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A HELLY NUMBER FOR UNIONS OF TWO BOXES IN R^2

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ABSTRACT. Let S be a polygonal region in the plane with edges parallel to the coordinate axes. If every 5 or fewer boundary points of S can be partitioned into sets A and B so that conv A $\cup$ conv B $\subseteq$ S, then S is a union of two convex sets, each a rectangle. The number 5 is best possible.

Without suitable hypothesis on edges of S, the theorem fails. Moreover, an example reveals that there is no finite Helly number which characterizes arbitrary unions of two convex sets, even for polygonal regions in the plane.

KEY WORDS AND PHRASES. Helly - type theorems, unions of convex sets.
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1. INTRODUCTION.

We begin with some preliminary definitions. Let S be a subset of R^d. A point x in S is called a <u>point of local convexity</u> of S if and only if there is some neighborhood N of x such that S $\cap$ N is convex. If S fails to be locally convex at some point q in S, then q is called a <u>point of local nonconvexity</u> (lnc point) of S. For points x and y in S, we say x <u>sees</u> y via S (x is <u>visible</u> from y via S) if and only if the corresponding segment [x,y] lies in S. Points $x_1,\dots,x_n$ in S are <u>visually independent</u> via S if and only if for $1 \leq i < j \leq n$, x_i does not see x_j via S. Set S is said to be <u>3-convex</u> if and only if for every 3 points in S, at least one of the corresponding line segments lies in S. Finally, S is called <u>star-shaped</u> if and only if there is some point p in S such that p sees each point of S via S. The set of all such points p is the (convex) kernel of S.

The following terminology will be used throughout the paper: Conv S, bdry S, and ker S will denote the convex hull, boundary, and kernel of set S, respectively, while lnc S will be the set of lnc points of S. For distinct points x and y, R(x,y) will represent the ray from x emanating through y. The reader is referred to Valentine [9] and to Lay [5] for a discussion of these concepts.

An interesting theorem by Lawrence, Hare, and Kenelly [4] provides the following characterization for unions of convex sets: For S a subset of a linear topological space, S is a union of k convex sets if and only if each finite subset F of S has a k-partition $F_1, \ldots, F_k$ with conv $F_i \subseteq S$, $1 \leq i \leq k$. A related problem concerns the existence of a finite Helly number to characterize such unions, and it is natural to ask when 'finite subset' in the hypothesis may be replaced by 'j-member subset' for an appropriate j. (See [3] for similar results.)

Unfortunately, the Lawrence, Hare, Kenelly Theorem cannot be improved for $k = 2$, even when S is a compact subset of the plane. (See Example 3.) However, the problem of obtaining a finite Helly number to characterize certain unions of convex sets remains open. Recent results suggest sets for which such a theorem might hold. In work by Danzer and Grünbaum [2], a kind of Helly number (called a piercing number) is found for certain families of boxes. Moreover, in [1] another kind of Helly number (a Krasnosel'skii number for unions of starshaped sets) is established for certain polygonal regions whose edges are parallel to the coordinate axes. Therefore, it seems reasonable to examine such polygonal regions. This is the situation studied here, and the following result is obtained:

Let S be a polygonal region in the plane with edges parallel to the coordinate axes. If every 5 or fewer boundary points of S can be partitioned into sets A and B so that conv A $\cup$ conv B $\subseteq$ S, then S is a union of two convex sets, each a rectangle. The number 5 is best possible. Without suitable restrictions on the edges of S, the result fails and in fact no finite Helly number exists.

2. THE RESULTS.

The following lemma will be useful.

LEMMA 1. Let S be a bounded subset of R^d, and let k be a fixed integer, $k \geq 3$. If every k or fewer boundary points of S can be partitioned into sets A and B such that conv A $\cup$ conv B $\subseteq$ S, then every k or fewer points of S can be partitioned in this way as well. Moreover, S is 3-convex.

PROOF. Clearly S is closed. If S is not connected, then it is easy to show that S has exactly two components, each convex. Hence without loss of generality we may restrict our attention to the case in which S is connected. Furthermore, we may assume that lnc S $\neq \emptyset$, for otherwise the closed, connected, locally convex set S will be convex by a theorem of Tietze [6].

We begin the proof by showing that lnc S $\subseteq$ ker S. Let $q \in$ lnc S $\neq \emptyset$. We assert that for every z in bdry S, $[q,z] \subseteq S$: Otherwise, since S is closed, there would be a convex neighborhood N of q with no point of $N \cap S$ seeing z via S. Since $q \in$ lnc S, $N \cap S$ is not convex, and standard arguments produce points x,y in $N \cap$ bdry S with $[x,y] \not\subseteq S$. However, then points x,y,z would be visually independent points in bdry S, contradicting our hypothesis. We conclude that $[q,z] \subseteq S$, and the assertion is established.

Now it is easy to show that $q \in$ ker S: If not, then for some w in S, $[q,w] \not\subseteq S$, and $[q,w] \sim S$ would contain an open interval (a,b) with $a,b \in$ bdry S and $q \leq a < b < w$. However, $[q,b] \subseteq S$. Again we have a contradiction, and $q \in$ ker S, the desired result.

Finally, to establish the lemma, let $x_i \in S$, $1 \leqslant i \leqslant k$, and select $q \in \text{lnc } S \neq \emptyset$. Without loss of generality, assume $q \notin \{x_i : 1 \leqslant i \leqslant k\}$, and let ray $R(q,x_i)$ meet bdry S at y_i with $q < x_i \leqslant y_i$. By our hypothesis, the points $y_1,\ldots,y_k$ can be partitioned into sets A and B such that conv A $\cup$ conv B $\subseteq$ S. Relabeling if necessary, assume that $A = \{y_1,\ldots,y_j\}$, $B = \{y_{j+1},\ldots,y_k\}$. Since $q \in \text{lnc } S$, conv $(\{q\} \cup A) \subseteq S$ and conv $\{x_1,\ldots,x_j\} \subseteq$ conv $(\{q\} \cup A) \subseteq S$. Similarly conv $\{x_{j+1},\ldots,x_k\} \subseteq S$. Since $k \geqslant 3$, it is easy to see that S is 3-convex, and the lemma is proved.

Without the requirement that set S be bounded, Lemma 1 fails, as Example 1 illustrates.

EXAMPLE 1. Let S be the set of points in R^2 which lie either in the first quadrant or on one of the coordinate axes. (See Figure 1.) The set bdry S may be partitioned into sets A and B satisfying the hypothesis of the lemma. However, the conclusion of the lemma fails for every $k \geqslant 3$.

We are ready to establish the following Helly-type theorem for unions of rectangles in the plane.

THEOREM 1. Let S be a polygonal region in the plane with edges parallel to the coordinate axes. If every 5 or fewer boundary points of S can be partitioned into sets A and B so that conv A $\cup$ conv B $\subseteq$ S, then S is a union of two convex sets, each a rectangle. The number 5 is best possible.

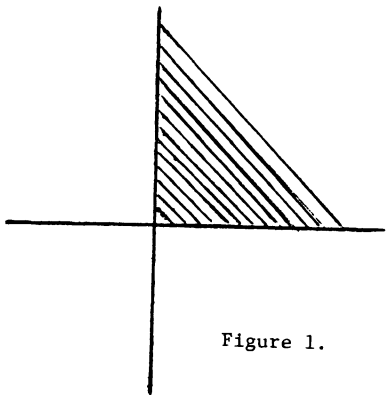


Figure 1.

PROOF. We begin with some preliminary observations. By Lemma 1, set S is 3-convex. Moreover, using [8, Theorem 1] , S is starshaped and lnc S $\subseteq$ ker S. Now in case S has 0, 1, or 2 lnc points, we may use [8, Theorem 3] to conclude that S is a union of two or fewer closed convex sets, the desired result. Therefore, throughout the argument it suffices to assume that S has at least 3 lnc points.

As in [7] and [1], it is helpful to order bdry S in a clockwise direction. This in turn induces a natural order on the edges of S, and we may classify each edge as 'right', 'left', 'up', or 'down' according to the order it inherits from bdry S. For convenience of notation, we label the lnc points of S by $q_1,q_2,\ldots,q_n$, again according to the clockwise order on bdry S. Recall that $n \geqslant 3$ by a previous assumption. The proof of the theorem will require some results concerning these lnc points.

It is easy to see that for each lnc point of S, exactly one of the following must occur relative to our order: The lnc point is either preceded by a 'right' edge and followed by an 'up' edge, preceded by a 'left' edge and followed by a 'down' edge, preceded by a 'down' edge and followed by a 'right' edge, or preceded by an 'up' edge and followed by a 'left' edge. Moreover, using this classification together with the 3-convexity of S, it is not hard to show that no three

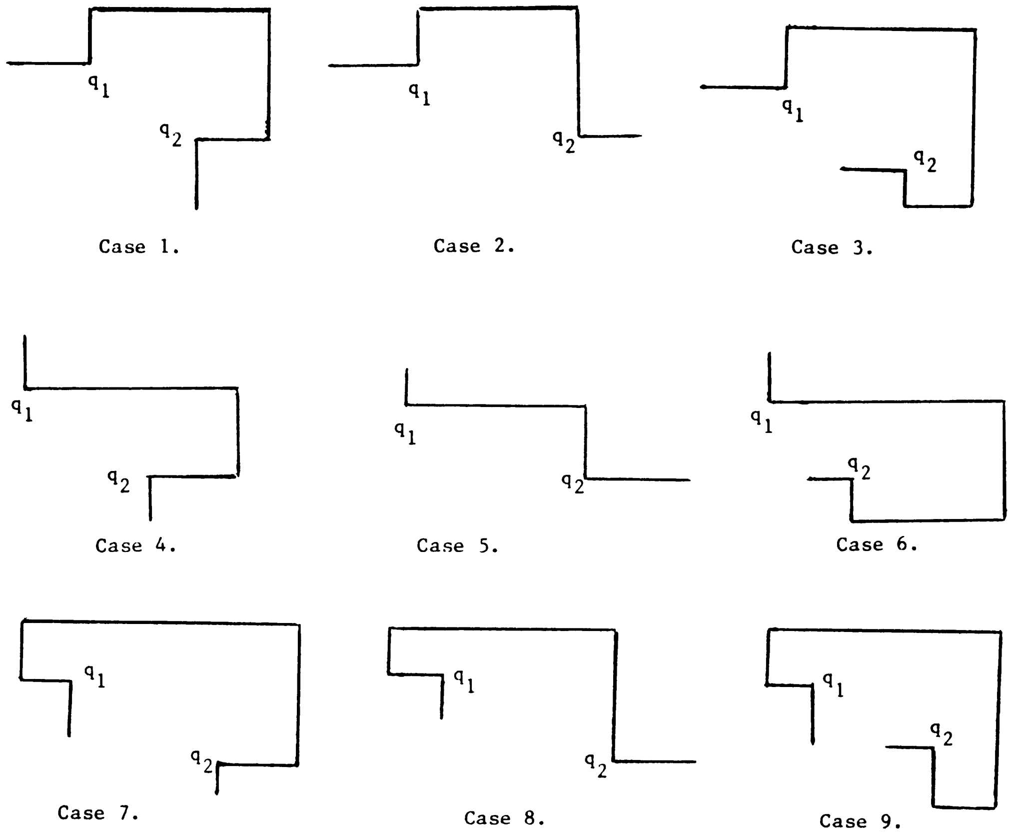


Figure 2.

consecutive lnc points of S (in our order) can be collinear: Otherwise, for an appropriate orientation of S, this would yield a 'right' edge whose predecessor is a 'down' edge and whose successor is an 'up' edge, producing three visually independent points of S, impossible.

Furthermore, we assert that every two consecutive lnc points of S determine a segment parallel to one of the coordinate axes: Suppose on the contrary that the condition fails for q_1 and q_2. Without loss of generality, assume that S is oriented in the plane so that q_1 is to the left of q_2 and above q_2. We examine possible classifications for q_1 and q_2. Using the fact that q_2 follows q_1 in our clockwise order, it is not hard to see that q_1 cannot be preceded by a 'left' edge and followed by a 'down' edge. For convenience of terminology, we say that q_1 cannot be classified as 'left-down'. Similarly, q_2 cannot be classified as 'right-up'.

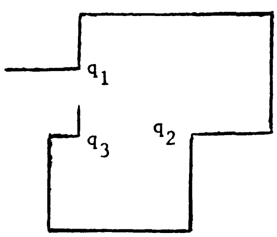
Examine the remaining 9 possibilities. (See Figure 2, Cases 1-9, for corresponding illustrations.)

1. q_1 is 'right-up' and q_2 is 'left-down'
2. q_1 is 'right-up' and q_2 is 'down-right'
3. q_1 is 'right-up' and q_2 is 'up-left'
4. q_1 is 'down-right' and q_2 is 'left-down'
5. q_1 is 'down-right' and q_2 is 'down-right'
6. q_1 is 'down-right' and q_2 is 'up-left'
7. q_1 is 'up-left' and q_2 is 'left-down'
8. q_1 is 'up-left' and q_2 is 'down-right'
9. q_1 is 'up-left' and q_2 is 'up-left'

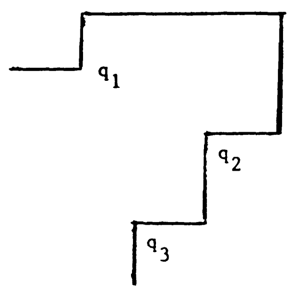
It is not hard to prove that each of Cases 2 through 9 above violates the condition that $q_1, q_2 \in \ker S$, so none of these can occur. Thus Case 1 is the only possibility.

Moreover, by a similar argument involving the classification of lnc point q_3, q_3 must be 'up-left': Classification 'down-right' is impossible in our clockwise order. Classifications 1)'right-up' and 2)'left-down' violate the facts that $q_1 \in \ker S$ and $q_2 \in \ker S$, respectively. (See Figure 3, Cases 1-2.)

However, this forces S to have 5 boundary points p_i, $1 \leqslant i \leqslant 5$, for which there is no partition satisfying our hypothesis: Select p_1 near q_1 on the edge preceding q_1, p_2 above q_1 and right of q_2, p_3 near q_2 on the edge



Case 1.



Case 2.

Figure 3.

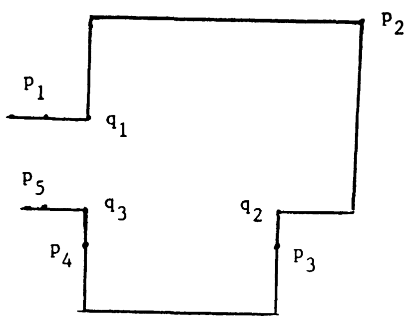


Figure 4.

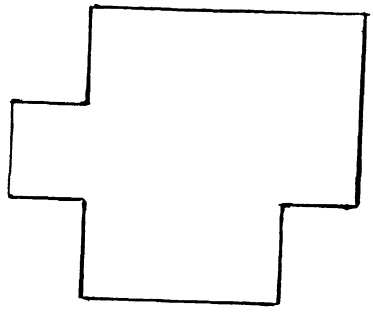


Figure 5.

following q_2, p_4 near q_3 on the edge preceding q_3, and p_5 near q_3 on the edge following q_3. (see Figure 4.) We have a contradiction, our supposition is false, and every two consecutive lnc points of S must determine a segment parallel to one of the coordinate axes. Thus the assertion is established.

The rest of the argument is easy. Using the assertion above together with the fact that no three consecutive lnc points of S are collinear, it is not hard to show that S has at most 4 lnc points. Similarly, S cannot have exactly 3 lnc points, and since we are assuming that $n \geq 3$, we conclude that $n = 4$. Hence by [8, Theorem 3], S is a union of two closed convex sets.

Finally, using decomposition techniques from [8, Theorem 3], it is easy to show that S may be expressed as a union of two rectangles, and the proof of Theorem 1 is complete.

To see that the number 5 in Theorem 1 is best possible, consider the following example.

EXAMPLE 2. Let S be the set in Figure 5. Every 4 points of S may be partitioned into sets A and B such that $\text{conv } A \cup \text{conv } B \subseteq S$. However, S is not a union of fewer than three convex sets.

In conclusion, it is interesting to observe that the result in Theorem 1 fails without the restriction on edges of S. In fact, there is no finite Helly number which characterizes unions of two convex sets, even for polygonal regions in the plane, and the Lawrence, Hare, Kenelly Theorem is best possible in this case. This is illustrated in Example 3.

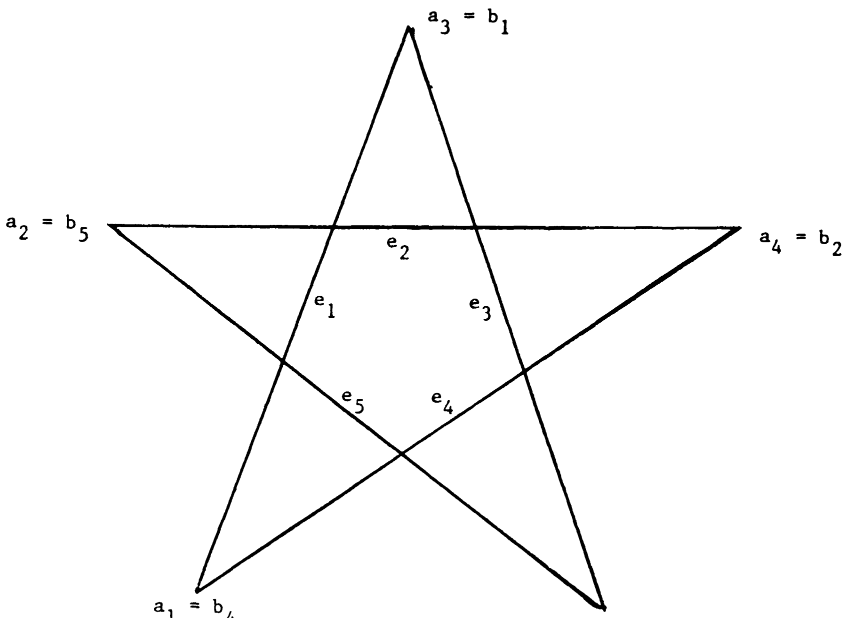


Figure 6.

EXAMPLE 3. For $n \geq 2$, let $P(n)$ be a regular $(2n + 1)$-gon whose edges are labeled in a clockwise direction by $e_1, e_2, \ldots, e_{2n+1}$. Adjusting our subscripts modulo $2n + 1$, for $1 \leq i \leq 2n+1$, Let $s_i \equiv [a_i, b_i]$ be a segment containing e_i such that s_i meets s_{i-2} at a_i and s_i meets s_{i+2} at b_i. Finally, let

$S(n)$ be the simply connected region determined by $\cup\{s_i : 1 \leqslant i \leqslant 2n+1\}$. (The case for $n = 2$ is illustrated in Figure 6.) Every $2n$ points of $S(n)$ may be partitioned into sets A and B so that conv A $\cup$ conv B $\subseteq$ S. However, the $2n + 1$ points $\{a_i : 1 \leqslant i \leqslant 2n + 1\}$ have no such partition, and $S(n)$ is not a union of fewer than three convex sets.

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