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ON MENNICKE GROUPS OF DEFICIENCY ZERO I

MUHAMMAD A. ALBAR

Department of Mathematical Sciences
University of Petroleum and Minerals
Dhahran, Saudi Arabia

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ABSTRACT. The Mennicke group $M(m,n,r) = \langle x,y,z \mid x^y = x^m,\ y^z = y^n,\ z^x = z^r\rangle$ is one of the few known 3-generator groups of deficiency zero. Several cases of $M(m,n,r)$ are studied.

KEY WORDS AND PHRASES. Presentation, Reidemeister-Schreier method, relation matrix.
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Mennicke [1] has given a class of three generator three relation groups defined by $M(m,n,r) = \langle x,y,z \mid x^y = x^m,\ y^z = y^n,\ z^x = z^r\rangle$ which he proves to be finite for $m = n = r \geq 3$ (see also Higman [2].) Macdonald [3] has shown that the above group is finite provided that neither $m^2 = 1$, $n^2 = 1$, nor $r^2 = 1$. For general m,n,r the above group is difficult to consider. Wamsley [3] discussed the group for some cases with $m = n = r$. The aim of this paper is to consider the group for several cases with general m,n,r.

a) The group $M = M(3,3,3) = \langle x,y,z \mid x^y = x^3,\ y^z = y^3,\ z^x = z^3\rangle$. Wamsley has shown that M' is abelian and $|M|$ divides 2^{11}. We use his result that M' is abelian and prove:

THEOREM 1. $|M| = 2^{11}$.

PROOF. We notice that $\frac{M}{M'} = Z_2 \times Z_2 \times Z_2$. A straightforward application of the Reidemeister-Schreier rewriting process can be used to find the order of M'. We suppress the details and merely notice that the relation matrix for M' is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Therefore $M' = Z_8 \times Z_8 \times Z_4$ and $|M| = 2^3(2^3 \times 2^3 \times 2^2) = 2^{11}$.

REMARK 1. Another group of deficiency zero is Johnson's group [4],

$$J(m,n,r) = \langle x,y,z \mid x^y = y^{n-2}\, x^{-1}\, y^{n+2},\ y^z = z^{r-2}\, y^{-1}\, z^{r+2},\ z^x = x^{m-2}\, z^{-1}\, x^{m+2}\rangle .$$

The order of $J = J(2,2,2)$ is 7.2^{11}, [4]. A question could be raised here if M and the 2-Sylow subgroup of J are isomorphic. To answer this question let $H = \langle x^{-1} y^2, y^{-1} z^2, z^{-1} x^2 \rangle < J$. We find that $H \triangleleft J$ and $\frac{J}{H} = Z_7$. Therefore H is the 2-Sylow subgroup of J. Using the Redemeister-Schreier process we write a presentation for H which gives $\frac{H}{H'} = Z_2 \times Z_2 \times Z_2 = \frac{M}{M'}$. A student K. F. Lee of David L. Johnson showed that M and H are different.

b) The group $M = M(m,n,0) = \langle x,y \mid x^y = x^m, y^{n-1} = e\rangle$, $m > 2$, $n > 2$. The relations $x^y = x^m$ and $y^{n-1} = e$ imply that the order of x is $(m^{n-1} - 1)$. We consider $H = \langle x \mid x^{(m^{n-1} - 1)}\rangle = Z(m^{n-1} - 1)$, $\frac{M}{H} = Z_{n-1}$. Therefore M is metacyclic and it is the split extension of Z_{n-1} by $Z(m^{n-1} - 1)$.

THEOREM 2. $M' = Z_d$ where $d = \frac{m^{n-1} - 1}{m - 1}$.

PROOF: We consider $H = \langle a = x^{m-1}\rangle$. The relations $a^x = a$ and $a^y = a^m$ imply that $H \triangleleft M$. $\frac{M}{H}$ is abelian implies that $H \supseteq M'$. But $a = x^{-1} y^{-1} xy \in M' \Longrightarrow H \subseteq M'$. Therefore $H = M'$.

The order of a is $\frac{m^{n-1} - 1}{(m-1,\ m^{n-1} - 1)} = \frac{m^{n-1} - 1}{m - 1} = m^{n-2} + m^{n-3} + \dots + m^2 + m + 1$.

REMARK 2. The above theorem could be proved using the Reidemeister-Schreier process.

REMARK 3. $\left|\frac{M}{M'}\right| = (m-1)(n-1)$ implies that $|M| = (n-1)(m^{n-1} - 1)$.

REMARK 4. The above theorem implies that M is a finite metabilian group.

REMARK 5. It is easy to see that $M(a, b, c) \cong M(b, c, a) \cong M(c, a, b)$ and $M(a,b,c) \not\cong M(a,c,b)$ in general.

REMARK 6. In working with Mennicke's group we find the commutator identity (known as the Witt identity)

$$[x, y, z^x][z, x, y^z][y, z, x^y] = e$$

quite helpful. This identity holds for any x, y and z in any group. We define $[x, y, z] = [[x,y], z]$ and $[x,y] = x^{-1} y^{-1} xy$.

c) $M = M(2,2,2) = \langle x, y, z \mid x^y = x^2, y^z = y^2, z^x = z^2\rangle$. Using the Witt identity we get $[x, z^2][z, y^2][y, x^2] = e$. We use the relations of M to get $x^2y^2z^2 = e$. Thus $z^2 = y^{-2}x^{-2}$ which together with $z^x = z^2$ gives $z = xy^{-2}x^{-3}$. We substitute in $y^z = y^2$ and use $x^y = x^2$ to get $y = x^{17}$. Finally $y = x^{17}$ and $x^y = x^2$ imply that $x = e$. The relations of M give $z = y = e$. Therefore, $M = E$.

d) $M(-1, -1, -1) = \langle x, y, z \mid x^y = x^{-1}, y^z = y^{-1}, z^x = z^{-1}\rangle$. $\frac{M}{M'} \cong Z_2 \times Z_2 \times Z_2$. A straightforward application of the Reidemeister-Schreier process gives that $M' = Z \times Z$ generated by $z x z^{-1}x^{-1}$ and $z y z^{-1}y^{-1}$. Therefore, we have proved:

THEOREM 3. M is an infinite metabilian group.

e) $M(2, 2, -1) = \langle x, y, z \mid x^y = x^2, y^z = y^2, z^x = z^{-1}\rangle$. Using the Witt identity we get $z^{-1}y^{-1}z^{-2}yz = x$. We use this relation together with the relations of M to get

$x = z^{-4}$. Substituting in $z^x = z^{-1}$ we get $z^2 = e$ and so $x = e$. We notice that $y = y^{z^2} = (y^z)^z = y \quad y^3 = e$. The relation $y^z = y$ becomes $(yz)^2 = e$. Thus $M = \langle y, z | y^3 = z^2 = (yz)^2 = e\rangle = S_3$.

f) $M(-1, -1, 0) = \langle x, y, z | x^y = x^{-1}, y^2 = e\rangle$. $\frac{M}{M'} = Z_2 \times Z_2$. Using the Reidemeister-Schreier process we get that M' is infinite cyclic generated x^2.

THEOREM 4. M is an infinite metabilian group.

REMARK 7. It is possible to find M' as follows. Let $H = \langle x^2|\rangle$. It is easy to see that $H \quad M$ and $\frac{M}{H} = z_2 \times z_2$. Therefore, $H \supset M'$. But $x^2 = y^{-1}x^{-1}yx \in M' \quad H \subset M'$. Thus $H = M'$.

g) $M(1, 0, -1) = \langle x, z | z^x = z^{-1}\rangle$. It is easy to see that $H = \langle z|\rangle$ is normal in M and $\frac{M}{H} = \langle x|\rangle$. Therefore M is the split extension of $\langle x|\rangle$ by $\langle z|\rangle$ where the action is given by $z^x = z^{-1}$, see [5]. We also notice that $(z^2)^x = z^{-2}$ and $xz^2x^{-1} = z^{-2}$ Therefore $K = \langle z^2\rangle \triangleleft M$. $\frac{M}{K} = Z \times Z_2 \Longrightarrow K \supset M'$. $z^2 = x^{-1}z^{-1}xz \Longrightarrow K \subset M'$.

Thus $K = M'$.

THEOREM 5. M is an infinite metabilian group.

h) It is easy to show the following cases:

(i) $M(1, 1, 1) = Z \times Z \times Z$ (ii) $M(1, 1, 0) = Z \times Z$

(iii) $M(1, 0, 0) = Z = M(1, 2, 0)$ (iv) $M(3, 2, 0) = Z_2$

(v) $M(0, 0, 0) = M(2, 2, 0) = M(2, 0, 0) = E$ (vi) $M(2, 3, 0) = S_3$.

(vii) $M(1, n, 0) = Z \times Z_{n-1}$ for $n > 1$.

(viii) $M(m, 2, 0) = M(m, 0, 0) = Z_{m-1}$ for $m > 2$.

(ix) $M(1, m, n)$ is infinite because $\frac{M(1, m, n)}{M'(1,m,n)}$ is infinite.

(x) $M(1, -1, 0) = Z \times Z_2$ (xi) $M(-m, 0, 0) = Z_{m+1}$, $m > 0$

(xii) $M(-m, 2, 0) = Z_{m+1}$, $m > 0$.

Mennicke's group was a generalization of a group given by Higman [2]. Another generalization of Higman's group was considered by Fluch [6] as

$$H = \langle a,b,c | b^{-\alpha}ab^{\alpha} = a^m, c^{-\beta}bc^{\beta} = b^n, a^{-\gamma}ca^{\gamma} = c^r\rangle.$$

We notice that when $\alpha = \beta = \gamma = 1$ then $H = M(m, n, r)$.

Another generalization of Mennicke's group was given by Post [7] as follows:

$$G(m,n,r,s,t) = \langle a,b,c | ab^m a^{-1} = b^n, bc^r b^{-1} = c^s, cac^{-1} = a^t\rangle.$$

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