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RESEARCH NOTES

LOCAL ENERGY DECAY FOR WAVES GOVERNED BY A SYSTEM OF NONLINEAR SCHRÖDINGER EQUATIONS IN A NONUNIFORM MEDIUM

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ABSTRACT. We show that the local energy of a smooth localized solution to a system of coupled nonlinear Schrödinger equations in a certain nonuniform medium decays to zero as the time approaches infinity.

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1. INTRODUCTION.

Consider a system of m coupled nonlinear Schrödinger equations in a nonuniform meduum

$$i(\partial/\partial t)U_n - (\partial^2/\partial x^2)U_n + F_n(|U_1|^2, \ldots, |U_n|^2, \ldots, |U_m|^2)U_n + k_n(x)U_n = 0 \quad (1.1)$$

where $n = 1, 2, \ldots, m$, k_n's are real-valued functions of x only and F_n's are real-valued functions. We will show that under certain conditions of F_n's and k_n's, namely,

$$F_n(|U_1|^2, \ldots, |U_n|^2, \ldots, |U_m|^2) = C_n|U_n|^2 + \sum_{h=1}^{n-1} |U_h|^2 + \sum_{h=n+1}^{m} |U_h|^2 \quad (1.2)$$

with positive constant C_n, for all $n = 1, 2, \ldots, m$, and

$$k_n(x) = 1/(1 + a^2x^2) \text{ with } 0 < a \le (2/3)^{\frac{1}{2}} \quad (1.3)$$

for all $n = 1, 2, \ldots, m$, the local energy $\sum_{n=1}^{m} \int_{-r}^{r} |U_n|^2(x, t)\,dx$ for the smooth and localized solution $(U_1, \ldots, U_m)$ decays to zero as t approaches infinity.

Eq. (1.1) with one component and in a linear type of nonuniform medium was derived by Chen and Liu [1-3] in the study of solitons in a nonuniform medium. See also Newell [4]. Gupta et al [5], Gupta [6] and Gupta and Ray [7] studied Eq. (1.1) with one component and a parabolic type of nonuniformity for its exact solution and the inverse scattering method. Eq. (1.1) with two components and $k_n = 0$, for all n, was derived for envelope waves with different circular polarizations in an isotropic nonlinear medium by Berkhoer and Zakharov [8] and also used by Elphick [9] for the quantum version of the one-component nonlinear Schrödinger model. Kaiser [10] discussed the well-posedness of it for an initial-value and boundary-value problem.

Our method in this work consists of an exploration of the conservation laws which Eq. (1.1) possesses and is a generalization of author's previous work [11] for the one-component nonlinear Schrödinger equation. In the following, we shall denote $(\partial/\partial x)w_n$ by $w_{n,x}$, etc., and the solution $(U_1, \ldots, U_m)$ will be assumed to be smooth and localized, i.e., U_n and all its partial derivatives approach zero as $|x|$ approaches infinity, for each t and for all n = 1, . . ., m.

2. METHOD.

Multiplying Eq. (1.1) by V_n, where V_n is the complex conjugate of U_n, and taking the imaginary part, we get

$$(|U_n|^2)_t = i(V_{n,x}U_n - U_{n,x}V_n)_x \tag{2.1}$$

Hence

$$\int_{-\infty}^{\infty} |U_n|^2(x,t)\,dx = \text{constant} \tag{2.2}$$

Next, multiplying Eq. (1.1) by $V_{n,t}$, taking the real part of it, making the use of (1.2) and integrating in x from $-\infty$ to ∞, we get

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} (|U_{n,x}|^2 + k_n|U_n|^2 + (1/2)C_n|U_n|^4)\,dx < \text{constant} \tag{2.3}$$

where the constant on the right-hand side is independent of t.

Now, taking the real part of $[(L_nU_n)_xV_n - (L_nU_n)V_{n,x}]$, where $L_nU_n = iU_{n,t} - U_{n,xx} + F_n(|U_1|^2, \ldots, |U_m|^2)U_n + k_nU_n$ and making the use of (1.2), we get

$$(1/2i)\sum_{n=1}^{m}(V_{n,x}U_n - U_{n,x}V_n)_t - (1/2)\sum_{n=1}^{m}(|U_n|^2)_{xxx} + 2\sum_{n=1}^{m}(|U_{n,x}|^2)_x$$
$$+ (1/2)\sum_{n=1}^{m}C_n(|U_n|^4)_x + (1/2)((\sum_{n=1}^{m}|U_n|^2)^2)_x - (1/2)\sum_{n=1}^{m}(|U_n|^4)_x$$
$$+ \sum_{n=1}^{m}k'_n|U_n|^2 = 0 \tag{2.4}$$

Now, making the use of the assumption (1.3) on k_n, multiplying (2.4) by $A(x) = \arctan(ax)$, where a is from the assumption on k_n, integrating in x from $-\infty$ to ∞, using the technique of integration by part and making the use of (2.2) and (2.3), we get

$$\int_0^{\infty}\int_{-r}^{r}\sum_{n=1}^{m}(|U_n|^2 + |U_{n,x}|^2 + |U_n|^4)\,dx\,dt < \infty \tag{2.5}$$

Let $r > 0$ and B be smooth such that $B(x) = 1$ for $|x| \le r$, $B(x) = 0$ for $|x| \ge 2r$ and $0 \le B \le 1$. Multiplying (2.1) by B and integrating in x from $-2r$ to $2r$, we get

$$\left|\int_{-2r}^{2r} B(|U_n|^2)_t\,dx\right| \le b\int_{-2r}^{2r}(|U_n|^2 + |U_{n,x}|^2)\,dx$$

for some positive constant b.

Let $0 < t_1 < t$, then

$$(t - t_1)\int_{-r}^{r}|U_n|^2\,dx \le (t - t_1)\int_{-2r}^{2r}B|U_n|^2\,dx$$
$$\le \int_{t_1}^{t}\int_{-2r}^{2r}B|U_n|^2\,dx\,ds + \int_{t_1}^{t}(s - t_1)\left|\int_{-2r}^{2r}B(|U_n|^2)_t\,dx\right|ds.$$

Let $t_1 = t - 1$, then

$$\int_{-r}^{r}|U_n|^2dx \le (b+1)\int_{t-1}^{t}\int_{-2r}^{2r}(|U_n|^2 + |U_{n,x}|^2)\,dx\,ds$$

Hence, by (2.5), $\int_{-r}^{r}|U_n|^2(x,t)\,dx \to 0$ as $t \to \infty$. Q.E.D.

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