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DYNAMICAL PROPERTIES OF MAPS DERIVED FROM MAPS WITH STRONG NEGATIVE SCHWARZIAN DERIVATIVE

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ABSTRACT. A strong Schwarzian derivative is defined, and it is shown that the convolution of a function with a map from an interval into itself having negative strong Schwarzian derivative is a function with negative Schwarzian derivative. Such convolutions have 0 as a stable periodic point and at most one other stable periodic orbit in the interior of the domain.

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1. INTRODUCTION.

Let $f:[0,1]\to[0,1]$ be a C^3 map, i.e., it has 3 continuous derivatives. The Schwarzian derivative at a point x is given by

$$\{f,x\} = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 . \tag{1.1}$$

This derivative was first formulated by H.A. Schwarz and has been used in the theory of differential equations [1]. Recently, it has found important application in the study of bifurcation of periodic orbits [2]. In [3,4], the Schwarzian derivative is used to study the limiting behaviour of dynamical systems.

The main result of [2] is

THEOREM 1. Let $f:[0,1]\to[0,1]$ be a C^3 map and let it satisfy

(i) $f(0) = f(1) = 0$

(ii) f has a unique critical point c in $(0,1)$

and (iii) $\{f,x\} < 0$ for all $x\in[0,1]-c$.

Then f has at most one stable orbit in $(0,1)$. If it exists it is the ω-limit set of c .

Note that the wording of Theorem 1 allows for the possibility that 0 is also a stable fixed point of f. Indeed, Theorem 1 holds even if the slope at 0 (and at 1) is equal to 0. Furthermore, the requirement that $f'''(x)$ is continuous at c can be relaxed. In [4, p. 100] it is shown that Theorem 1 is true even if only $f'(x)$ is continuous at c. We collect these observations in a version of Theorem 1 which we shall need in the sequel.

THEOREM 2. Let $f:[0,1]\to[0,1]$ be a C^3 map everywhere in $(0,1)$ except possibility at c where it is C^1. Assume conditions (i), (ii) and (iii) of Theorem 1. Then f has at most one stable periodic orbit in $(0,1)$. If it exists it is the ω-limit set of c.

In this note we define a strong Schwarzian derivative and show that the convolution of a function with a map having negative Schwarzian derivative is a function with negative Schwarzian derivative.

In practice one is concerned with the structural stability of the map f, i.e. with what happens to the dynamical properties of f once it is perturbed. Since $\{f,x\}<0$ is an open condition, it is clear that for maps which are C^3 and close to f, this condition will be retained. Thus, for small, smooth perturbations of f, the negative Schwarzian property is not destroyed. However, for any given perturbation, the Schwarzian derivative must be computed to verify that $\{f,x\}<0$. In this note we consider a class of large perturbations of f derived by convoluting f with a known function, g. We define a new derivative of f called the strong Schwarzian derivative, denoted by $\bar{S}f$. The main result of this note shows that if $\bar{S}f<0$, then the map $F=f*g$ has negative Schwarzian derivative. Thus, we can draw dynamical conclusions about maps which are large perturbations of f.

2. A SIMPLE LEMMA.

Let $(Sf)(x) = f'''(x)f'(x) - \frac{3}{2}(f''(x))^2$. We define the strong Schwarzian derivative $\bar{S}f$ at points a and b in $[0,1]$, $a<b$, by

$$(\bar{S}f)(a,b) = f'(a)f'''(b) - \frac{3}{2}f''(a)\,f''(b). \tag{2.1}$$

Clearly, if $(\bar{S}f)(a,b)<0$ for all $a<b$, then $(Sf)(x)<0$ for all x in $[0,1]$.

LEMMA 1. Let $f:[0,1]\to[0,1]$ be a unimodal map with 3 continuous derivatives such that $f'(x)>0$ on $[0,c)$ and $f'(x)<0$ on $(c,1]$. Furthermore, assume:

1) $(Sf)(x)<0$

2) $f''(x)<0$, $f'''(x)\geq 0$, $f^{IV}(x)\leq 0$ for all $x\in(0,1)$.

Then $(\bar{S}f)(a,b)<0$ for all $a,b\in(0,1)$.

PROOF. Let $a<b$. Since $(Sf)(a)<0$ and $f'''(a)>0$

$$f'(a) < \frac{3}{2}\frac{[f''(a)]^2}{f'''(a)}.$$

Then

$$f'(a)f'''(b) < \frac{3}{2}f''(a)f''(b)\,\frac{f''(a)}{f''(b)}\,\frac{f'''(b)}{f'''(a)}. \tag{2.2}$$

Since $f'''(a)\geq 0$, $f''(a)\leq f''(b)$, and since $f^{IV}(x)\leq 0$, $f'''(a)\geq f'''(b)$. Hence

$$f'(a)f'''(b) < \frac{3}{2}f''(a)f''(b), \tag{2.3}$$

i.e., $(\bar{S}f)(a,b)$ for all $a < b$, $a \in (0,1)$.

To complete the proof, we must show that

$$f'''(a)f'(b) < \frac{3}{2} f''(a)f''(b) , \tag{2.4}$$

where $a < b$. There are two cases.

CASE 1. $f'(b) > 0$. Since $a < b$, $f'(a) > 0$. Then

$$f'''(a) < \frac{3}{2} \frac{[f''(a)]^2}{f'(a)}$$

and

$$f'(b)f'''(a) < \frac{3}{2} [f''(a)f''(b)] \frac{f''(a)}{f''(b)} \frac{f'(b)}{f'(a)} . \tag{2.5}$$

Since $f'''(x) \geq 0$, $\frac{f''(a)}{f''(b)} \leq 1$, and since $f''(x) < 0$, $\frac{f'(b)}{f'(a)} < 1$. Hence the inequality (2.4) holds.

CASE 2. $f'(b) < 0$. Since $f'''(a) > 0$,

$$f'(b)f'''(a) - \frac{3}{2} f''(a)f''(b) < 0 . \tag{2.6}$$

This completes the proof. Q.E.D.

EXAMPLE 1. $f(x) = rx(1-x)$, $0 \leq r \leq 4$. Since $f_r'''(x) = 0$, and $f_r''(x) < 0$, $\bar{S}f_r(a,b) < 0$ for all (a,b) in $(0,1)$.

EXAMPLE 2. $f_\alpha(x) = xe^{-2x}$, $\alpha > 0$, defined on $\left[0,\frac{2}{\alpha}\right]$, where the critical point $c = \frac{1}{\alpha}$ and the point of inflection is at $x = 2/\alpha$. It is easy to verify that $f_\alpha(x)$ satisfies all the conditions of Lemma 1.

3. MAPS DEFINED BY CONVOLUTION.

Let $f:[0,1]\to[0,1]$ satisfy $f(0) = f(1) = 0$ and let $g:[0,1]\to[0,\infty)$ be a map such that $g(x) \geq 0$ and

$$\int_0^1 g(x)dx \leq 2 . \tag{2.7}$$

We extend both f and g to $(-\infty, \infty)$ by letting $f(x) = g(x) = 0$ outside of $[0,1]$ and use the same symbols to denote these extended functions on $(-\infty,\infty)$. The convolution of f and g is given by

$$F(x) = \int_{-\infty}^{\infty} g(x-t)f(t)dt = g*f(x) . \tag{2.8}$$

The support of F is $[0,2]$ and (2.7) guarantees that the range of F is contained in $[0,2]$. Hence $F:[0,2]\to[0,2]$ is a well-defined map.

LEMMA 2. Let $f:[0,1]\to[0,1]$, $f(0) = f(1) = 0$, be in C^3 and assume $(\bar{S}f)(a,b) < 0$ for all points a,b in $[0,1]$. Let $g(0,1)\to[0,\infty]$ be in C^3, let (2.7) be satisfied, and assume $g(0^+)$ and $g(1^-)$ exist. Then $F(x)$ has a continuous third derivative everywhere except possibly at $x = 1$, where at least $F'(x)$ is continuous, and $F(x)$ has negative Schwarzian derivative for all $x \in (0,2)$.

PROOF. Notice that in (2.8), $0 \leq x-t \leq 1$ and $0 \leq t \leq 1$. Hence for $0 \leq x < 1$, $0 \leq t \leq x$, and we have

$$F(x) = \int_0^x g(x-t)f(t)dt \ . \tag{2.9}$$

Similarly, for $1 < x \leq 2$,

$$F(x) = \int_{x-1}^x g(x-t)f(t)dt \ . \tag{2.10}$$

Using Leibnitz's Rule, we obtain

$$F'(x) = \int_0^x g'(x-t)f(t)dt + g(0^+)f(x), \quad 0 \leq x < 1 \tag{2.11}$$

and

$$F'(x) = \int_{x-1}^x g'(x-t)f(t)dt + g(0^+)f(x) - g(1^-)f(x-1) \ , \tag{2.12}$$

where $1 < x \leq 2$. Now $F'(1^-) = F'(1^+)$ if

$$g(0^+)f(1^-) = g(0^+)\,f(1^+) - g(1^-)f(0^+) \ , \tag{2.13}$$

which is so since $f(0) = f(1) = 0$. Hence $F'(x)$ is continuous on $(0,2)$. That $F'''(x)$ is continuous on $(0,2)$, except possibly at $x = 1$, follows from the fact that $f,g \in C^3$.

To prove $(SF)(x) < 0$, we differentiate (2.8) to get

$$F'(x) = \int_{-\infty}^{\infty} g'(x-t)f(t)dt \ . \tag{2.14}$$

Integrating by parts,

$$F'(x) = -\,g(x-t)f(t)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} g(x-t)f'(t)dt$$
$$= \int_{-\infty}^{\infty} g(x-t)f'(t)dt \ . \tag{2.15}$$

Similarly,

$$F''(x) = \int_{-\infty}^{\infty} g(x-t)f''(t)dt \tag{2.16}$$

and

$$F'''(x) = \int_{-\infty}^{\infty} g(x-t)f'''(t)dt \ . \tag{2.17}$$

Thus,

$$(SF)(x) = \int_{-\infty}^{\infty} g(x-t)f'(t)dt \int_{\infty}^{\infty} g(x-y)f'''(y)dy$$
$$- \frac{3}{2}\int_{-\infty}^{\infty} g(x-t)f''(t)dt \int_{-\infty}^{\infty} g(x-y)f''(y)dy \tag{2.18}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-t)g(x-y)\left[f'(t)f'''(y) - \frac{3}{2}f''(t)f''(y)\right]dtdy \ .$$

Since $g(\xi) \geq 0$ and $(\bar{S}f)(a,b) < 0$ for all $a,b \in (0,1)$ by Lemma 1, we have $(SF)(x) < 0$ and hence $\{F,x\} < 0$ for all $x \in (0,2)$. Q.E.D.

Note that if f and g can be extended smoothly at the endpoints 0 and 1, then boundary conditions can be derived which will ensure that $F \in C^3$ everywhere. Under such conditions it would be possible to avoid $f(0) = 0$, $f(1) = 0$.

From (2.9) we see that $F(0) = 0$ and from (2.11) it follows that $F'(0^+) = g(0^+)f(0) < 1$. Hence 0 is a stable fixed point.

THEOREM 3. Let f and g be as in Lemma 2 and assume that f is a unimodal map, strictly increasing on $[0,c)$ and strictly decreasing on $(c,1]$. Then $F(x) = g*f(x)$ has 0 as a stable periodic point and at most one other stable periodic orbit in $(0,2)$.

PROOF. $F(x)$ is unimodal on $[0,2]$ with critical point $\bar{c} = 1$, since f is unimodal on $[0,1]$. By Lemma 2, $F'''(x)$ is continuous everywhere except possibly at 1, where $F'(x)$ is continuous, and $\{F,x\} < 0$. Clearly 0 is a stable periodic point. Applying Theorem 2 we obtain the desired result. Q.E.D.

Since $F'(0^+) = 0$ there exists a point $0 < a < 1$, such that $F(a) = a$. Let $b = F^{-1}(a) \cap (1,2)$. Then clearly $[0,2) \cup (b,1]$ is in the domain of attraction of a.

COROLLARY. If $F(c) > b$, then F has 0 as a stable periodic point and no other stable periodic orbits.

PROOF. If F has a stable periodic orbit in $(0,2)$, Theorem 2 implies that it is the ω-limit set of $c = 1$. Here, clearly, the ω-limit set is equal to $\{0\}$. Q.E.D.

EXAMPLE 1. Let $f(x) = rx(1-x)$ and $g(x) = k$. Then, for $0 \leq x \leq 1$,

$$F(x) = (g*f)(x) = k\int_0^x rt(1-t)dt$$

$$= krx^2\left(\frac{1}{2} - \frac{x}{3}\right). \tag{2.19}$$

By symmetry, we have

$$F(x) = F(2-x),\ 1 \leq x \leq 2. \tag{2.20}$$

Thus,

$$F(x) = \begin{cases} krx^2\left(\frac{1}{2} - \frac{x}{3}\right), & 0 \leq x \leq 1 \\ kr(x-2)^2\left(\frac{x}{3} - \frac{1}{6}\right), & 1 \leq x \leq 2 \end{cases}. \tag{2.21}$$

For $F:[0,2] \to [0,2]$ to be well-defined we need

$$F(1) = \frac{kr}{6} \leq 2. \tag{2.22}$$

By Theorem 3, we know that $F(x)$ can have at most one stable periodic orbit in $(0,2)$. For $kr = 6$, the point $c = 1$ is a stable fixed point.

Note that in this example $F'''(x)$ is not continuous at 1, but $F'(x)$ is.

EXAMPLE 2. Let $f(x) = rx(1-x)$ on $[0,1]$, $0 \leq r \leq 4$, and $g(x) = px(1-x)$

on $[0,1]$. Then $F(x) = g*f(x)$ is given by

$$F(x) = \begin{cases} \frac{rp}{6}(x^3 - x^4 + \frac{x^5}{5}) & \text{on } [0,1] \\ F(2-x) & \text{on } [1,2] \end{cases} \tag{2.23}$$

For F to be well-defined, we require

$$F(1) = \frac{rp}{30} \le 2 ,$$

i.e., $rp \le 60$. For $rp = 42$, $F(x) = x$ at $x_o = .51$. Hence, by the symmetry of F, $[0, .51) \cup (1.49, 2.00]$ is in the domain of attraction of 0. Since $F(1) = 1.4$, $c = 1$ is not in the domain of attraction of 0, and hence it is possible for a stable periodic orbit to exist in $(.51, 1.49)$. However, for $rp = 48$, $[0, .46) \cup (1.56, 2.00]$ is in the domain of attraction of 0. Since $F(1) = 1.6$, $c = 1$ is in the domain of attraction of 0. Hence, by Theorem 3, 0 is the only stable periodic orbit. The same is true for $48 \le rp \le 60$.

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