Vol. 9 No. 1 (1986) 111-122

SINGULARITY METHODS FOR MAGNETOHYDRODYNAMICS

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(Received October 4, 1984)

ABSTRACT. Singular solutions for linearized MHD equations based on Oseen approximations have been obtained such as Oseenslet, Oseenrotlet, mass source, etc. By suitably distributing these singular solutions along the axes of symmetry of an axially symmetric bodies, we derive the approximate values for the velocity fields, the force and the momentum for the case of translational and rotational motions of such bodies in a steady flow of an incompressible viscous and magnetized fluid.

KEY WORDS AND PHRASES. Fundamental solutions, Oseen approximation, Oseenslet, Doublet, Oseens rotlet, mass source, Distribution of Singularities, force, momentum, spheroid.
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 73D30, 46F.

1. INTRODUCTION.

The motion of a body in a steady flow of an incompressible viscous and magnetized fluid is governed by a set of nonlinear equations known as magnetohydrodynamic (MHD) equations. Exact solutions for these equations have been obtained only for a few very specific problems. However, for many applications these equations can be linearized by using two linearization schemes known as Oseen and Stokes approximations [1,2].

Different analytical techniques have been applied to solve these linearized forms for simple configurations, such as the classification separation of variables method [3,4], matched asymptotic expansions, and integral equation techniques [2].

A method of singularities has been developed recently to solve various boundary value problems in mathematical physics displines such as potential theory, scattering theory [5], hydrodynamics [6,7], and elasticity [8]. Our aim in this paper is to extend this method to solve some boundary value problems in MHD, using Oseen's approximation of the MHD equations. In Section 2 we present the mathematical formulation of the equations on the basis of this approximation. In Section 3, we present the fundamental solution (singularity) of Oseens equations and construct other singularities needed in our analysis, including Oseens rotlet, Oseens doublet, Oseens stresslet. In the last two sections we solve two types of motion problems for axially symmetric bodies by suitably distributing singularities about their axes of symmetry: First, the steady rotation of the bodies about their longitudinal axis; second, the uniform translational motion of those bodies in the direction of their axis of symmetry. Configurations of interest in this study are prolate and oblate spheriods and their limiting cases including the sphere, the circular disc, and the slender body. For these problems we derive formulae for the velocity fields along with the physical quantities, the drag and the force.

Using the analogy between the MHD and classicial hydrodynamics, the results for similar problems in the latter are deduced.

2. MATHEMATICAL FORMULATION.

The non-dimensional equations governing the steady flow of an incompressible, viscous, electrically conducting fluid are

$$R\bar{u}\cdot\nabla\,\bar{u} = -\nabla p + \frac{M^2}{R_m}\ \nabla \times \bar{H} \times \bar{H}\ ,$$

$$\nabla\cdot\bar{u} = 0\ ,\ \nabla\cdot\bar{H} = 0\ ,\ \nabla \times \bar{E} = 0\ , \tag{2.1}$$

$$R_m\bar{J} = \nabla \times \bar{H} = R_m(\bar{E} + \bar{u} \times \bar{H})$$

where $R = aU/\nu$ is the Reynolds number, $R_m = \sigma\mu_e aU$ is the magnetic Reynolds number, and $M = \mu_e H_0 a\ (\frac{\sigma}{\rho\nu})^{\frac{1}{2}}$ is the Hartmann number.

The vectors $\bar{u}$, $\bar{J}$, $\bar{H}$, and $\bar{E}$ are the velocity field, the electric current density, the magnetic field strength, and the electricl field strength, respectively. The constants ρ, p, ν, μ_e, and σ are the fluid density, pressure, kinematic viscosity, magnetic permeability, and the electric conductivity. The constants U, a, and H_0 are the typical velocity, characteristic length, and the uniform magnetic field. It is assumed that the magnetic field is oriented in the $\bar{e}_x$ direction i.e., along the x-axis. Furthermore, for the steady rotation problem the typical velocity is $U = a\,\Omega$ where Ω is the uniform angular velocity, and for the translational motion U is the velocity of the uniform flow in the $\bar{e}_x$ direction.

The Oseens approximation replaces the convective (non-linear) terms in equations (2.1) by convection due to the uniform velocity and uniform magnetic fields at infinity. Furthermore, because of the symmetry conditions, the electric field $\bar{E}$ is taken to be zero. So writing the velocity field $\bar{u}$ and the magnetic field $\bar{H}$ as

$$\bar{u} = \bar{e}_x + \bar{u}' \quad \text{and} \quad \bar{H} = \bar{e}_x + \bar{H}'$$

in equations (2.1), neglecting the quadratic terms, and dropping the primes, we obtain the following linear system [2]:

$$\bar{u} = \frac{1}{R_{-1}-R_1}\,[(R_{-1} - R)\bar{u}_1 - (R_1 - R)\bar{u}_{-1}] \tag{2.2}$$

$$\bar{H} = \frac{R_m}{R_{-1}-R_1}(\bar{u}_1 - \bar{u}_{-1}) \tag{2.3}$$

$$\nabla \cdot \bar{u}_i = 0, \quad i = \pm 1 \tag{2.4}$$

$$R_i \frac{\partial \bar{u}_i}{\partial x} + \nabla P - \nabla^2 \bar{u}_i = 0 \tag{2.5}$$

where R_1 and R_{-1} are the roots of the equation

$$R_i^2 - (R + R_m) R_i + RR_m - M^2 = 0, \text{ and } P = p + \frac{M^2}{R_m} \bar{H}\cdot\bar{e}_x . \tag{2.6}$$

Oseens approximation is valid for $R_m \ll 1$ and $R/M^2 \ll 1$, that is, when the magnetic field dominates over the inertia forces; however, it is valid for small and large Hartmann number.

In the presence of a solid, the boundary conditions associated with the above system of equations, in addition to the no-slip condition ($\bar{u} = 0$ on the surface), are $\bar{H} = 0$ inside and on the solid since it is insulated, and all the perturbations must vanish at infinity.

Two important special cases emerge from the above system of equations: Firstly when $R_i = R = 0$ equations (2.4) and (2.5) reduce to Stokes flow of non-conducting fluids. Secondly, the case $(R_i = k \neq 0)$ gives the equations of Oseens flow.

3. THE FUNDAMENTAL SOLUTIONS.

The solutions of the equations

$$R_i \frac{\partial \bar{u}_i}{\partial x} + \nabla P - \nabla^2 \bar{u}_i = \bar{g} \tag{3.1a}$$

and

$$\nabla \cdot \bar{u}_i = 0 , \tag{3.1b}$$

where $\bar{g}$ is a forcing function having some singular behavior in an infinite medium, are called fundamental solutions. The primary fundamental solution is called the Oseenslet and it corresponds to a forcing function

$$\bar{g} = 8\pi \bar{\alpha} \delta(\bar{x}) , \tag{3.2}$$

where $\bar{\alpha}$ is a constant vector, and $\delta(\bar{x})$ is the three dimensional delta function. The velocity and pressure for the Oseenslet are

$$\bar{U}_i^{os}(\bar{x},\bar{\alpha}) = \frac{2e^{k_i(x-r)}}{r} - \frac{1}{k_i} \nabla(\bar{\alpha}\cdot\nabla) \int_0^{-k_i(x-r)} \frac{1-e^{-t}}{t} dt , \tag{3.3}$$

$$P_i^{os}(\bar{x},\bar{\alpha}) = 2 \frac{\bar{\alpha}\cdot\bar{x}}{r^3} , \tag{3.4}$$

where $r = |\bar{x}|$, and $k_i = \frac{1}{2} |R_i|$.

The net force experienced by a control volume containing the Ossenselt is given by

$$\bar{F} = 8 \pi \mu \bar{\alpha} . \tag{3.5}$$

The linearity of Oseen's equation implies that derivatives of any order of the Oseenslet in an arbitrary fixed direction is again a solution of (3.1), with a forcing function having the same derivative of $\bar{g}$. These derivatives can be obtained easily by considering the Taylor series expansions of the velocity and the pressure of the Oseenslet about a fixed point $\bar{\beta} \neq 0$, that is,

$$\overline{U}_i^{os}(\overline{x}-\overline{\beta},\overline{\alpha}) = \overline{U}_i^{os}(\overline{x},\overline{\alpha})-(\overline{\beta}\cdot\nabla)\overline{U}_i^{os}(\overline{x},\overline{\alpha})+\frac{1}{2}(\overline{\beta}\cdot\nabla)^2\overline{U}_i^{os}(\overline{x},\overline{\alpha})+\ \ldots \tag{3.6}$$

with a similar expansion for the pressure $P_i^{os}\ (\overline{x}-\overline{\beta},\overline{\alpha})$. The first term in (3.6) is the Oseenslet itself. The second term is the "Oseensdoublet" and the third one is the "Oseensquadropole".

The Oseensdoublet is given by

$$\overline{U}_i^{od}(\overline{x},\overline{\alpha},\overline{\beta}) = -2\overline{\alpha}(\overline{\beta}\cdot\nabla)\frac{e^{k_i(x-r)}}{r}+\frac{1}{k_i}\nabla(\overline{\beta}\cdot\nabla)(\overline{\alpha}\cdot\nabla)\int_0^{k_i(x-r)}\frac{1-e^{-t}}{t}\,dt\ , \tag{3.7}$$

$$P_i^{od}\ (\overline{x},\overline{\alpha},\ \overline{\beta}) = 2(\overline{\beta}\cdot\nabla)\frac{\overline{\alpha}\cdot\overline{x}}{r^3}\ , \tag{3.8}$$

and the corresponding forcing function

$$\overline{g}_{od} = -8\ \pi\{\overline{\beta}\ \cdot\ \nabla\delta(\overline{x})\}\ \overline{\alpha}\ . \tag{3.9}$$

The Oseensdoublet can be written as a sum of antisymmetric and symmetric (with respect to interchanging $\overline{\alpha}$ and $\overline{\beta}$) terms, respectively called "Oseensrotlet" and "Oseensstresslet" as in hydrodynamics Stokes flow [7]. The velocity and the pressure of the Oseensrotlet are give by

$$\overline{U}_i^{or}\ (\overline{x},\overline{\gamma}) = -\ \nabla\ x\ \frac{e^{k_i(\underline{x}-r)}}{r}\,\gamma \qquad (\overline{\gamma} = \overline{\alpha}\ x\ \overline{\beta}) \tag{3.10}$$

$$P_i^{or}\ (\overline{x},\overline{\gamma}) = 0 \tag{3.11}$$

with the corresponding forcing function

$$\overline{g}_{or} = -4\ \pi\ \nabla\ x\ \delta(\overline{x})\ \overline{\gamma}\ . \tag{3.12}$$

The net torque exerted by an Oseens rotlet enclosed by a control volume on the surrounding fluid is

$$\overline{M} = -4\pi\ \mu\ \overline{\gamma}\ . \tag{3.13}$$

The velocity vector, the pressure, and the forcing function of the Oseens stresslet are

$$\overline{U}_i^{ss}(\overline{x},\overline{\alpha},\overline{\beta}) = -\ [\overline{\alpha}(\overline{\beta}\cdot\nabla)\ +\ \overline{\beta}(\overline{\alpha}\cdot\nabla)]\ \frac{e^{k_i(x-r)}}{r} + \frac{1}{k_i}\nabla(\overline{\beta}\cdot\nabla)(\overline{\alpha}\cdot\nabla)\int_0^{k_i(x-r)}\frac{1-e^{-t}}{t}\,dt \tag{3.14}$$

$$P_i^{ss}\ (\overline{x},\overline{\alpha},\overline{\beta}) = 2[(\overline{\beta}\cdot\nabla)\overline{\alpha}\ +\ (\overline{\alpha}\cdot\nabla)\overline{\beta}]\ \cdot\ \frac{\overline{x}}{r^3}\ , \tag{3.15}$$

$$\overline{g}_{ss} = -4\pi[(\overline{\beta}\cdot\nabla)\overline{\alpha}\delta(\overline{x})\ +\ (\overline{\alpha}\cdot\nabla)\overline{\beta}\delta(\overline{x})]\ . \tag{3.16}$$

Due to the symmetry property this singularity contributes neither a net force nor a momentum to the surrounding medium.

Another singularity which is useful in the present study is called "mass source". Its velocity, pressure and forcing functions are

$$\overline{U}^{ms}\ (\overline{x}) = \nabla\left(\frac{1}{r}\right)\ , \tag{3.17}$$

$$P^{ms}\ (\overline{x}) = \frac{x}{r^3}\ , \tag{3.18}$$

$$\overline{g}_{ms}(\overline{x}) = -\ \frac{1}{2k_i}\ \nabla\delta(\overline{x})\ . \tag{3.19}$$

Solutions of various boundary value problems in MHD involving the motion of axially symmetric bodies can be obtained by superposition of flows due to a suitable distribution of some of these singular solutions along the axis of symmetry of the body. This will be demonstrated in the following sections.

4. STEADY ROTATION OF PROLATE SPHEROID.

Let us assume that the prolate spheroid

$$\frac{x^2}{a^2} + \frac{\rho^2}{b^2} = 1 \ , \ \rho^2 = y^2 + z^2 \ , \quad a \geq b \tag{4.1a}$$

is rotating around the x-axis with angular velocity Ω in a viscous and electrically conducting flow. The focal length $2c$ and the eccentricity e of the spheroid are related by

$$c = (a^2 - b^2)^{\frac{1}{2}} = ea \ , \ 0 \leq e \leq 1 \ . \tag{4.1b}$$

The velocity vector of the spheroid is

$$\bar{U} = \Omega \ \bar{e}_x \ x\bar{X} = \Omega \ (-z\bar{e}_y + y\bar{e}_z) \ . \tag{4.2}$$

Now we construct the required solution of (2.4) and (2.5) by taking a line distribution of Oseensrotlets along the x-axis, between the foci, that is,

$$\bar{u}_i(\bar{x}) = \bar{U} - \int_{-c}^{c} g(t)(c^2-t^2)\bar{U}_i^{or} \ (\bar{x}-\bar{t}, \ \bar{e}_x)dt \tag{4.3}$$

where $\bar{t} = t \ \bar{e}_x$, and $g(t)(c^2 - t^2)$ is the strength of the distribution. This solution satisfies the boundary condition at infinity. Applying the no-slip condition $(\bar{u}_i = 0)$ we obtain the following Fredholm equation for the function $g(t)$

$$\Omega = \int_{-c}^{c} \frac{g(t)(c^2-t^2)\epsilon^{k_i(x-t-r)}(1+k_i r)}{r^3} dt \ . \tag{4.4}$$

The solution of this equation will be obtained using a perturbation technique for small values of k_i or equivalently, for small Hartmann number M. For this purpose we write $g(t)$ as a power series in k_i, that is

$$g(t)=g_0(t) + g_1(t)k_i + g_2(t)k_i^2 + g_3(t)k_i^3 + g_4(t)k_i^4 + 0(k_i^5) \ . \tag{4.5}$$

Expanding the exponential function in the integrand and then equating the coefficients of different powers of k_i leads to the following system of integral equations:

$$\int_{-c}^{c} \frac{g_0(t)(c^2-t^2)}{r^3} dt = \Omega \tag{4.6}$$

$$\int_{-c}^{c} \frac{g_1(t)(c^2-t^2)}{r^3} dt = - \int_{-c}^{c} \frac{g_0(t)(c^2-t^2)(x-t)}{r^3} dt \tag{4.7}$$

$$\int_{-c}^{c} \frac{g_2(t)(c^2-t^2)}{r^3} dt = - \int_{-c}^{c} \frac{(c^2-t^2)}{r^3} (x-t)g_1(t) + \frac{1}{2} \{(x-t)^2 - r^2\}g_0(t) \ dt \tag{4.8}$$

$$\int_{-c}^{c} \frac{g_3(t)(c^2-t^2)}{r^3}\, dt = - \int_{-c}^{c} \frac{(c^2-t^2)}{r^3} (x-t)g_2(t) + \frac{1}{2} \{(x-t)^2 - r^2\}g_1(t)$$

$$+ \frac{1}{6} \{(x-t)^3 - 3(x-t)r^2 + 2r^3\}g_0(t)\, dt \tag{4.9}$$

and

$$\int_{-c}^{c} \frac{g_4(t)(c^2-t^2)}{r^3}\, dt = - \int_{-c}^{c} \frac{(c^2-t^2)}{r^3} (x-t)g_3(t) + \frac{1}{2} \{(x-t)^2 - r^2\}g_2(t)$$

$$+ \frac{1}{6} \{(x-t)^3-3(x-t)r^2 + 2r^3\}g_1(t)$$

$$+ \frac{1}{24} \{(x-t)^4-6(x-t)^2r^2 + 8(x-t)r^3- 3r^4\}g_0(t)\, dt. \tag{4.10}$$

Equation (4.6) is the same integral equation which appears in the rotational motion of prolate spheroids in Stokes flow. Thus its solution is

$$g_0(t) = \Omega \left[\frac{2e}{1 - e^2} - L \right]^{-1} , \tag{4.11}$$

where

$$L = \log \frac{1 + e}{1 - e} . \tag{4.12}$$

Next, substitution of (4.11) into (4.9) yields

$$\int_{-c}^{c} \frac{g_1(t)(c^2-t^2)}{r^3}\, dt = 2\, g_0(2e - L)x . \tag{4.13}$$

To solve (4.13) we set

$$g_1(t) = A_1 t . \tag{4.14}$$

So the integrated form of (4.13) is

$$A_1(c^2B_{3,1} - B_{3,3}) = 2g_0(2e - L)x \tag{4.15}$$

where the functions $B_{m,n}$ are defined by

$$B_{m,n} = \int_{-c}^{c} \frac{t^n}{r^m}\, dt \quad (n = 0,1,2,3,\ldots,\ m = -1,1,3,5,\ldots) . \tag{4.16a}$$

They satisfy the recurrence relation

$$B_{m,n} = - \frac{c^{n-1}}{m-2} \left(\frac{1}{r_2^{\,m-2}} + \frac{(-1)^n}{r_1^{\,m-2}} \right) + \frac{n-1}{m-2} B_{m-2,n-2} + x B_{m,n-1} , \quad n \geq 2 \tag{4.16b}$$

$$B_{1,0} = \log \frac{r_2 - (x-c)}{r_1 - (x+c)} \tag{4.16c}$$

$$B_{3,0} = \frac{1}{\rho^2} \left(\frac{n+c}{r_1} - \frac{x-c}{r_2} \right) \tag{4.16d}$$

$$B_{1,1} = r_2 - r_1 + x B_{1,0} \tag{4.16e}$$

$$B_{3,1} = \left(\frac{1}{r_1} - \frac{1}{r_2} \right) + B_{3,0} \tag{4.16f}$$

where

$$r_1 = \sqrt{(x+c)^2 + \rho^2} , \quad r_2 = \sqrt{(x-c)^2 + \rho^2} . \tag{4.16h}$$

On the spheroid surface, equation (4.15) takes the form

$$A_1 \left[\frac{2e}{1 - e^2} + 4e - 3L \right] x = 2g_0(2e - L)x \ .$$

Therefore,

$$A_1 = 2g_0(2e-L) \left[\frac{2e}{1 - e^2} + 4e - 3L \right]^{-1} \ . \tag{4.17}$$

By the same procedure we solve equations (4.8), (4.9) and (4.10). Curtailing the details, we obtain the following solutions:

$$g_2(t) = a^2C_0 + C_2t^2 \ , \tag{4.18}$$

$$g_3(t) = D_0a^3 + D_1a^2 + D_3t^3 \tag{4.19}$$

and

$$g_4(t) = E_0a^4 + E_1a^3t + E_2a^2t^2 + E_4t^4 \tag{4.20}$$

where

$$C_2 = \left[\frac{2e}{1 - e^2} + 13e - \frac{3(5-e^2)L}{2} \right]^{-1} \left[A_1\{9e - \frac{3(3-e^2)L}{2}\} - g_0\{e - \frac{(1-e^2)L}{2}\} \right] \tag{4.21a}$$

$$C_0 = \left[\frac{2e}{1 - e^2} - L \right]^{-1} \left[C_2(3e - \frac{3-e^2}{2} L) - A_1(3e - \frac{3-e^2}{2} L + g_0(e - \frac{(1-e^2)}{2} L \right] , \tag{4.21b}$$

$$D_0 = - \frac{4}{9} g_0e^3 \left[\frac{2e}{1 - e^2} - 1 \right]^{-1} , \tag{4.22a}$$

$$D_3 = \left[\frac{2e}{1 - e^2} + 33e - \frac{16}{3} e^3 - \frac{5(7-3e^2)}{2} L \right]^{-1} \left[\{20e - \frac{16}{3} e^3 -2(5-3e^2) L\}C_2 \right.$$

$$\left. - \{3e - 2e^3 - \frac{3(1-e^2)}{2} L\} A_1 - \frac{2}{9} e^3g_0 \right] , \tag{4.22b}$$

$$D_1 = \left[\frac{2e}{1-e^2} + 4e - 3L \right]^{-1} [H_1 - D_3\{-15e + 4e^3 + \frac{3(5-3e^2)}{2} L\}] \tag{4.22c}$$

with

$$H_1 = \{4e^3 - 12e + 2(3-2e^2)L\} C_2 + (4e - 2L) C_0$$

$$+ 3e - 2e^3 - \frac{3(1-e^2)}{2} L \quad A_1 + \frac{2}{3} e^3g_0 \ , \tag{4.22d}$$

and

$$E_4 = [\frac{2e}{1-e^2} + \frac{307}{4} e - \frac{113}{4} e^3 - \frac{15}{8} (21 - 14e^2 + e^4)L]^{-1} L_4 \ , \tag{4.23a}$$

$$E_2 = [\frac{2e}{1-e^2} + 13e - \frac{3(5-e^2)}{2} L]^{-1} [L_3 - E_4 \{\frac{-105}{2} e + \frac{55}{2} e^3$$

$$+ \frac{3}{4} (35 - 30e^2 + 3e^4)L \}] \ . \tag{4.23b}$$

$$E_1 = [\frac{2e}{1-e^2} + 4e - 3L]^{-1} L_2 \ , \tag{4.23c}$$

$$E_4 = \left[\frac{2e}{1-e^2} - L \right]^{-1} \left[L_1 - E_4 \left\{ \frac{15}{4} e - \frac{13e^3}{4} - \frac{3(1-e^2)(5-e^2)}{8} L \right\} - E_2 \left\{ -3e + \frac{3-e^2}{2} L \right\} \right] , \quad (4.23d)$$

where

$$L_1 = D_3 \left\{ \frac{15}{4} e - \frac{13}{4} e^3 - \frac{3(1-e^2)(5-e^2)}{8} L \right\} + D_1 \left\{ -3e + \frac{3-e^2}{2} L \right\}$$
$$+ C_2 \left\{ \frac{-3}{2} e + \frac{3}{2} e^3 + \frac{1}{4} (1-e^2)(3-e^2) L \right\} + C_0 \left\{ e - \frac{1-e^2}{2} L \right\}$$
$$+ \frac{A_1}{6} \left\{ \frac{3}{2} e - \frac{5}{2} e^3 - \frac{3}{4}(1-e^2)^2 L \right\} + \frac{1}{6} e^3 g_0 , \quad (4.24a)$$

$$L_2 = D_0 (4e - 2L) - \frac{4}{9} e^3 g_0 , \quad (4.24b)$$

$$L_3 = D_3 \left\{ -\frac{75}{2} e + \frac{47}{2} e^3 + \frac{3(25-24e^2+3e^4)}{4} L \right\} + D_2 \left\{ 9e - \frac{9-3e^2}{2} L \right\}$$
$$+ C_2 \left\{ 9e - 8e^3 - \frac{1}{2} (1-e^2)(9 - 2e^2 L) \right\} + C_0 \left\{ -e + \frac{1-e^2}{2} L \right\}$$
$$+ \frac{A_1}{6} \left\{ -3e + 5e^3 + \frac{3}{2} (1-e^2)^2 L \right\} + \frac{1}{3} e^3 g_0 \quad (4.24c)$$

and

$$L_4 = D_3 \left\{ \frac{175}{4} e - \frac{275}{12} e^3 - \frac{5}{8} (35 - 30e^2 + 3e^4) L \right\}$$
$$+ C_2 \left\{ -\frac{15}{2} e + \frac{13}{2} e^3 + \frac{3}{4} (1-e^2)(5-e^2) L \right\}$$
$$+ \frac{A_1}{6} \left\{ \frac{3}{2} e - \frac{5}{2} e^3 - \frac{3}{4} (1-e^2)^2 L \right\} - \frac{e^3 g_0}{18} . \quad (4.24d)$$

The value of the torque $\bar{M}$ experienced by the spheroid is obtained by adding the torques exerted by the distributed Oseensrutlets, that is,

$$\bar{M}_i = -8\pi\mu \, \bar{e}_x \int_{-c}^{c} (c^2 - t^2) g(t) \, dt$$
$$= -\frac{32}{3} \pi \mu a^3 \bar{e}_x [e^3 g_0 + (e^3 C_0 + \frac{1}{5} e^5 C_2)(ak_i)^2$$
$$+ e^3 D_0 (ak_i)^3 + (e^3 E_0 + \frac{1}{5} e^5 E_2 + \frac{3}{35} e^7 E_4)(ak_i)^4 + O(k_i^5)] . \quad (4.25)$$

The results for a sphere can be obtained from the above by taking the limit as e approaches zero, and those for an oblate spheroid by replacing c by ic and e by $ie(1-e^2)^{\frac{1}{2}}$. Then by taking the limit as e approaches one in the latter, the results for a circular disc are obtained. Thus the torques exerted by a sphere of radius a and by a circular disc of radius b are, respectively, given by

$$\bar{M}_i = -8 \pi\mu\Omega a^3 \bar{e}_x [1 + \frac{4}{15} (ak_i)^2 - \frac{1}{3} (ak_i)^3 + \frac{8}{25} (ak_i)^4] \quad (4.26)$$

and

$$\bar{M}_i = -\frac{32}{3} \pi\Omega b^3 \bar{e}_x [1 + \frac{1}{5} (bk_i)^2 - \frac{4}{9} (bk_i)^3 + \frac{5}{21} (bk_i)^4] . \quad (4.27)$$

Formulae for the torques on the rotating sphere and rotating circular disc about its diameter have been obtained previously by several authors [9,2] using different techniques, up to the third order. Those results are special cases of (4.26) and (4.27) when $ak_i = bk_i = M/2$, while the fourth order term appear to be new.

5. TRANSLATION OF PROLATE SPHEROID.

In this section the prolate spheroid (4.1) is assumed to have a uniform velocity $U\bar{e}_x$ directed along its axis of symmetry. In this case the velocity will be obtained by employing a line distribution of Oseenslets in the $\bar{e}_x$ direction with strength $f(x)$, and a line distribution of mass sources with strength $h(x)$ between the foci of the spheriod. Thus the solution will have the following functional expression

$$\bar{u}_i(\bar{x}) = U\,\bar{e}_x + \int_{-c}^{c} f(t)\,\bar{U}_i^{os}(\bar{x}-\bar{t},\,\bar{e}_x)\,dt - \int_{-c}^{c} h(t)\,\bar{U}_i^{ms}(\bar{x}-\bar{t})\,dt\,. \tag{5.1}$$

On the surface of the spheroid the no-slip condition gives the following integral equation for $f(t)$ and $h(t)$:

$$-U\,\bar{e}_x = \int_{-c}^{c} f(t)\,\bar{U}_i^{os}(\bar{x}-\bar{t},\,\bar{e}_x)\,dt - \int_{-c}^{c} h(t)\,\bar{U}_i^{ms}(\bar{x}-\bar{t})\,dt\,. \tag{5.2}$$

Again, for small values of k_i we have

$$\bar{U}_i^{os}\,(\bar{x},\,e_x) = \bar{U}_0\,(\bar{x},\,\bar{e}_x) + \bar{U}_1(\bar{x},\,\bar{e}_x)\,k_i + \bar{U}_2(\bar{x},\,e_x)k_i^2 + O(k_i^3)\,, \tag{5.3a}$$

where

$$\bar{U}_0\,(\bar{x},\,\bar{e}_x) = \frac{1}{r}\,\bar{e}_x + \frac{x\,\bar{x}}{r^3}\,,\quad r = |\bar{x}| \tag{5.3b}$$

$$\bar{U}_0(\bar{x},\,\bar{e}_x) = \frac{x-r}{r}\,\bar{e}_x + \frac{x^2-r^2}{2r^3}\,\bar{x} \tag{5.3c}$$

$$\bar{U}_2(\bar{x},\,\bar{e}_x) = \frac{(x-r)^2}{2r}\,\bar{e}_x + \frac{(x-r)^2(x+2r)}{6r^3}\,\bar{x}\,. \tag{5.3d}$$

The strengths $f(x)$ and $h(x)$ are assumed to have the Maclaurin series

$$f(x) = f_0(x) + f_1(x)k_i + f_2(x)k_i^2 + O(k_i^3) \tag{5.4a}$$

$$h(x) = h_0(x) + h_1(x)k_i + h_2(x)k_i^2 + O(k_i^3)\,. \tag{5.4b}$$

Substituting (5.3), (5.4), and (5.5) into equation (5.2), and equating the coefficients of like powers of k_i, we obtain the following system of equations:

$$\int_{-c}^{c} [f_0\bar{U}_0(\bar{x}-\bar{t},\,\bar{e}_x) - h_0(t)\,\bar{U}^{ms}(\bar{x}-\bar{t})]\,dt = -U\,\bar{e}_x \tag{5.6}$$

$$\int_{-c}^{c} [f_1(t)\bar{U}_0(\bar{x}-\bar{t},\,\bar{e}_x) - h_1(t)\,\bar{U}^{ms}(\bar{x}-\bar{t})]\,dt = -\int_{-c}^{c} f_0(t)\,\bar{U}_1(\bar{x}-\bar{t},\,\bar{e}_x)\,dt \tag{5.7}$$

$$\int_{-c}^{c} [f_2(t)\bar{U}_0(\bar{x}-\bar{t},\,\bar{e}_x) - h_2(t)\,\bar{U}^{ms}(\bar{x}-\bar{t})]\,dt = -\int_{-c}^{c} [f_1(t)\bar{U}_1(\bar{x}-\bar{t},\,\bar{e}_x) + f_0(t)\,\bar{U}_2(\bar{x}-\bar{t},\,\bar{e}_x)]\,dt\,. \tag{5.8}$$

To solve equation (5.6), we set

$$f_0(t) = F_0 \ , \ h_0(t) = H_0 t \tag{5.9a,b}$$

where F_0 and H_0 are constants to be found. Substituting (5.9) into (5.6) and using the function $B_{m,n}$, equation (5.6) has the following form

$$[F_0(B_{1,0} + {}^2B_{3,0} - 2\,B_{3,1} + B_{3,2}) + H_0(xB_{3,1} - B_{3,2})]\,\bar{e}_x$$

$$+ [F_0(x\,B_{3,0} - B_{3,1}) + H_0\,B_{3,1}]\,\rho\bar{e}_\rho = -\,U\bar{e}_x\ . \tag{5.10}$$

By making use of the recurrence relation (4.16b) and the values of $B_{m,n}$ on the surface of the prolate spheroid, equation (5.10) takes the form

$$\left(\frac{2}{e}F_0 + \frac{2e}{1-e^2}H_0\right)\left(\frac{b^2\bar{e}_x + e^2 x\rho e_\rho}{a^2 - e^2x^2}\right) + [(2L - \tfrac{1}{e})F_0 - L_0H_0]\,\bar{e}_x = -\,U\,\bar{e}_x\ . \tag{5.11}$$

This equation is satisfied if

$$F_0 = \frac{-e^2}{(1-e^2)}H_0 = Ue^2\left[\,2e - (1+e^2)\,L\,\right]^{-1}. \tag{5.12}$$

Following the same procedure for equation (5.7) we find, after some computations, that

$$f_1(t) = F_{10} + F_{11}\,t \tag{5.13a}$$

and

$$h_1(t) = H_{10} + H_{11}t + H_{12}t^2 \tag{5.13b}$$

where

$$F_{10} = \frac{-e^2}{1-e^2}\,H_{11} = \frac{-2ea}{U}\,F_0^2\ , \tag{5.13c}$$

$$H_{10} = \frac{1}{3}\,e^2a^2H_{12} = \frac{a^2(1-e^2)\{2e-(1-e^2)L\}}{12e - 2(3-e^2)L}\,F_0 \tag{5.13d}$$

and

$$F_{11} = \frac{-6e + 10e^3 + 3(1-e^2)^2\,L}{e^2[12e - 2(3-e^2)\,L]}\,F_0\ . \tag{5.13e}$$

The solution of equation (5.8) is

$$f_2(t) = F_{20} + F_{21}\,t + F_{22}\,t^2\ , \tag{5.14a}$$

$$h_2(t) = H_{20} + H_{21}\,t + H_{22}\,t^2 + H_{23}\,t^3\ , \tag{5.14b}$$

where

$$H_{22} = \frac{ae(1-e^2)\,\{-10e + (5-e^2)L\}F_0}{\{6e+(3-5e^2)L\}\{-2e+(1+3^2)L\}}\ , \tag{5.15a}$$

$$F_{21} = \frac{2ae^2\{38e^3 - 18e + (9-22e^2 + 5e^4)L\}}{3(1-e^2)\,\{-2e + (1+e^2)L\}\{6e-(3-e^2)L\}}\ , \tag{5.15b}$$

$$H_{23} = \left(\frac{1-e^2}{3e}\right)\frac{\{2e^3 - 6e + (3-2e^2-e^4)\,L\}\,F_0 + 3e\{-2e^2+(1-e^2)(L-3)L\}F_{11}}{6e^2 + 2e^4 - (6e - 5e^3 + e^5)L + \frac{3}{2}\,(1-e^2)^2L^2}\ , \tag{5.15c}$$

$$F_{22} = \frac{\{9e - \frac{(9-3e^2)L}{2}\}H_{23} + \frac{3e^2-1}{3e}F_0 + \{e-(1-e^2)L\}F_{11}}{8e - (4-2e^2)L} \quad ,(5.15d)$$

$$F_{20} = -\frac{1}{3}e^2a^2F_{22} + [\frac{a^2}{18e^2}\{-6e + 16e^3 - 12e^5 + 3(1-e^2)^3L\}F_0$$

$$-2e^3aF_{10} + \frac{a^2}{6}\{-6e + 2e^3 + (1-e^2)(3-e^2)L\}F_{11}][2e-(1+e^2)L]^{-1} \; ; \quad (5.15e)$$

$$H_{20} = -\frac{a^2}{4e}\{(1-e^2)^2F_{10} + 6e^3H_{22} + 4e(1-e^2)F_{21}\} \; , \quad (5.15f)$$

$$H_{21} = -\left(\frac{1-e^2}{6e^2}\right)\left[6F_{20} + \frac{a^2(1-e^2)^2}{e^2}F_0 + 3a^2(1-e^2)F_{11} + \frac{6a^2e^4}{(1-e^2)}H_{23} + 6e^2a^2F_{22}\right]. \quad (5.15h)$$

By super position of (3.5), the force experienced by the spheroid is

$$\bar{F}_i = -8\pi\mu \int_{-c}^{c} f(t)dt\,\bar{e}_x \quad (5.16)$$

$$= -16\pi\mu\, a[eF_0 + eF_{10}k_i + (eF_{20} + \frac{1}{3}a^2e^3F_{22})k_i^2 + O(k_i^3)]\,\bar{e}_x \; . \quad (5.17)$$

Following the procedure of section 4, the forces exerted by a sphere of radius a and a circular disk of radius b are given by

$$\bar{F}_i = 6\pi\mu\, Ua\{1 + \frac{3}{4}(ak_i) - \frac{43}{80}(ak_i)^2 + O(ak_i)^3\}\,\bar{e}_x \quad (5.18)$$

and

$$\bar{F}_i = 16\mu\, bU\{1 + \frac{2}{\pi}(ak_i) + \left(\frac{\pi^2+12}{12\pi^2}\right)(ak_i)^2 + O(ak_i)^3\}\,\bar{e}_x \; . \quad (5.19)$$

Results (5.18), (5.19) agree up to the first order with the known results [2], while the second order term appears to be new.

Another interesting limiting case is the elongated rod, in which the slenderness ration $\varepsilon = \frac{b}{2a}$ is small. In this case the force is given by

$$\bar{F}_i = \frac{-4\pi\mu\, Ua}{\ln\varepsilon}\left[1 - \{\frac{1}{2} + \frac{ak_i}{4} - \frac{(ak_i^2)}{18}\}\cdot\frac{1}{\ln\varepsilon} + O(ak_i)^3\right]\bar{e}_x \; . \quad (5.20)$$

6. NON-CONDUCTING FLUID FLOW.

For a non-conducting fluid we have $R_m = 0$, and therefore $M = 0$, hence the system of equations (2.4, 2.5) breaks down into two uncoupled systems of equations each associated with one of the roots (0,R) of equation (2.6).

The first system is

$$\nabla^2\bar{u} = P$$
$$\nabla\cdot\bar{u} = 0$$

which describes the steady Stokes flow, thus all the previous results in Section 4 and 5 reduce to that of CHWANG and WU [6,7], by putting $k_i = 0$.

Secondly, the system associated with the root R is

$$R \frac{\partial \bar{u}}{\partial x} = - \nabla P + \nabla^2 \bar{u}$$

$$\nabla \cdot \bar{u} = 0$$

which governs the steady Oseen flow.

For this type of flow the results of section 4 and 5 with $k_i = R/2$ are believed to be new, apart from the limiting formulas (4.26), (4.27), (5.18), and (5.19) which agree up to the first order with Lamb [10]. Finally, formula (5.20) for the slender bodies concides with the formula derived by Dorel [11].

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