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a*-FAMILIES OF ANALYTIC FUNCTIONS

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ABSTRACT. Using convolutions, a new family of analytic functions is introduced. This family, called a* - family, serves in certain situations to unify the study of many previously well known classes of analytic functions like multivalent convex, starlike, close-to-convex or prestarlike functions, functions starlike with respect to symmetric points and other such classes related to the class of univalent or multivalent functions. A necessary and sufficient condition on the Taylor series coefficients so that an analytic function with negative coefficients is in an a*-family is obtained and sharp coefficents bound for functions in such a family is deduced. The extreme points of an a*-family of functions with negative coefficients are completely determined. Finally, it is shown that Zmorvic conjecture is true if the concerned families consist of functions with negative coefficients.

KEY WORDS AND PHRASES. Univalent Functions, Multivalent Functions, Convolution, p-valent starlike functions, p-valent close-to-convex functions, p-valent prestarlike functions, starlike functions with respect to symmetric points, Coefficients bound, Extreme points etc.
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1. INTRODUCTION.

Let A_p, p = 1,2..., denote the family of functions f, analytic in $E = \{z : |z| < 1\}$ and having Taylor series expansion

$$f(z) = z^p + \sum_{k=1}^{\infty} a_k z^{k+p}. \tag{1.1}$$

In the present paper we introduce the concept of an a*-family of functions in A_p. It turns out that many familiar subfamilies of A_p, related to univalent and multivalent functions, are a*-families. We determine a sufficient condition, on the coefficients, such that a function f in A_p, given by (1.1), is an a*-family. Further, we show that such a condition is also necessary when f is in $A[p]$, the family of functions f in A_p having Taylor series expansion

$$f(z) = z^p - \sum_{k=1}^{\infty} |a_k| z^{k+p}. \tag{1.2}$$

Using these results we determine the extreme points of an a*-family in $A[p]$. Finally, we give some applications of our results in Section 4.

2. DEFINITION AND EXAMPLES.

The Hadamard product or convolution f * g of functions f and g, analytic in E and given by $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$, is defined as the analytic function $(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k$.

Definition. A family of functions f in A_p, p = 1,2..., is said to be an a*-family if there exist functions s_o and g_o analytic in E defined by

$$s_o(z) = \sum_{k=0}^{\infty} c_k z^{k+p}, \; c_o > 0, \; c_k \geq 0, \; k = 1,2... \tag{2.1}$$

and

$$g_o(z) = \sum_{K=0}^{\infty} d_k z^{k+p}, \; d_o > 0, \; d_k \geq 0, \; k = 1,2... \tag{2.2}$$

satisfying

$$\frac{c_k}{c_o} - \frac{d_k}{d_o} > 0, \; k = 1,2 ..; \tag{2.3}$$

and a number ρ, $0 \leq \rho \leq c_o / d_o$, such that for every f in F

$$(g_o * f)(z) \neq 0, \; 0 < |z| < 1 \tag{2.4}$$

and

$$\text{Re} \left(\frac{(s_o * f)(z)}{(g_o * f)(z)}\right) > \rho \tag{2.5}$$

for z in E.

We write-F is an a*-family with the touple (s_o, g_o, ρ)- when the touple (s_o, g_o, ρ) is explicitly needed. Further, we denote an a*-family in A[p] by [a*]-family.

The following well known families in A_p are some examples of a *-families.

Example 1 For $0 \leq \alpha < 1$ and p = 1,2 . , let $S^*(p,\alpha)$ denote the a*-family in A_p with the touple

$$\left(\left\{\frac{z}{(1-z)^2} - \sum_{m=1}^{p-1} m\, z^m\right\}, \frac{z^p}{1-z}, p\alpha\right) \tag{2 6}$$

and $K(p,\alpha)$ denote the a*-family in A_p with the touple

$$\left(\left\{\frac{z(1+z)}{(1-z)^3} - \sum_{m=1}^{p-1} m^2 z^m\right\}, \left\{\frac{z}{(1-z)^2} - \sum_{m=1}^{p-1} m\, z^m\right\}, p\alpha\right) \tag{2.7}$$

It is easily seen that $S^*(p,\alpha)$ is the family of p-valent starlike functions of order α i.e. family of functions f in A_p satisfying $\text{Re}(zf'(z)/f(z)) > p\alpha$. Also, $K(p,\alpha)$ is the family of p-valent convex functions of order α i e. family of functions f in A_p such that zf'/p is in $S^*(p,a)$ [4].

Example 2. Let K(n+p-1) denote the class of functions f in A_p satisfying the condition

$$\text{Re}\left(\frac{(z^n f)^{(n+p)}}{(z^{n-1} f)^{(n+p-1)}}\right) > \frac{n+p}{2}$$

where p = 1,2..., n = -p+1, -p+2, .., p; and z is in E. It is known [1] that f in A_p is in K(n+p-1), if and only if, Re G(z) > ½ where

$$G(z) = \frac{\dfrac{z^p}{(1-z)^{n+p+1}} * f(z)}{\dfrac{z^p}{(1-z)^{n+p}} * f(z)}$$

A brief calculation shows that, for $h(z) = (z^p/(1-z)^{n+p}) * f(z)$ and f in $K(n+p-1)$

$$\mathrm{Re}(zh'(z)/h(z)) = (p+n)\,\mathrm{Re}\,G(z) - n > (p-n)/2 > 0$$

Thus, $h(z) \neq 0$ in $0 < |z| < 1$ and it follows that $K(n+p-1)$ is an a*-family with the touple

$$\left(\frac{z^p}{(1-z)^{n+p+1}}, \frac{z}{(1-z)^{n+p}}, \tfrac{1}{2}\right). \tag{2.8}$$

Example 3. Let $J(p,\alpha)$ denote the a*family in A_p with touple

$$\left(\left\{\frac{z}{(1-z)^2} - \sum_{m=1}^{p-1} m z^m\right\}, z^p, p\alpha\right) \tag{2.9}$$

where $0 \le \alpha < 1$. It is easily seen that $J(p,\alpha)$ is a known subfamily of the family of p-valent close-to-convex functions [5]. The family $J(1,\alpha)$ is the family of univalent functions f in A_1 satisfying $\mathrm{Re}(f'(z)) > \alpha$ for z in E.

Example 4. The family $I(p,\alpha)$, $0 \le \alpha < 1$, $p = 1,2\ldots$, of functions f in A_p satisfying $\mathrm{Re}\,\{f(z)/z^p\} > p\alpha$, for z in E, is an a*-family with the touple

$$\left(\frac{z^p}{(1-z)}, z^p, p\alpha\right). \tag{2.10}$$

Example 5. The family $N(\alpha)$ of univalent functions starlike of order α, $0 \le \alpha < 1$, with respect to symmetric points i.e. the family of functions f in A_1 satisfying $\mathrm{Re}\,(zf'(z)/(f(z) - f(-z))) > \alpha$, is an a*-family with the touple

$$\left(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}, \alpha\right). \tag{2.11}$$

Example 6. A function f in A_p, $p = 1,2\ldots$, is said to be p-valent prestarlike of order α, $0 \le \alpha < 1$, if the function $h(z) = (z^p/(1-z)^{2p(1-\alpha)}) * f(z)$ is p-valent starlike of order α. A brief calculation shows that

$$F(z) \cdot \frac{p(1-2\alpha)}{2(1-\alpha)} + \frac{1}{2(1-\alpha)}\frac{zh'(z)}{h(z)}$$

$$= \frac{\left[\frac{p(1-2\alpha)}{2(1-\alpha)}\left\{\frac{z}{(1-z)} - \sum_{m=2}^{p-1} z^m\right\} + \frac{1}{2(1-\alpha)}\left\{\frac{z}{(1-z)^2} - \sum_{m=2}^{\infty} mz^m\right\}\right] * h(z)}{h(z)}$$

$$= \left[\sum_{m=0}^{\infty} \frac{2p(1-\alpha)+m}{2(1-\alpha)} z^{m+p}\right] * h(z)/h(z)$$

$$= \frac{p\,z^p}{(1-z)^{2p(1-\alpha)+1}} * f(z) \Big/ \frac{z^p}{(1-z)^{2p(1-\alpha)}} * f(z).$$

Further, a function f is p-valent prestarlike of order α, if and only if, $\mathrm{Re}\,F(z) > p/2$. Now, it follows that the family $PS^*(p,\alpha)$, of p-valent prestarlike functions of order α, is an a*-family with the touple

$$\left(\frac{z^p}{(1-z)^{2p(1-\alpha)+1}}, \frac{z^p}{(1-z)^{2p(1-\alpha)}}, \tfrac{1}{2}\right). \tag{2.12}$$

We note that $PS^*(1,\alpha)$ is the class of prestarlike functions of order α studied by Ruscheweyh [7].

3. NECESSARY AND SUFFICIENT CONDITIONS AND EXTREME POINTS.

We have the following sufficient condition on $|a_n|$ for a function f in A_p, given by (1.1), to be in an a*-family.

Theorem 1. Let f, in A_p, be given by (1.1). Let $\{c_k\}_{k=0}^{\infty}$ and $\{d_k\}_{k=0}^{\infty}$, the sequences of real numbers with $c_o > 0$, $d_o > 0$, $(c_k/c_o) - (d_k/d_o) \geq 0$; and ρ, $0 \leq \rho < (c_o/d_o)$ be such that

$$\sum_{k=1}^{\infty} (c_k - \rho d_k)\ |a_k| \leq (c_o - \rho d_o). \tag{3.1}$$

Then, f is in the a*-family with the touple (s_o, g_o, ρ), where s_o and g_o are defined by (2.1) and (2.2) respectively.

Proof. Let $h(z) = (g_o * f)(z)$. Then, for z in E,

$$\mathrm{Re}\ (h(z)/z^p) \geq d_o - \sum_{k=1}^{\infty} d_k\ |a_k|\ |z|^k. \tag{3.2}$$

Now, since $(c_k/c_o) - (d_k/d_o) \geq 0$ and (3.1) holds, we have

$$1 - \sum_{k=1}^{\infty} \frac{d_k}{d_o}\ |a_k| \geq 1 - \sum_{k=1}^{\infty} \frac{(c_k - \rho d_k)}{(c_o - \rho d_o)}\ |a_k| \geq 0. \tag{3.3}$$

Thus, (3.2) and (3.3) give that $h(z) \neq 0$ in $0 < |z| < 1$.

Next, for s_o and g_o defined by (2.1) and (2.2) and $|z| = r$,

$$\mathrm{Re}\ \left(\frac{(s_o * f)(z)}{(g_o * f)(z)} - \rho\right) = \mathrm{Re}\ \left(\frac{(c_o-\rho d_o) + \sum_{k=1}^{\infty} (c_k-\rho d_k) a_k z^k}{d_o + \sum_{k=1}^{\infty} d_k a_k z^k}\right)$$

$$= \frac{(c_o-\rho d_o) - \sum_{k=1}^{\infty} (c_k-\rho d_k)\ |a_k|\ r^k}{d_o - \sum_{k=1}^{\infty} d_k\ |a_k|\ r^k}.$$

Thus, by (3.1) and (3.3), we get

$$\mathrm{Re}\ \left(\frac{(s_o * f)(z)}{(g_o * f)(z)}\right) > \rho$$

Hence, f is in the a*-family with the touple (s_o, g_o, ρ) and this completes the proof of the theorem.

The following theorem gives a necessary and sufficient condition on $|a_n|$ for a function f in A[p], given by (1.2), to be in an [a*]-family.

Theorem 2. Let the sequences $\{c_k\}_{k=0}^{\infty}$, $\{d_k\}_{k=0}^{\infty}$ and the nonnegative number ρ be defined as in Theorem 1. Then a function f in A[p], given by (1.2), is in the [a*]-family with the touple (s_o, g_o, ρ), if and only if,

$$\sum_{k=1}^{\infty} (c_k - \rho d_k)\ |a_k| \leq (c_o - \rho d_o) \tag{3.4}$$

where s_o and g_o are defined by (2.1) and (2.2) respectively.

Proof. Let f be in the [a*]-family with the touple (s_o, g_o, ρ). Then, for $|z| = r < 1$,

$$\mathrm{Re}\left(\frac{(s_o*f)(z)}{(g_o*f)(z)} - \rho\right) = \mathrm{Re}\left(\frac{(c_o-\rho d_o) - \sum_{k=1}^{\infty}(c_k-\rho d_k)|a_k|\ z^k}{d_o - \sum_{k=1}^{\infty} d_k\ |a_k|\ z^k}\right) > 0\ . \tag{3.5}$$

Now, let $-1 < z < 1$. By the condition (2.4),
$h(z) = d_o z^p - \sum_{k=1}^{\infty} d_k |a_k| z^{k+p} \neq 0$ in $0 < |z| < 1$, so that
$d_o - \sum_{k=1}^{\infty} d_k |a_k| z^{k+p} > 0$ for $-1 < z < 1$. It follows now by (3.5) that

$$\sum_{k=1}^{\infty} (c_k - \rho d_k) |a_k| z^k \leq (c_o - \rho d_o). \tag{3.6}$$

Now, taking limit $z \to 1$ along the real axis in (3.6), we get (3.4). Thus, the proof of the theorem is complete in view of Theorem 1.

Corollary 1. Let f, given by (1.2), be in an ⌊a*⌋-family with the touple (s_o, g_o, ρ), where s_o, g_o and ρ are defined as in the definition in Section 2.
Then, for k = 1,2...

$$|a_k| \leq \frac{c_o - \rho d_o}{c_k - \rho d_k}. \tag{3.7}$$

The inequality (3.7) is sharp, the extremal function being

$$f_k(z) = z^p - ((c_o - \rho d_o)/(c_k - \rho d_k))\, z^{k+p} \tag{3.8}$$

for each k = 1,2,... .

Proof. The corollary is a direct consequence of the necessary and sufficient condition (3.4).

Corollary 2. Let F and G be two a*-families with the touples $(s_o, g_o, 0)$ and $(s_o, g_1, 0)$ respectively. Then,

$$F \cap A[p] = G \cap A[p]. \tag{3.9}$$

Remark 1. Choosing $\{c_k\} \equiv \{k+p\}$, $\{d_k\} \equiv 1$ and $\rho = p\alpha$; $k = 0,1,2,\ldots$; $p = 1,2,\ldots$; and $0 \leq \alpha < 1$, it follows from Theorem 1 that the condition

$$\sum_{k=1}^{\infty} (k+p - p\alpha) |a_k| \leq p(1-\alpha) \tag{3.10}$$

is sufficient for a function f, given by (1.1), to be in $S^*(p,\alpha)$. Further, by Theorem 2, it follows that the condition (3.10) is both necessary and sufficient for a function f, given by (1.2), to be in $S^*[p,\alpha] \equiv S^*(p,\alpha) \cap A[p]$.

If we choose $\{c_k\} \equiv \{(k+p)^2\}$, $\{d_k\} = \{k+p\}$ and $\rho = p\alpha$, $k = 0,1,2\ldots$, $p = 1,2\ldots$, $0 \leq \alpha < 1$, then Theorem 1 gives that the condition

$$\sum_{k=1}^{\infty} \frac{(k+p)}{p} (k+p-p\alpha) |a_k| \leq p(1-\alpha) \tag{3.11}$$

is sufficient for a function f, given by (1.1), to be in $K(p,\alpha)$. Further, by Theorem 2 we have that the condition (3.11) is both necessary and sufficient for a function f, given by (1.2), to be in $K[p,\alpha] \equiv K(p,\alpha) \cap A[p]$.

We note that the sufficient conditions (3.10) and (3.11) have been obtained by Ozaki [6], Goodman [2] and Schild [8] respectively in the particular cases $\alpha = 0$, $p \geq 1$; $\alpha = 0$, $p = 1$ and $\alpha = \frac{1}{2}$, $p = 1$. Silverman [9] has obtained similar necessary and sufficient conditions in the particular case $p = 1$ and $0 \leq \alpha < 1$.

Remark 2. Choosing $\{c_k\}_{k=0}^{\infty} \equiv \{k+p\}_{k=0}^{\infty}$, $d_o = 1$, $d_k = 0, k=1,2,\ldots$ and $\rho = p\alpha$, $p = 1,2\ldots$, $0 \leq \alpha < 1$, it follows from Theorem 1 that the condition

$$\sum_{k=1}^{\infty}(k+p) |a_k| \leq p(1-\alpha) \tag{3.12}$$

is sufficient for a function f, given by (1.1), to be in $J(p,\alpha)$. Further, by Theorem 2, it follows that the condition (3.12) is both necessary and sufficient for a function f, given by (1.2), to be in $J[p,\alpha] \equiv J[p,\alpha] \cap A[p]$.

<u>Remark 3</u>. With the choice $\{c_k\}_{k=0}^{\infty} \equiv \{1\}$, $d_o = 1$ and $d_k = 0$, $k = 1,2,\ldots$ and $\rho = p\alpha$, $p = 1,2\ldots$, $0 \le \alpha < 1$, we have that, if

$$\sum_{k=1}^{\infty} |a_k| \le p(1-\alpha) \tag{3.13}$$

then f, given by (1.1) is in $I(p,\alpha)$. Further, the condition (3.13) is both necessary and sufficient for a function f, given by (1.2), to be in $I[p,\alpha] \equiv I[p,\alpha] \cap A[p]$.

<u>Remark 4</u>. If we choose $\{c_k\}_{k=0}^{\infty} \equiv \{k\}_{k=0}^{\infty}$, $d_k = 1$ when k is odd and $d_k = 0$ when k is even; and $\rho = \alpha$, $0 \le \alpha < 1$, then it follows from Theorem 1 that the condition

$$\sum_{k=1}^{\infty} (2k+1-\alpha)\ |a_{2k+1}| + \sum_{k=1}^{\infty} 2k|a_k| \le (1-\alpha) \tag{3.14}$$

is sufficient for a function f, given by (1.1), to be in $N(\alpha)$. Further, it follows from Theorem 2 that the condition (3.14) is both necessary and sufficient for a function f, given by (1.2), to be in $N[\alpha] \equiv N(\alpha) \cap A[p]$.

<u>Remark 5</u>. Let $z^p/(1-z)^{2p(1-\alpha)} = \sum_{k=0}^{\infty} C_o(\alpha,k+p)z^{k+p}$ and $z^p/(1-z)^{2p(1-\alpha)+1} = \sum_{k=0}^{\infty} 0_1(\alpha,k+p)z^{k+p}$. Then, by choosing $\{c_k\}_{k=0}^{\infty} \equiv \{C_1(\alpha,k+p)\}_{k=0}^{\infty}, \{d_k\}_{k=0}^{\infty} \equiv \{C_o(\alpha,k+p)\}_{k=0}^{\infty}$ and $\rho = \frac{1}{2}$ it follows from Theorem 3.1 that the condition

$$\sum_{k=1}^{\infty} (k+p-p\alpha)\ C_o(\alpha,\rho+\alpha)\ |a_k| \le p(1-\alpha) \tag{3.15}$$

is sufficient for f, given by (1.1), to be in $PS^*(p,\alpha)$.

Further, from Theorem 2 it follows that the condition (3.15) is both necessary and sufficient for a function f, given by (1.2) to be in $PS^*[p,\alpha] \equiv PS^*(p,\alpha) \cap A[p]$.

We note that (3.15) includes a recent result of Silverman and Silvia [10].

In view of Theorem 2, it follows that an [a*]-family is a closed convex subset of the space of analytic functions in E with the compact open topology. Thus, the closed convex hull of an [a*]-family F is equal to itself. In the next theorem we determine the extreme points of an [a*]-family.

<u>Theorem 3</u>. Let the sequences $\{c_k\}_{k=0}^{\infty}$, $\{d_k\}_{k=0}^{\infty}$, the nonnegative number ρ and the functions s_o and g_o be defined as in the definition in Section 2. Further, let $f_o(z) = z^p$, $f_k(z) = z^p - (c_o - \rho d_o)/(c_k - \rho d_k)\ z^{k+p}$, $k = 1,2\ldots$. Then, the extreme points of the [a*]-family with the touple (s_o,g_o,ρ) are precisely the set of functions $\{f_k\}_{k=0}^{\infty}$.

<u>Proof</u>. We show that a function f is in the [a*]-family with the touple (s_o,g_o,ρ), if and only if, it can be written in the form $\sum_{k=0}^{\infty} t_k f_k(z)$, where $t_k \ge 0$ and $\sum_{k=0}^{\infty} t_k = 1$. The conclusion in the theorem about the extreme points is equivalent to this result.

First, let $f(z) = \sum_{k=0}^{\infty} t_k f_k(z)$, where $t_k \ge 0$ and $\sum_{k=0}^{\infty} t_k = 1$. Then

$$f(z) = z^p - \sum_{k=1}^{\infty} t_k \frac{(c_o - \rho d_o)}{(c_k - \rho d_k)} z^{k+p}.$$

Now,

$$\sum_{k=1}^{\infty} \frac{(c_k - \rho d_k)}{(c_o - \rho d_o)} t_k \frac{(c_o - \rho d_o)}{(c_k - \rho d_k)} = \sum_{k=1}^{\infty} t_k = 1 - t_o \le 1.$$

Thus, by Theorem 2, f is in the [a*]-family with the touple (s_o, g_o, ρ).

Conversely, let f, defined by the Taylor series (1.2), be in the [a*]-family with the touple (s_o, g_o, ρ). Then, by Corollary 1, for k = 1,2,...,

$$|a_k| \le \frac{c_o - \rho d_o}{c_k - \rho d_k}.$$

We let, for k = 1,2,..., $t_k = ((c_k - \rho d_k)/(c_o - \rho d_o))\,|a_k|$ and $t_o = 1 - \sum_{k=1}^{\infty} t_k$. Note that $0 \le t_k \le 1$ for k = 1,2,3,... . Now with this choice of t_k, we can write $f(z) = \sum_{k=0}^{\infty} t_k f_k(z)$ where $\sum_{k=0}^{\infty} t_k = 1$. This completes the proof of the theorem.

Remark 6. The extreme points of individual [a*]-families can be obtained from Theorem 3 by substituting appropriate values of c_k, d_k and ρ as in Remarks 1-5.

4. APPLICATIONS.

In our next theorem, using Theorem 2, we determine the sharp values of $\beta \equiv \beta(\alpha)$ and $\gamma \equiv \gamma(\alpha)$ such that $K[p,\alpha] \subset S^*[p,\beta]$ and Re $\{f(z)/z^p\} > \gamma$ for f in $K[p,\alpha]$ and z in E. It is to be noted that, in general, there does not exist $\beta = \beta(\alpha) > \alpha$ such that $K(p,\alpha) \subset S^*(p,\alpha)$, p = 2,3... . Further, the value of Re $\{f(z)/z^p\}$ can well be negative for $|z| < 1$ and f in $K(p,\alpha)$, p = 2,3,... [3].

Theorem 4. Let f, given by (1.2), be in K $[p,\alpha]$. Then,

(i) f is in $S^*[p,\beta]$ where

$$\beta \equiv \beta(\alpha) = \frac{p+1}{2p+1-p\alpha} \tag{4.1}$$

and

(ii) $\mathrm{Re}\{f(z)/z^p\} > \gamma$, for z in E, where

$$\gamma \equiv \gamma(\alpha) = \frac{p^2(1-\alpha)+p+1}{(p+1)(p+1-p\alpha)}. \tag{4.2}$$

Both the results in (i) and (ii) are sharp.

Proof. Let f be in K $[p,\alpha]$. Then, by the necessary condition (3.11), we have

$$\sum_{k=1}^{\infty} \frac{(k+p)(k+p-p\alpha)}{p^2(1-\alpha)} |a_k| \le 1. \tag{4.3}$$

In view of the sufficient condition (3.10), we first determine the maximum value of β such that (4.3) implies

$$\sum_{k=1}^{\infty} \frac{(k+p-p\beta)}{p(1-\beta)} |a_k| \le 1. \tag{4.4}$$

Again, it is sufficient to determine the maximum value of β such that for k = 1,2,...,

$$\frac{(k+p)(k+p-p\alpha)}{p(1-\alpha)} \ge \frac{(k+p-p\beta)}{1-\beta}$$

or, equivalently, $\beta \le (k+p)/(2p+k-p\alpha)$. Since the sequence $\{\phi_k\} = \{(k+p)/(2p+k-p\alpha)\}$ is increasing in k, we choose

$$\beta \equiv \beta(\alpha) = (p+1)/(2p+1-p\alpha). \tag{4.5}$$

Further, by using (3.10) and (3.11), it is easily seen that the function

$$h(z) \equiv z^p - \frac{p^2(1-\alpha)}{(p+1)(p+1-p\alpha)} z^{p+1} \tag{4.6}$$

in $K[p,\alpha]$ is in $S^*[p,\beta]$ with β defined by (4.1) but is not in $S^*[p,\beta_1]$ for any $\beta_1 > \beta$. This shows that the value of $\beta \equiv \beta(\alpha)$ is precise.

In view of (3.13), to prive (ii), we first find the maximum value of $\gamma = \gamma(\alpha)$ such that (4.2) implies

$$\sum_{k=1}^{\infty} \frac{|a_k|}{p(1-\gamma)} \le 1 \tag{4.7}$$

Adopting the proof for part (i), we get

$$\gamma \equiv \gamma(\alpha) = \frac{p^2(1-\alpha)+p+1}{(p+1)(p+1-p\alpha)}.$$

Further, the function $h(z)$ given by (4.6) shows that $\mathrm{Re}\{h(z)/z^p\} > p\gamma$ but $\mathrm{Re}\{h(z)/z^p\} \le p\gamma_1$ for $\gamma_1 > \gamma$ and $z = r < 1$ where γ is defined by (4.2). This completes the proof of the theorem.

We, next, consider Zmorovic conjecture for functions in A[1]. Zmorovic [11] conjectured that $J(1,0) \subset S^*(1,0)$. Subsequently, this conjecture was proved to be false, by Zmorovic himself [12] among others, by showing the existence of a function in $J(1,0)$ which is not in $S^*(1,0)$. On the other hand, $K(1,0) \not\subset J(1,0)$, for the function $h_1(z) = z/(1-z)$ is in $K(1,0)$ and $\mathrm{Re}\, h'(z) < 0$ in a region $-\frac{1}{2} < 1,0 < \arg z < \pi/2$. Thus, it follows that there is no inclusion relation between the three classes $K(1,0)$, $S^*(1,0)$ and $J(1,0)$. However, it follows easily from Corollary 2 that

$$S^*[1,0] = J[1,0] = N[0].$$

Further, for $p = 1$, (3.11) ==> (3.12) ==> (3.14) ==> (3.10) ==> (3.13). Thus, we have

$$K[1,\alpha] \subsetneq J[1,\alpha] \subsetneq N[\alpha] \subsetneq S^*[1,\alpha] \subsetneq I[1,\alpha].$$

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