)}}{k(1 + \beta k^2) - c(1+s)} \cos kx\, dk, \qquad (6.2)$$

for a wave source in the lower fluid and

$$\phi' = \frac{2s}{1+s}\log R + \frac{2s}{1+s}\int_0^\infty \frac{(1+\beta k^2)e^{-k(y+\eta)}}{k(1+\beta k^2) - c(1+s)}\cos kx\, dk, \tag{6.3}$$

$$\phi' = \log R + \frac{1-s}{1+s}\log R' - \frac{2s}{1+s}\int_0^\infty \frac{(1+\beta k^2)e^{k(y-\eta)}}{k(1+\beta k^2) - c(1+s)}\cos kx\, dk, \tag{6.4}$$

for a wave source in the upper fluid.

(ii) Multipoles singularities.

The velocity potentials are

$$\phi = \frac{\cos(n+1)\theta}{R^{n+1}} + \frac{(-1)^{n+1}}{n!}\int_0^\infty \frac{[k(1+\beta k^2)+K]k^n e^{-k(y+\eta)}}{k(1+\beta k^2) - c(1+s)}\cos kx\, dk, \tag{6.5}$$

$$\phi' = \frac{2(-1)^n c}{n!}\int_0^\infty \frac{k^n e^{k(y-\eta)}}{k(1+\beta k^2) - c(1+s)}\cos kx\, dk, \tag{6.6}$$

if the singularity is in the lower fluid and

$$\phi = \frac{-2sc}{n!}\int_0^\infty \frac{k^n e^{-k(y+\eta)}}{k(1+\beta k^2) - c(1+s)}\cos kx\, dk, \tag{6.7}$$

$$\phi' = \frac{\cos(n+1)\theta}{R^{n+1}} + \frac{1}{n!}\int_0^\infty \frac{k^n[k(1+\beta k^2) - K]}{k(1+\beta k^2) - c(1+s)} e^{k(y-\eta)}\cos kx\, dk, \tag{6.8}$$

if the singularity is in the upper fluid.

(b) Point singularities

If the singularity is in the lower fluid

$$\phi = \frac{P_n(\cos\theta)}{R^{n+1}} + \frac{(-1)^n}{n!}\int_0^\infty \frac{[k(1+\beta k^2)+K]k^n e^{-k(y+\eta)}}{k(1+\beta k^2) - c(1+s)} J_0(kr)dk, \tag{6.9}$$

$$\phi' = \frac{2c(-1)^{n+1}}{n!}\int_0^\infty \frac{k^n e^{k(y-\eta)}}{k(1+\beta k^2) - c(1+s)} J_0(kr)dk \tag{6.10}$$

and if it is in the upper fluid,

$$\phi = \frac{-2\,sc}{n!}\int_0^\infty \frac{k^n e^{-k(y+\eta)}}{k(1+\beta k^2) - c(1+s)} J_0(kr)\, dk, \tag{6.11}$$

$$\phi' = \frac{P_n(\cos\theta)}{R^{n+1}} + \frac{1}{n!}\int_0^\infty \frac{k^n[k(1+\beta k^2) - K]}{k(1+\beta k^2) - c(1+s)} e^{k(y-\eta)} J_0(kr)dk. \tag{6.12}$$

7. SINGULARITIES AT THE SURFACE OF SEPARATION.

Clearly the results of the previous sections are not valid for $\eta = 0$. Here we use coordinates based on the singularity at the orgin. Then it may be shoen that the potentials are as follows

(a) Line singularities

(i) Wave source

Both fluids of finite depth

$$\left.\begin{aligned}\phi &= \int_0^\infty \frac{1}{k\Delta}[2cs + (k(1+\beta k^2)\sinh kh' - 2cs\cosh kh')\cosh k(h-y)\cos kx]dk,\\ \phi' &= \int_0^\infty \frac{1}{k\Delta}[2c + (k(1+\beta k^2)\sinh kh - 2c\cosh kh)\cosh k(h'+y)\cos kx]dk\end{aligned}\right\} \tag{7.1}$$

where Δ is given by (3.2).

Lower fluid of finite depth

$$\left.\begin{aligned}
\phi &= 2s \log R + \int_0^\infty \frac{\delta}{\Delta}(1 + \beta k^2)(\cosh kh - s \sinh kh)\cosh k(h-y)\cos kx\, dk \\
&\quad + 2s \int_0^\infty \frac{1}{k}[e^{-ky} - \delta\cosh k(h-y)]\cos kx\, dk, \\
\phi' &= 2 \log R - \int_0^\infty \frac{\delta}{\Delta}(1 + \beta k^2)(\cosh kh - s \sinh kh)\sinh kh\, e^{ky}\cos kx\, dk \\
&\quad + 2s \int_0^\infty \frac{\delta}{k} \sinh kh\, e^{ky} \cos kx\, dk
\end{aligned}\right\} \quad (7.2)$$

where $\delta^{-1} = \cosh kh + s \sinh kh$, and Δ is given by (5.1).

Both fluids infinite

$$\left.\begin{aligned}
\phi &= \frac{2s}{1+s} \log R - \frac{2}{1+s} \int_0^\infty \frac{(1 + \beta k^2)e^{-ky}}{k(1 + \beta k^2) - c(1+s)} \cos kx\, dk, \\
\phi' &= \frac{2}{1+s} \log R + \left(\frac{1-s}{1+s}\right) \int_0^\infty \frac{(1 + \beta k^2)e^{ky}}{k(1 + \beta k^2) - c(1+s)} \cos kx\, dk,
\end{aligned}\right\} \quad (7.3)$$

(ii) Multipoles

Both fluid of finite depth

For multipoles corresponding to $n = 1, 3, 5, \ldots$ (even multipoles)

$$\left.\begin{aligned}
\phi &= \frac{1}{(2m+1)!} \int_0^\infty \frac{k^{2m+1}}{\Delta} [2sc \cosh kh' - k(1 + \beta k^2)\sinh kh']\cosh k(h-y)\cos kx\, dk, \\
\phi' &= \frac{1}{(2m+1)!} \int_0^\infty \frac{k^{2m+1}}{\Delta} [2c \cosh kh - k(1 + \beta k^2)\sinh kh]\cosh k(h'+y)\cos kx\, dk,
\end{aligned}\right\} \quad (7.4)$$

and for the multipoles corresponding to $n = 0, 2, 4, \ldots$ (odd multipoles)

$$\left.\begin{aligned}
\phi &= \frac{c(1+s)}{(2m)!} \int_0^\infty \frac{k^{2m}}{\Delta} \sinh kh' \cosh k(h-y)\cos kx\, dk, \\
\phi' &= \frac{-c(1+s)}{(2m)!} \int_0^\infty \frac{k^{2m}}{\Delta} \sinh kh \cosh k(h'+y)\cos kx\, dk, \quad (m = 0, 1, 2, \ldots)
\end{aligned}\right\} \quad (7.5)$$

where Δ is given by (3.2).

Lower fluid of finite depth

Similarly for even multipoles $(m = 0, 1, 2, \ldots)$

$$\left.\begin{aligned}
\phi &= \frac{1}{(2m+1)!} \int_0^\infty \frac{k^{2m+1}}{\Delta} [2sc - k(1 + \beta k^2)]\cosh k(h-y)\cos kx\, dk, \\
\phi' &= \frac{1}{(2m+1)!} \int_0^\infty \frac{k^{2m+1}}{\Delta} [2c \cosh kh - k(1 + \beta k^2)\sinh kh]e^{ky}\cos kx\, dk,
\end{aligned}\right\} \quad (7.6)$$

and for odd multipoles

$$\left.\begin{aligned}
\phi &= \frac{c(1+s)}{(2m)!} \int_0^\infty \frac{k^{2m}}{\Delta} \cosh k(h-y)\cos kx\, dk, \\
\phi' &= \frac{-c(1+s)}{(2m)!} \int_0^\infty \frac{k^{2m}}{\Delta} \sinh kh\, e^{ky}\cos kx\, dk,
\end{aligned}\right\} \quad (7.7)$$

where Δ is given by (5.1).

Both fluids infinite

For multipoles corresponding to $n = 1, 3, 5, \ldots$ (even multipoles), we have

$$\left.\begin{aligned}
\phi &= \frac{1}{(2m+1)!} \int_0^\infty \frac{k^{2m+1}[k(1+ \beta k^2) - 2\, sc]e^{ky}}{k(1+\beta k^2) - c(1+s)} \cos kx\, dk, \\
\phi' &= \frac{1}{(2m+1)!} \int_0^\infty \frac{k^{2m+1}[k(1+ \beta k^2) - 2c]e^{my}}{1k(1+\beta k^2) - c(1+s)} \cos kx\, dk,
\end{aligned}\right\} \quad (7.8)$$

and for odd multipoles

$$\left.\begin{aligned} \phi &= \frac{-c(1+s)}{(2m)!} \int_0^\infty \frac{k^{2m} e^{-ky}}{k(1+\beta k^2) - c(1+s)} \cos kx \, dk, \\ \phi' &= \frac{c(1+s)}{(2m)!} \int_0^\infty \frac{k^{2m} e^{ky}}{k(1+\beta k^2) - c(1+s)} \cos kx \, dk, \quad (m = o, 1, 2, \ldots) \end{aligned}\right\} \tag{7.9}$$

(b) Point singularities

Both fluids of finite depth

For multipoles corresponding to n = o, 2, 4, ...(even multipoles)

$$\left.\begin{aligned} \phi &= \frac{1}{(2m)!} \int_0^\infty \frac{k^{2m}}{\Delta} [2sc \cosh kh' - k(1+\beta k^2)\sinh kh'] \cosh k(h-y)\, J_o(kr)\, dk, \\ \phi' &= \frac{1}{(2m)!} \int_0^\infty \frac{k^{2m}}{\Delta} [2sc \cosh kh - k(1+\beta k^2)\sinh kh] \cosh k(h'+y)\, J_o(kr)\, dk, \end{aligned}\right\} \tag{7.10}$$

and for multipoles corresponding to n = 1, 3, 5, ...(odd multipoles)

$$\left.\begin{aligned} \phi &= \frac{c(1+s)}{(2m+1)!} \int_0^\infty \frac{k^{2m+1}}{\Delta} \sinh kh' \cosh k(h-y)\, J_o(kr)dk, \\ \phi' &= \frac{-c(1+s)}{(2m+1)!} \int_0^\infty \frac{k^{2m+1}}{\Delta} \sinh kh \cosh k(h'+y)\, J_o(kr)dk, \quad (m = o, 1, 2, \ldots) \end{aligned}\right\} \tag{7.11}$$

where Δ is given by (3.2).

Lower fluid of finite depth

For even multipoles (m = o,1,2,...)

$$\left.\begin{aligned} \phi &= \frac{1}{(2m)!} \int_0^\infty \frac{k^{2m}}{\Delta} [2sc - k(1+\beta k^2)] \cosh k(h-y)\, J_o(kr)dk, \\ \phi' &= \frac{1}{(2m)!} \int_0^\infty \frac{k^{2m}}{\Delta} [2c \cosh kh - k(1+\beta k^2)\sinh kh] e^{ky} \cos kx \, dk, \end{aligned}\right\} \tag{7.12}$$

and for odd multipoles

$$\left.\begin{aligned} \phi &= \frac{c(1+s)}{(2m+1)!} \int_0^\infty \frac{k^{2m+1}}{\Delta} \cosh k(h-y)\, J_o(kr)dk, \\ \phi' &= \frac{-c(1+s)}{(2m+1)!} \int_0^\infty \frac{k^{2m+1}}{\Delta} \sinh kh\, e^{ky}\, J_o(kr)\, dk, \end{aligned}\right\} \tag{7.13}$$

where Δ is given by (5.1).

Both fluids infinite

For multipoles corresponding to n = o, 2, 4, ... (even multipoles), we have

$$\left.\begin{aligned} \phi &= \frac{1}{(2m)!} \int_0^\infty k^{2m} \frac{[k(1+\beta k^2) - 2sc] e^{ky}}{k(1+\beta k^2) - c(1+s)} J_o(kr)\, dk, \\ \phi' &= \frac{1}{(2m)!} \int_0^\infty k^{2m} \frac{[k(1+\beta k^2) - 2c] e^{-ky}}{k(1+\beta k^2) - c(1+s)} J_o(kr)\, dk, \end{aligned}\right\} \tag{7.14}$$

and for odd multipoles

$$\left.\begin{aligned} \phi &= \frac{-c(1+s)}{(2m+1)} \int_0^\infty k^{2m+1} \frac{e^{ky}}{k(1+\beta k^2) - c(1+s)} J_o(kr)\, dk, \\ \phi' &= \frac{c(1+s)}{(2m+1)} \int_0^\infty k^{2m+1} \frac{e^{-ky}}{k(1+\beta k^2) - c(1+s)} J_o(kr)\, dk. \quad (m = o, 1, 2, \ldots). \end{aligned}\right\} \tag{7.15}$$

It should be noted here that there is a non-uniqueness for $\beta > o$ to the extent that any multiple of a slope potential may be added. The forms given above correspond to a continuous interface slope at the orgin, where the interface elevation is always finite.

8. CONCLUSION.

A complete survey for all the basic singularities that can be used in two fluids problems with surface tension is presented. Results of Gorgui and Kaseem [2] and Kaseem [3] can be recovered by putting $\beta = o$ in the appropriate forms and also those of Rhodes-Robinson [4] can be obtained by putting $s = o$.

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