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AN OPERATOR INEQUALITY

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ABSTRACT. An inequality is proved in abstract separable Hilbert space H where A and B are bounded self-adjoint positive operators defined in H such that $R(A) = R(B)$ and $R(A)$ is closed.

KEY WORDS AND PHRASES. Hilbert space, positive operators, generalized inverse.
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1. INTRODUCTION AND PRELIMINARIES.

Let H be an abstract separable Hilbert space and T a linear bounded operator from H into H.
We denote the null space of T by $N(T)$, the range of T by $R(T)$ and assume that $R(T)$ is closed.
We define the generalized inverse (or Moore-Penrose inverse) operator T^+ of T as the unique linear extension of $(T/N(T)^{\perp})^{-1}$ to H such that $N(T^+)=R(T)^{\perp}$.
The linear bounded operator T^+ fulfills the following "Moore-Penrose equations" :

$$T\,T^+\,T = T$$
$$T^+\,T\,T^+ = T^+$$
$$(T\,T^+)^* = T\,T^+$$
$$(T^+\,T)^* = T^+\,T$$

which could also be used as a definition of T^+.
Penrose used these relations to define the generalized inverse of a matrix in [1].
For a systematic treatment of generalized inverses and their properties in an operator - theoretic setting, we refer to Nashed and Votruba [2]. For an extensive annotated bibliography on the theory and applications of generalized inverses, see [3]. Kaffes in [4] proved the following inequality:

$$[\lambda A+(1-\lambda)B]^+ \leq \lambda A^+ + (1-\lambda)B^+ \,.$$

where A,B are positive semi-definite matrices of the same order and $R(A) = R(B)$. In this paper we shall prove that the above inequality holds in an abstract separable Hilbert space H, where A,B are bounded self-adjoint positive operators defined in H such that $R(A) = R(B)$ and $R(A)$ is closed.

2. PROOF OF THE INEQUALITY.

THEOREM 2.1: Let A and B be bounded self-adjoint positive operators from a Hilbert space H into H. Assume that $R(A)$ is closed and $R(A) = R(B)$. Then for $0 \le \lambda \le 1$ we have

$$[\lambda A+(1-\lambda)B]^{+} \le \lambda A^{+} + (1-\lambda)B^{+} \tag{2.1}$$

PROOF. The above inequality (2.1) is trivial when $\lambda=0$ or $\lambda=1$. Let $0 < \lambda < 1$ and $A_\lambda = \lambda A+(1-\lambda)B$. Then it is not difficult to prove that $R(A) = R(A_\lambda) = R(B)$. From theorem 2 of [5] we can deduce that $A^{+} = A^{+*}$ if $A=A^{*}$, and that if Range A is closed so is Range A^{+} (when $A=A^{*}$, Dom $A=H$). From these we can prove that $R(A) = R(A^{+})$. To prove (2.1) it suffices to show that

$$(A_\lambda^{+} f,f) \le \lambda\ (A^{+} f,f) + (1-\lambda)\ (B^{+}f,f) \qquad \forall\ f\epsilon H$$

where (,) means scalar multiplication in H.

Let $f\epsilon H$. Then $f=f_1 + f_2$ with $f_1\ \epsilon R(A)$ and $f_2\ \epsilon N(A)$.

Since
$$R(A)=R(B)=R(A_\lambda)$$

and
$$(Af,f)=(Af_1,f_1)\ ,\ (Bf,f)=(Bf_1,f_1),(A_\lambda f,A_\lambda)=(A_\lambda f_1,f_1),$$

it is enough to prove that:

$$(A_\lambda^{+}f,f)\le\lambda(A^{+} f,f)+(1-\lambda)(B^{+}f,f)\ \forall\ f\epsilon R(A). \tag{2.2}$$

Given that $f\epsilon R(A)=R(B)=R(A_\lambda)$ there are $g_1\epsilon H$, $g_2\epsilon H$, $g_3\epsilon H$ such that

$$Ag_1=f,Ag_2=f,\ A_\lambda g_3=f \tag{2.3}$$

By means of relation (2.3) the above inequality assumes the following form

$$\begin{aligned}
&\Leftrightarrow\ (A_\lambda^{+}A_\lambda g_3\ ,\ A_\lambda g_3) \le \lambda(A^{+}Ag_1,Ag_1)+(1-\lambda)(B^{+}Bg_2,Bg_2) \Leftrightarrow\\
&\Leftrightarrow\ (A_\lambda A_\lambda^{+}A_\lambda g_3,g_3)\le\lambda(AA^{+}Ag_1,g_1)+(1-\lambda)\ (BB^{+}Bg_2,g_2) \Leftrightarrow\\
&\Leftrightarrow\ (A_\lambda g_3,g_3)\le\lambda(Ag_1,g_1)+(1-\lambda)\ (Bg_2,g_2)
\end{aligned} \tag{2.4}$$

Provided that the operators A,B are positive we have

$$(A(g_1-g_3),g_1,-g_3)=(Ag_1,g_1)+(Ag_3,g_3)-2(Ag_3,g_1) \ge 0 \tag{2.5}$$

$$(B(g_2-g_3),g_2-g_3)=(Bg_2,g_2) + (Bg_3,g_3)-2(Bg_3,g_2) \ge 0 \tag{2.6}$$

and
$$(Ag_3,g_1)=(A_\lambda g_3,g_3),(Bg_3,g_2)=A_\lambda g_3,g_3) \tag{2.7}$$

If we multiply (2.5) by λ, (2.6) by $(1-\lambda)$ and add the resulting equations and if we take (2.7) under consideration then the desired inequality (2.4) is obtained.

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