Internat. J. Math. & Math. Sci.
Vol. 7 No. 1 (1984) 65-73

ON COEFFICIENTS OF PATH POLYNOMIALS

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(Received March 1, 1982 and in revised form December 20, 1982)

ABSTRACT. Explicit formulae, in terms of sugraphs of the graph, are given for the first six coefficients of the simple path polynomial of a graph. From these, explicit formulae are deduced for the number of hamiltonian paths in graphs with up to six nodes. Also simplified expressions are given for the number of ways of covering the nodes of some families of graphs with k paths, for certain values of k.

KEY WORDS AND PHRASES. Path polynomials, simple path polynomials, hamiltonian path, path cover.
1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 05A15, 05C99.

1. INTRODUCTION

The graphs considered here will be finite, and will have no loops nor multiple edges. Let G be such a graph. With every path α in G, let us associate an indeterminate or *weight* w_α. With every *path cover* (a spanning forest whose elements are all paths) C of G, let us associate the monomial

$$w(C) = \prod_{\alpha} w_\alpha \quad ,$$

where the product is taken over all the elements of C. Then the *path polynomial* of G is

$$\Sigma\, w(C) \quad ,$$

where the summation is taken over all the path covers in G.

Let us assign a weight w to every path in G. Then the path polynomial of G would be a polynomial in w. If we denote it by P(G;w), then

$$P(G;w) = \sum_{k=0}^{p-1} a_k\, w^{p-k} \quad ,$$

where p is the nodes in G and a_k is the number of path covers of G with p-k components. P(G;w) is called the *simple path polynomial* of G, because of the simple manner in which weights are assigned. The basic properties of path polynomials are given in Farrell [1].

In this paper, we will use a simple combinatorial technique in order to derive expressions for the coefficients a_k , in terms of subgraphs of G . From these results, we will deduce explicit formulae for the number of hamiltonian paths in graphs with up to 6 nodes. We will also give formulae for the number of ways of covering the nodes of some families of graphs with node-disjoint paths, for certain values of k , (the results parallel those given in Farrell [2]).

2. PRELIMINARY RESULTS

We will refer to the path with one node (an isolated node) as the *trivial path*.

LEMMA 1. Let G be a connected graph with p nodes. Let C be a path cover of G , with p-n components. Then C contains exactly n edges.

PROOF. The result holds trivially for n=0 . We will therefore consider the case when $n > 0$. It is clear that the number of edges in C will be maximum, when the number of non trivial components is minimum, and vice versa.

The minimum number of non trivial components that C can have is 1 . Since C has p-n components, it must contain p-n-1 isolated nodes. But C has p nodes in all. Therefore the non trivial path must contain n + 1 nodes, and hence n edges. It follows that the maximum number of edges in C is n .

C will have the maximum number of non trivial components when they are all independent edges. Suppose that there are r independent edges. Then C must contain p-n-r isolated nodes. Therefore

$$(p-n-r) + 2r = p .$$

Hence n = r , and therefore the minimum number of edges in C is n . The result then follows.

THEOREM 1. Let G be a graph with p nodes and q edges. Then

$$P(G;w) = \sum_{k=0}^{p-1} a_k w^{p-k} ,$$

with

$$a_k = \binom{q}{k} - \gamma_k ,$$

where γ_k is the number of subgraphs of G with k edges and containing a component which is not a path.

PROOF. By definition, a_k is the number of path covers of G with p-k components. From Lemma 1, any path cover with p-k components, must contain k edges. The number of sets of k edges in G is $\binom{q}{k}$. Those which form graphs with components that are not paths, are counted by γ_k . The result therefore follows.

3. THE COEFFICIENTS OF P(G;w)

By a *nonpath graph* we will mean a graph containing a component which is not a path. Let G be a graph. We *combine* a graph H with G by either (i) identifying a node of H with a node of G or (ii) making H a component of the graph originally consisting of G only. We will use Theorem 1 in order to establish our results.

The First Three Coefficients

Let G be a graph with p nodes and q edges. It is clear that in $P(G;w)$, $a_0 = 1$, and $a_1 = q$. Since every spanning subgraph of G with two edges, must be a path cover, $\gamma_2 = 0$. Hence

$$a_2 = \binom{q}{2} .$$

The Fourth Coefficient

The nonpath graphs with 3 edges are the triangle and the 3-star (the graph consisting of 3 edges joined to a common node). Let A be the number of triangles in G. Any choice of 3 edges at a node will yield a 3-star. Therefore the number of 3-stars in G is

$$\beta = \sum_{i=1}^{p} \binom{\nu_i}{3} ,$$

where ν_i is the valency of node i in G. Therefore

$$\gamma_3 = A + \beta .$$

Hence, from Theorem 1,

$$a_3 = \binom{q}{3} - A - \beta .$$

The Fifth Coefficient

The nonpath graphs with 4 edges are (i) the triangle with an edge attached to it, (ii) the triangle together with an independent edge, (iii) the quadrilateral, (iv) the 4-star, (v) the 3-star with an edge attached to a terminal node and (vi) the 3-star together with an independent edge. See Harary [3] (Appendix 1).

The graphs (i) and (ii) are the only graphs that can be formed by combining an edge with a triangle. Therefore the number of such subgraphs in G is $(q-3)A$. Let B be the number of quadrilaterals in G. Then the number of graphs in category (iii) is B. The graphs (iv), (v) and (vi) can be formed by combining an edge with a 3-star. The number of different ways of combining an edge with a 3-star is $(q-3)\beta$. Now, there is only one other graph that can be formed by combining an edge with a 3-star. It is the triangle with an edge attached to it. The number Δ of subgraphs of G consisting of a triangle, together with an independent edge is counted, by finding the number of edges δ_{ijk} in $G-\{i,i,k\}$, the graph obtained from G by removing the nodes i, j and k of triangle T_{ijk}. Therefore

$$\Delta = \Sigma\, \delta_{ijk} ,$$

with

$$\delta_{ijk} = q + 3 - (\nu_i + \nu_j + \nu_k) , \tag{3.1}$$

where ν_i, ν_j and ν_k are the valencies of the nodes of a triangle T_{ijk} in G, and the summation is taken over all the triangles in G. Hence the number of subgraphs which are triangles with an edge attached, is $(q-3)A-\Delta$. Any choice of 4 edges at a node yields a 4-star. Therefore the number of 4-stars is

$$\Lambda = \sum_{i=1}^{p} \binom{\nu_i}{4} .$$

The quantity $(q-3)\beta$ counts each 4-star four times, since any of the 4 possible sets of 3 edges of a 4-star can be attached to the fourth edge to yield the *same* 4-star. In order to compensate for this, we must subtract 3Λ from $(q-3)\beta$. Thus we have

$$\gamma_4 = (q-3)A + B + (q-3)\beta - 3\Lambda - [(q-3)A - \Delta]$$

$$= (q-3)\beta + B + \Delta - 3\Lambda \quad .$$

Hence from Theorem 1,

$$a_4 = \binom{q}{4} - (q-3)\beta + 3\Lambda - B - \Delta .$$

The Sixth Coefficient

The nonpath graphs with 5 edges are shown below in Figure 1. The graphs have all been numbered. We will refer to the graph numbered k , as *graph (k)*. The number of different graph (k)'s in G will be denoted by *N(k)*. We refer the reader to [3] for a list of these graphs.

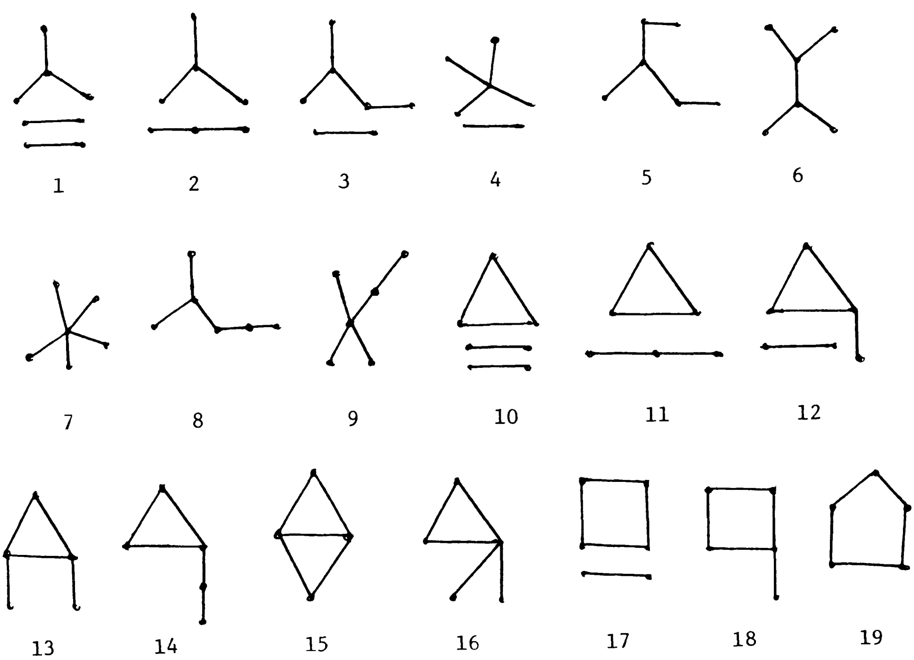

Figure 1

Let *Class 1* be the set containing graphs (1), (2),..., (9), (12), (13), (14), (15), (16) and (18). It is clear that Class 1 contains all the graphs that can be formed by combining two edges with a 3-star. The number of ways of combining two edges in G, with a 3-star in G, is $\binom{q-3}{2}\beta$. However, in this quantity, each of graphs (4), (9) and (16) will be counted 4 times. Each of graphs (6), (13) and (15) will be counted twice and graph (7) will be counted 10 times. Therefore the number of subgraphs of G of the types described in Class 1 is

$$|\text{Class } 1| = \binom{q-3}{2}\beta - 3[N(4) + N(9) + N(16)] - [N(6) + N(13) + N(15)] - 9\,N(7)\ .$$

The number of 5-stars in G is

$$N(7) = \sum_{i=1}^{p} \binom{\nu_i}{5} = \mu\ .$$

Graphs (4), (7), (9) and (16) represent all the possible graphs that can be formed by combining an edge with a 4-star (N.B. Graph (7) will be counted 5 times). Therefore

$$N(4) + 5N(7) + N(9) + N(16) = (q-4)\Lambda\ .$$

$$N(4) + N(9) + N(16) = (q-4)\Lambda - 5\mu\ .$$

Let

$$\Psi = \Sigma\binom{\nu_i-1}{2}\binom{\nu_j-1}{2}\ , \tag{3.2}$$

where i and j are adjacent nodes of valencies $\nu_i(\geq 3)$ and $\nu_j(\geq 3)$ in G, and the summation is taken over all pairs of adjacent nodes of valencies ≥ 3 in G. Then Ψ counts the graphs formed by combining two edges taken at each of a pair of adjacent nodes in G. These graphs will be precisely graphs (6), (13) and (15). Hence

$$N(6) + N(13) + N(15) = \Psi\ .$$

Therefore

$$|\text{Class } 1| = \binom{q-3}{2}\beta - 3(q-4)\Lambda + 6\mu - \Psi\ .$$

Graphs (10) and (11) are formed by combining with a triangle T_{ijk}, any two edges of the graph obtained from G by removing the nodes of T_{ijk} i.e. G-{ i,j,k }. Therefore

$$N(10) + N(11) = \Sigma\binom{\delta_{ijk}}{2} = \eta\ , \tag{3.3}$$

where δ_{ijk} and the summation are as defined above in Equation (3.1).

N(17) is counted by the number of edges in the graphs obtained from G by removing the nodes of each of its quadrilaterals. Let S be a subset of the nodes of G. Let us denote by G-S, the graph obtained from G by removing the nodes in S. Let E(G-S) be the number of edges in G-S. Then

$$N(17) = \Sigma\,[q + 4 - (\nu_h+\nu_i+\nu_j+\nu_k)] = \omega\ , \tag{3.4}$$

Where h,i,j and k are the nodes of a quadrilateral in G, ν_h, ν_i, ν_j and ν_k and their valencies, and the summation is taken over all the quadrilaterals in G .

N(19) = C , the number of pentagons in G .

Thus we have,

$$\gamma_5 = \binom{q-3}{2}\beta - 3(q-4)\Lambda + 6\mu - \Psi + \eta + \omega + C .$$

Hence from Theorem 1, we get

$$a_5 = \binom{q}{5} - \binom{q-3}{2}\beta + 3(q-4)\Lambda + \Psi - 6\mu - \eta - \omega - C .$$

The following Theorem summarizes our discussions.

THEOREM 2. Let G be a graph with p nodes and q edges. Let

$$P(G;w) = \sum_{k=0}^{p-1} a_k w^{p-k}$$

be the simple path polynomial of G . Then

(i) $a_0 = 1$,

(ii) $a_1 = q$,

(iii) $a_2 = \binom{q}{2}$,

(iv) $a_3 = \binom{q}{3} - A - \beta$,

(v) $a_4 = \binom{q}{4} - (q-3)\beta + 3\Lambda - B - \Delta$,

and

(vi) $a_5 = \binom{q}{5} - \binom{q-3}{2}\beta + 3(q-4)\Lambda + \Psi - 6\mu - \eta - \omega - C$,

where A, B and C are the numbers of triangles, quadrilaterals and pentagons respectively in G, β, Λ and μ are the numbers of 3-stars, 4-stars and 5-stars respectively in G , and Δ , Ψ , η and ω are the summations defined in above in Equations (3.1), (3.2), (3.3) and (3.4).

4. AN ILLUSTRATION

Let G be the complete graph with 6 nodes. We will use Theorem 2 in order to find its simple path polynomial. In this case,

$$p = 6,\ q = 15,\ A = 20,\ B = 45 \text{ and } C = 72$$

Also

$$\beta = 60,\ \Lambda = 30,\ \mu = 6 .$$

$$\Delta = 60,\ \Psi = 540,\ \eta = 60 \text{ and } \omega = 45 .$$

From Theorem 2,

$$a_0 = 1,\ a_1 = 15,\ a_2 = 105,\ a_3 = \binom{15}{3} - 20 - 60 = 375 .$$

$$a_4 = \binom{15}{4} - 12(60) + 3(30) - 45 - 60 = 630 .$$

$$a_5 = \binom{15}{5} - \binom{12}{2}60 + 3.11.30 + 546 - 6.6 - 60 - 45 - 72$$

$$= 3003 - 3960 + 990 + 540 - 213$$

$$+ 4533 - 4173 = 360 .$$

Hence we get

$$P(K_6;w) = w^6 + 15w^5 + 105w^4 + 375w^3 + 630w^2 + 360w \ .$$

5. HAMILTONIAN PATHS IN SMALL GRAPHS

It is clear that the coefficient of w in $P(G;w)$ is the number of hamiltonian paths in G. The following corollaries of Theorem 2 are therefore immediate.

COROLLARY 2.1. The number of hamiltonian paths in a graph G with 5 nodes and q edges is

$$\binom{q}{4} - (q-3)\beta + 3\Lambda - B - \Delta \ .$$

COROLLARY 2.2. The number of hamiltonian paths in a graph G with 6 nodes and q edges is

$$\binom{q}{5} - \binom{q-3}{2}\beta + 3(q-4)\Lambda + \Psi - 6\mu - \eta - \omega - C \ .$$

6. APPLICATIONS

We define the *wheel* W_p to be the graph obtained by joining an isolated node to all the nodes of a circuit with $p-1$ nodes. The *fan* F_p is the graph obtained by joining an isolated node to all the nodes of a path with $p-1$ nodes. The *short ladder* S_n is the graph obtained by joining the corresponding nodes of two equal paths with n nodes. The *long ladder* L_n, is the graph obtained by joining the corresponding nodes of two equal circuits with n nodes. S_n and L_n will therefore contain $2n$ nodes each. S_n will contain $3n-2$ edges and L_n, $3n$ edges.

THEOREM 3. The number of ways of covering the nodes of the wheel W_p, with $p-3$ node disjoint paths, is

$$\binom{2p-2}{3} - \frac{1}{6}(p-1)(p^2-5p+18) \ .$$

With $p-r$ node disjoint paths, it is

$$\binom{2p-2}{4} - \frac{1}{6}(2p-5)(p-1)(p^2-5p+12) + 3\binom{p-1}{4} - (p-1)(p-3) \ ,$$

and with $p-5$ node disjoint paths, it is

$$\binom{2p-2}{5} - \frac{1}{6}(p-3)(2p-5)(p^3-6p^2+17p-12) + 3(2p-6)\binom{p-1}{4} - 6\binom{p-1}{4} + (p-1)(p-2) \ .$$

PROOF. W_p has $2p-2$ edges, $p-1$ subgraphs that are triangles, quadrilaterals and pentagons. (Provided of course, that the rim itself is not one of these subgraphs.) It has $[(p-1) + \binom{p-1}{3}]$ 3-stars, $\binom{p-1}{4}$ 4-stars, and $\binom{p-1}{5}$ 5-stars. Also

$$\Delta = (p-1)\ (p-4) \ ,$$

$$\Psi = (p-1) + (p-1)\ \binom{p-2}{2} \ ,$$

$$\eta = (p-1)\binom{p-4}{2} \ ,$$

and

$$\omega = (p-1)\ (p-5) \ .$$

The result follows by making the appropriate substitutions into Theorem 2.

THEOREM 4. The number of ways of covering the nodes of F_p with p-3 node disjoint paths is

$$\binom{2p-3}{3} - \binom{p-1}{3} - (2p-5) .$$

With p-4 node disjoint paths, it is

$$\binom{2p-3}{4} - \frac{1}{3}(p-3)^2 (p^2-3p+8) - 3\binom{p-1}{4} - (p^2-6p+9) ,$$

and with p-5 node disjoint paths, it is

$$\binom{2p-3}{5} - \frac{1}{6}(p-3)(2p-7)(p^2-3p+8) + 3(2p-7)\binom{p-1}{4} - 6\binom{p-1}{5}$$
$$+ 2p^2 - 34p + 21 .$$

PROOF. F_p has 2p-3 edges, p-2 triangles, p-3 quadrilaterals and p-4 pentagons. It also has $\left[\binom{p-1}{3} + (p-3)\right]$ 3-stars, $\binom{p-1}{4}$ 4-stars and $\binom{p-1}{5}$ 5-stars. In this case,

$$\Delta = (p-5)(p-4) + 2(p-4) = (p-3)(p-4) ,$$
$$\Psi = (p-4) + \binom{p-2}{2}(p-3) = \frac{1}{2}(p^3-8p^2-19p-26) ,$$
$$\eta = 2\binom{p-4}{2} + (p-5)\binom{p-4}{2} = \frac{1}{2}(p-4)(p-5)^2 ,$$

and

$$\omega = 2(p-5) + (p-6)(p-5) = p^2 - 9p + 20 .$$

The result then follows by making the appropriate substitutions in Theorem 2.

THEOREM 5. The number of ways of covering the nodes of S_n with p-3 node disjoint paths, where p=2n , is

$$\binom{3n-2}{3} - 2n + 4 .$$

With p-4 node disjoint paths, it is

$$\binom{3n-2}{4} - (6n^2 - 21n + 19) ,$$

and with p-5 node disjoint paths, it is

$$\binom{3n-2}{5} - 9n^3 + 48n^2 - 80n - 38 .$$

PROOF. S_n has 3n-2 edges, no triangles, n-1 quadrilaterals and no pentagons. It has 2n-4 3-stars, no 4-stars and no 5-stars. Therefore

$$\Delta = \eta = 0 , \Psi = 3n-8 \text{ and } \omega = 3n^2-13n+14 .$$

The result follows by substituting into Theorem 2.

THEOREM 6. The number of ways of covering the long ladder L_n with p-3 node disjoint paths, where p=2n , is

$$\binom{3n}{3} - 2n .$$

With p-4 node disjoint paths, it is

$$\binom{3n}{4} - 6n^2 + 5n ,$$

and with p-5 node disjoint paths, it is

$(^{3n}) \quad (9n^3 \; 18n^2$ [illegible]

PROOF. L_n has $3n$ edges, n quadrilaterals $(n\neq 4)$, no triangles, and no pentagons $(n\neq 5)$. It has $2n$ 3-stars, no 4-stars, and no 5-stars. Also,

$$\Delta = 0 ,$$

$$\Psi = 3n ,$$

$$\eta = 0 ,$$

and

$$\omega = (3n-8)n .$$

The result follows by substituting into Theorem 2.

The main results obtained in this paper are analogous to those given for chromatic polynomials in Farrell [2].

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