VALUE DISTRIBUTION AND THE LEMMA OF
THE LOGARITHMIC DERIVATIVE ON POLYDISCS

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ABSTRACT.   Value distribution is developed on polydiscs with the special emphasis

that the value distribution function depend on a vector variable. A Lemma of the

logarithmic derivative for meromorphic functions on polydiscs is derived. Here the

Bergman boundary of the polydiscs is approached along cones of any dimension and

exceptional sets for such an approach are defined.

1. INTRODUCTION.

Value distribution for polydisc exhaustions has been studied by Ronkin [1],

Stoll [2] and others. They emphasized the growth of entire holomorphic and

meromorphic functions and the representation of canonical functions to a given

divisor in $\mathbb{C}^n$. For applications to mathematical logic, Lee A. Rubel and

C. Ward Henson [3], [4] inquired if the Lemma of the logarithmic derivative could

be established for a meromorphic function on a fixed given polydisc.

In the classical one variable theory, the Lemma of the logarithmic derivative

has been one of the basic tools for a long time. In several variables, the

analogous Lemma was proved only recently. For ball exhaustions of $\mathbb{C}^n$, Vitter [5]

proved the Lemma for differential operators with constant coefficients and derived

the defect relation for meromorphic maps $f : \mathbb{C}^n \to \mathbb{P}_m$. See also Stoll [6].

Vitter's Lemma extends easily to differential operators with polynomial or, with

proper modifications, to differential operators with entire coefficients.

Earlier, a weak version of the Lemma was proved by Gauthier and Hengartner [7] for

the special operator $D' = z_1 \, \partial/\partial z_1 + \ldots + z_n \, \partial/\partial z_n$, which does not permit the

extension to a general differential operator. The differential operator D' suffices

for meromorphic functions, but imposes unnecessary restrictions for meromorphic maps

$f : \mathbb{C}^n \to \mathbb{P}_m$. Recently, Shiffman has shown how to derive Vitter's Lemma from the

result of Griffiths and King [8], which also can be interpreted as a Lemma of the

logarithmic derivative. Biancofiore and Stoll [9] gave an elementary proof of

Vitter's result. The same method will be used to obtain the Lemma of the

logarithmic derivative for polydiscs.

A theory should reflect the intrinsic algebraic, geometric and analytic

structures of the mathematical landscape under consideration. Our development of

value distribution theory on polydiscs will adhere to this principle. This is

an important feature of this paper, which is mostly self contained and requires only

a minimal knowledge of several complex variables.

Let us outline the main result. The euclidean space $\mathbb{R}^n$ is partially ordered

by its coordinates. Denote by $\|r\|$ the length of r in $\mathbb{R}^n$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$

define $|z| = (|z_1|, \ldots, |z_n|)$. For $0 < \mathfrak{h} \in \mathbb{R}^n$ define

$$\overset{+}{\mathbb{R}^n}(\mathfrak{h}) = \{r \in \mathbb{R}^n \mid 0 < r < \mathfrak{h}\}$$

$$\mathbb{D}(\mathfrak{h}) = \{z \in \mathbb{C}^n \mid |z| < \mathfrak{h}\}$$

$$\mathbb{D}\langle\mathfrak{h}\rangle = \{z \in \mathbb{C}^n \mid |z| = \mathfrak{h}\}.$$

Let Ω_n be the rotation invariant measure on the torus $\mathbb{D}\langle r\rangle$ for $r \in \overset{+}{\mathbb{R}^n}(\mathfrak{h})$ such that

$\mathbb{D}\langle r\rangle$ has total volume 1. In $\overset{+}{\mathbb{R}^n}(\mathfrak{h})$, we could approach $\mathfrak{h}$ via $t\mathfrak{h}$ with $t \to 1$ only.

However Rubel is interested in the more general approach $r \to \mathfrak{h}$ with $0 < r < \mathfrak{h}$. We

introduce admissible approach cones M with $1 \le \dim M \le n$ which contain both cases.

On M we introduce a measure Λ_M such that $\Lambda_M(M) = \infty$. If $E \subseteq M$ with $\Lambda_M(E) < \infty$,

then $\mathfrak{h}$ can be approached from $M - E$.

Let f be a non-constant meromorphic function on $\mathbb{D}(\mathfrak{h})$. The characteristic $T_f(\mathfrak{r},\mathfrak{q})$ of f is a function of two points $\mathfrak{r}$ and $\mathfrak{q}$ in $\mathbb{R}^n_+(\mathfrak{h})$. Take $\tau \in \mathbb{N}[1,n]$ and define $f'_\tau = \partial f/\partial z_\tau$. Keep $\mathfrak{q} \in \mathbb{R}^n_+(\mathfrak{h})$ fixed. Then there is an open subset E of the manifold M with $\Lambda_M(E) < \infty$ such that for all $\mathfrak{r} \in M - E$ with $\mathfrak{r} \geq \mathfrak{q}$ we have

$$\int_{\mathbb{D}\langle r\rangle} \log^+ |f'_\tau/f| \Omega_n \leq 17 \log^+ T_f(\mathfrak{r},\mathfrak{q}) + 19 \log^+ \frac{1}{\|\mathfrak{h} - \mathfrak{r}\|} .$$

2. POLYDISCS.

For any set S let S^n be the n-fold cartesian product. If $S \subseteq \mathbb{R}$ and if $-\infty \leq a < b \leq +\infty$, define

$$S[a,b] = \{x \in S \mid a \leq x \leq b\} \qquad S(a,b) = \{x \in S \mid a < x < b\}$$

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$$\overset{+}{S} = S(0,\infty) = \{x \in S \mid x > 0\} \qquad S_+ = S[0,\infty) = \{x \in S \mid x \geq 0\}.$$

If $S \subseteq \mathbb{C}^n$, define $S_* = S - \{0\}$. If $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ are vectors in $\mathbb{R}^n$, define $x \leq y$ if $x_j \leq y_j$ for all $j \in \mathbb{N}[1,n]$ and $x < y$ if $x_j < y_j$ for all $j \in \mathbb{N}[1,n]$. If $S \subseteq \mathbb{R}^n$, define

$$S[\mathfrak{r}] = \{x \in S \mid x \leq \mathfrak{r}\} \qquad S(\mathfrak{r}) = \{x \in S \mid x < \mathfrak{r}\}.$$

For $\mathfrak{z} = (z_1,\ldots,z_n)$ and $\mathfrak{w} = (w_1,\ldots,w_n)$ in $\mathbb{C}^n$, and $p \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ define

$$\mathfrak{z} + \mathfrak{w} = (z_1 + w_1, \ldots, z_n + w_n) \qquad \mathfrak{z} \cdot \mathfrak{w} = (z_1 w_1,\ldots,z_n w_n)$$

$$\lambda\mathfrak{z} = (\lambda z_1,\ldots,\lambda z_n) \qquad \mathfrak{z}^p = (z_1^p,\ldots,z_n^p)$$

$$|\mathfrak{z}| = (|z_1|,\ldots,|z_n|) \qquad \bar{\mathfrak{z}} = (\bar z_1,\ldots,\bar z_n)$$

$$\|\mathfrak{z}\| = (|z_1|^2 + \ldots + |z_n|^2)^{1/2} \qquad e^{\mathfrak{z}} = (e^{z_1},\ldots,e^{z_n})$$

$$\log z = (\log z_1, \ldots, \log z_n) \qquad\qquad z^\lambda = e^{\lambda \log z} \quad \text{if } z \in (\mathbb{C}_*)^n$$

$$0 = 0_n = (0, \ldots, 0) \in \mathbb{C}^n \qquad\qquad 1_n = (1, \ldots, 1) \in \mathbb{C}^n.$$

Here log is the principal value. For $r \in \mathbb{R}^n_+$, define the polydiscs

$$\mathbb{D}(r) = \{z \in \mathbb{C}^n \mid |z| < r\} \qquad\qquad \mathbb{D}[r] = \{z \in \mathbb{C}^n \mid |z| \le r\}.$$

The Bergman-Shilov boundary is given by

$$\mathbb{D}\langle r\rangle = \{z \in \mathbb{C}^n \mid |z| = r\}.$$

A surjective, real analytic map $\alpha_r : \mathbb{R}^n \to \mathbb{D}\langle r\rangle$ is defined by $\alpha_r(\phi) = r \cdot e^{i\phi}$ for

all $\phi \in \mathbb{R}^n$. If $r > 0$, then $\mathbb{D}\langle r\rangle$ is an n-dimensional, oriented, real analytic

manifold such that α_r is an orientation preserving, local diffeomorphism which is

bijective on $\mathbb{R}[0, 2\pi)^n$.

For $r = 1 \in \mathbb{R}$, we obtain the unit disc $\mathbb{D} = \mathbb{D}(1)$ with $\overline{\mathbb{D}} = \mathbb{D}[1]$ and $\partial\mathbb{D} = \mathbb{D}\langle 1\rangle$.

For $r \in \mathbb{R}^n_+$, define the diagonal manifold

$$\Delta(r) = \bigcup_{t \in \mathbb{R}(0,1)} \mathbb{D}\langle tr\rangle \qquad\qquad \Delta[r] = \bigcup_{t \in \mathbb{R}[0,1]} \mathbb{D}\langle tr\rangle.$$

A surjective, real analytic map $\beta_r : \mathbb{R}(0,1) \times \mathbb{R}^n \to \Delta(r)$ is defined by

$\beta_r(t, \phi) = t\alpha_r(\phi)$ for all $t \in \mathbb{R}(0,1)$ and $\phi \in \mathbb{R}^n$. Moreover, $\Delta(r)$ is an

$(n+1)$-dimensional, oriented, real analytic manifold such that β_r is an orientation

preserving local diffeomorphism which is bijective on $\mathbb{R}(0,1) \times \mathbb{R}[0, 2\pi)^n$.

On $(\mathbb{C}_*)^n$, we introduce the holomorphic differential forms

$$\Omega_n = \frac{1}{(2\pi i)^n} \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n} \qquad\qquad (2.1)$$

$$\xi_n = \left(\frac{1}{2\pi i}\right)^{n-1} \sum_{\lambda=1}^{n} (-1)^{\lambda-1} \left(\bigwedge_{\mu=1}^{\lambda-1} \frac{dz_\mu}{z_\mu}\right) \wedge \left(\bigwedge_{\mu=\lambda+1}^{n} \frac{dz_\mu}{z_\mu}\right). \tag{2.2}$$

For $j \in \mathbb{N}[1,n]$ we have

$$\frac{dz_j}{2\pi i z_j} \wedge \xi_n = \Omega_n. \tag{2.3}$$

Let $j_r : \mathbb{D}\langle r\rangle \to (\mathbb{C}_*)^n$ be the inclusion map. Then

$$\alpha_r^* j_r^*(\Omega_n) = \frac{1}{(2\pi)^n} d\phi_1 \wedge \ldots \wedge d\phi_n > 0 \tag{2.4}$$

$$\int_{\mathbb{D}\langle r\rangle} \Omega_n = 1. \tag{2.5}$$

Hence $j_r^*(\Omega_n)$ is a positive measure on $\mathbb{D}\langle r\rangle$ of total mass 1 such that $j_r^*(\Omega_n)$ is invariant under the action of $\mathbb{D}\langle 1_n\rangle$ on $\mathbb{D}\langle r\rangle$ defined by $\zeta(z) = \zeta \cdot z$ for $\zeta \in \mathbb{D}\langle 1_n\rangle$ and $z \in \mathbb{D}\langle r\rangle$. In particular, if $F : \mathbb{D}\langle r\rangle \to \mathbb{C}$ is integrable, then

$$\int_{\mathbb{D}\langle r\rangle} F\Omega_n = \int_{z \in \mathbb{D}\langle r\rangle} \left(\frac{1}{2\pi} \int_0^{2\pi} F(e^{i\phi}z)\, d\phi\right) \Omega_n(z). \tag{2.6}$$

Let χ be a form of bidegree $(1,1)$ on an open subset $U \neq \emptyset$ of $\mathbb{C}^n$ where

$$\chi = \frac{i}{2\pi} \sum_{\mu,\nu=1}^{n} \chi_{\mu\bar\nu}\, dz_\mu \wedge d\bar z_\nu. \tag{2.7}$$

Then χ is said to be __non-negative__ (respectively __positive__) at $z \in U$ if for all $0 \neq x \in \mathbb{C}^n$ we have

$$\sum_{\mu,\nu=1}^{n} \chi_{\mu\bar\nu}(z) x_\mu \bar x_\nu \geq 0 \quad \text{(respectively} > 0). \tag{2.8}$$

If $S \subseteq U$ and χ is non-negative (respectively positive) at every point of S we say

that χ is non-negative on S (respectively positive on S) in signs $\chi \geq 0$ on S

(respectively $\chi > 0$ on S). On $U \cap (\mathbb{C}_*)^n$ we have

$$\chi \wedge \xi_n = \sum_{\mu,\nu=1}^{n} \chi_{\mu\bar{\nu}} z_\mu \; d\bar{z}_\nu \wedge \Omega_n. \tag{2.9}$$

Take $r \in \overset{+}{\mathbb{R}}{}^n$ and assume that $U \supset \Delta(r)$. Let $\iota_r : \Delta(r) \to U$ be the inclusion map. For

$t \in \mathbb{R}(0,1)$ and $\phi = (\phi_1,\ldots,\phi_n) \in \mathbb{R}^n$ we have

$$\beta_r^* \iota_r^* (\chi \wedge \xi_n)(t,\phi) = \left(\frac{1}{2\pi}\right)^n \sum_{\mu,\nu=1}^{n} \chi_{\mu\bar{\nu}}(tr \cdot e^{i\phi}) r_\mu r_\nu e^{i(\phi_\mu - \phi_\nu)} 2t \; dt \wedge d\phi \tag{2.10}$$

where $d\phi = d\phi_1 \wedge \ldots \wedge d\phi_n$. In particular

$$\iota_r^* (\chi \wedge \xi_n) \geq 0 \quad \text{if } \chi \geq 0 \text{ on } \Delta(r)$$

$$\iota_r^* (\chi \wedge \xi_n) > 0 \quad \text{if } \chi > 0 \text{ on } \Delta(r).$$

For $0 \neq z \in \mathbb{C}^n$, define $\ell_z : \mathbb{C} \to \mathbb{C}^n$ by $\ell_z(u) = uz$. If $z \in \mathbb{D}\langle r \rangle$, then

$\ell_z(\mathbb{C}(1) - \{0\}) \subseteq \Delta(r)$. The identities (2.6) and (2.10) and Fubini's theorem imply

easily:

LEMMA 2.1. Take $r \in \overset{+}{\mathbb{R}}{}^n$. Let χ be a form of bidegree (1,1) on an open

neighborhood U of $\Delta(r)$. Assume that $\iota_r^*(\chi \wedge \xi_n)$ is integrable over $\Delta(r)$. Then

$$\int_{\Delta(r)} \chi \wedge \xi_n = \int_{z \in \mathbb{D}\langle r \rangle} \left(\int_{\mathbb{D}} \ell_z^*(\chi) \right) \Omega_n(z). \tag{2.11}$$

COROLLARY 2.2. Take $r \in \overset{+}{\mathbb{R}}{}^n$ and $r \in \overset{+}{\mathbb{R}}$. Let χ be a form of bidegree (1,1) on an

open neighborhood U of $\Delta(rr)$. Assume that $\iota_{rr}^*(\chi \wedge \xi_n)$ is integrable over $\Delta(rr)$. Then

$$\int_{\Delta(rr)} \chi \wedge \xi_n = \int_{z \in \mathbb{D}\langle r \rangle} \left(\int_{\mathbb{D}(r)} \ell_z^*(\chi) \right) \Omega_n(z).$$

Take $n \in \mathbb{N}$ and $p \in \mathbb{Z}[0,n]$. Let $\mathcal{T}(p,n)$ be the set of all injective increasing maps $\tau : \mathbb{N}[1,p] \to \mathbb{N}[1,n]$. If $p = 0$, then $\mathcal{T}(p,n) = \emptyset$. If $p \in \mathbb{N}[1,n]$, let $\iota_p : \mathbb{N}[1,p] \to \mathbb{N}[1,n]$ be the inclusion map. If $p = n$, then $\mathcal{T}(n,n) = \{\iota_n\}$. For $\tau \in \mathcal{T}(p,n)$, we assign various operators.

The complement $\tilde{\tau} \in \mathcal{T}(n-p, n)$ is uniquely defined by $\text{Im } \tau \cap \text{Im } \tilde{\tau} = \emptyset$ or by $\text{Im } \tau \cup \text{Im } \tilde{\tau} = \mathbb{N}[1,n]$. Clearly $\tilde{\tilde{\tau}} = \tau$ and $\tilde{\iota}_n = \emptyset$.

A surjective linear map $\pi_\tau : \mathbb{C}^n \to \mathbb{C}^p$ called the τ-$\underline{\text{projection}}$ is defined by

$$\pi_\tau(z) = (z_{\tau(1)}, \ldots, z_{\tau(p)}) \in \mathbb{C}^p. \qquad (2.12)$$

Let δ_{jk} be the Kronecker symbol. Define $\mathbf{e}_j = (\delta_{j1}, \ldots, \delta_{jn})$. For each $z \in \mathbb{C}^{n-p}$ an injective, affine map $\tau_z : \mathbb{C}^p \to \mathbb{C}^n$ called the τ-$\underline{\text{injection}}$ $\underline{\text{at}}$ z is defined by

$$\tau_z(w) = \sum_{\mu=1}^{p} w_\mu \mathbf{e}_{\tau(\mu)} + \sum_{\nu=1}^{n-p} z_\nu \mathbf{e}_{\tilde{\tau}(\nu)}. \qquad (2.13)$$

If $z = 0 \in \mathbb{C}^{n-p}$, then τ_0 is linear. If $\lambda > 0$ and $\mathfrak{v}$ and $\mathfrak{w}$ belong to $\mathbb{C}^p$, then

$$\tau_0(\mathfrak{v} \cdot \mathfrak{w}) = \tau_0(\mathfrak{v}) \cdot \tau_0(\mathfrak{w}) \qquad \qquad \tau_0(\mathfrak{v}^\lambda) = \tau_0(\mathfrak{v})^\lambda. \qquad (2.14)$$

If $z \in \mathbb{C}^n$, then one and only one number $p \in \mathbb{Z}[0,n]$ and map $\sigma_z \in \mathcal{T}(p,n)$ exist such that $z_{\sigma_z(j)} \neq 0$ for $j = 1, \ldots, p$ and $z_{\tilde{\sigma}_z(j)} = 0$ for $j = 1, \ldots, n-p$. The map σ_z is called the $\underline{\text{support}}$ of z. Obviously $p = 0$ iff $z = 0$. Also

$$\text{Im } \sigma_{x \cdot y} = \text{Im } \sigma_x \cap \text{Im } \sigma_y. \qquad (2.15)$$

Take $r \in \mathbb{R}_+^n$. Let F be a function on $\mathbb{D}\langle r \rangle$. Define

$$\mathfrak{M}(\mathbf{r},F) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} F(r_1 e^{i\phi_1},\ldots,r_n e^{i\phi_n})\, d\phi_1 \cdots d\phi_n \qquad (2.16)$$

provided the integral exists (perhaps in the sense of summability where $+\infty$ or $-\infty$ are permitted). If $\mathbf{r} > 0$, then

$$\mathfrak{M}(\mathbf{r},F) = \int_{\mathbb{D}\langle \mathbf{r}\rangle} F\Omega_n. \qquad (2.17)$$

If $0 \le \mathbf{r} \ne 0$, then $\sigma_{\mathbf{r}} \in \mathfrak{C}(p,n)$ with $p \in \mathbb{N}[1,n]$. Take $0 \in \mathbb{C}^{n-p}$. Abbreviate $\sigma = \sigma_{\mathbf{r}}$. Then

$$\mathfrak{M}(\mathbf{r},F) = \int_{\mathbb{D}\langle \pi_\sigma(\mathbf{r})\rangle} (F \circ \sigma_0)\Omega_p. \qquad (2.18)$$

If $\mathbf{r} = 0$, then

$$\mathfrak{M}(0,F) = F(0). \qquad (2.19)$$

Take $p \in \mathbb{N}[1,n]$ and $\tau \in \mathfrak{C}(p,n)$. Take $\mathbf{z} \in \mathbb{D}\langle \pi_{\widetilde{\tau}}(\mathbf{r})\rangle$, then $\tau_{\mathbf{z}}(\mathbb{D}\langle \pi_\tau(\mathbf{r})\rangle) \subseteq \mathbb{D}\langle \mathbf{r}\rangle$. Hence $F \circ \tau_{\mathbf{z}}$ is defined on $\mathbb{D}\langle \pi_\tau(\mathbf{r})\rangle$. By Fubini's theorem $\mathfrak{M}(\pi_\tau(\mathbf{r}), F \circ \tau_{\mathbf{z}}) = H(\mathbf{z})$ exists for almost all $\mathbf{z} \in \mathbb{D}\langle \pi_{\widetilde{\tau}}(\mathbf{r})\rangle$ and we have

$$\mathfrak{M}(\mathbf{r},F) = \mathfrak{M}(\pi_{\widetilde{\tau}}(\mathbf{r}),H). \qquad (2.20)$$

More explicitly, this is written as

$$\mathfrak{M}(\mathbf{r},F) = \mathfrak{M}(\pi_{\widetilde{\tau}}(\mathbf{r}), \mathfrak{M}(\pi_\tau(\mathbf{r}), F \circ \tau_{\mathbf{z}})). \qquad (2.21)$$

If $\mathbf{r} > 0$, then

$$\int_{\mathbb{D}\langle \mathbf{r}\rangle} F\Omega_n = \int_{\mathbf{z} \in \mathbb{D}\langle \pi_{\widetilde{\tau}}(\mathbf{r})\rangle} \left(\int_{\mathbb{D}\langle \pi_\tau(\mathbf{r})\rangle} F \circ \tau_{\mathbf{z}}\Omega_p \right) \Omega_{n-p}(\mathbf{z}). \qquad (2.22)$$

Sometimes we shall write $\mathfrak{M}(\mathbf{r},F)$ as in (2.18) even if $\mathbf{r}$ has some zero coordinates. Then (2.21) writes as in (2.22) which is more instructive.

3. PLURISUBHARMONIC FUNCTIONS.

Let B be a subset of $\mathbb{R}^n_+$. A function $g : B \to \mathbb{R} \cup \{-\infty\}$ is said to be __increasing__

if $g(x) \leq g(y)$ whenever $x \in B$ and $y \in B$ with $x \leq y$. The set B is said to be

__logarithmic convex__ if $x \in B$ and $y \in B$ and $\lambda \in \mathbb{R}(0,1)$ implies $x^\lambda y^{1-\lambda} \in B$. The

function g is said to be logarithmic convex, if B is logarithmic convex and if $x \in B$

and $y \in B$ and $\lambda \in \mathbb{R}(0,1)$ imply

$$g(x^\lambda y^{1-\lambda}) \leq \lambda g(x) + (1 - \lambda)g(y). \tag{3.1}$$

Here we are mostly interested in the case where B is a polydisc. Take $\mathfrak{h} \in \mathbb{R}^n_+$.

Then $\mathbb{R}^n_+(\mathfrak{h})$ is __logarithmic convex__. Take $p \in \mathbb{N}[1,n]$ and $\tau \in \mathfrak{T}(p,n)$. Then $\pi_\tau(\mathfrak{h}) \in \mathbb{R}^p_+$.

If $0 \in \mathbb{C}^{n-p}$, an injective linear map $\tau_0 : \mathbb{C}^p \to \mathbb{C}^n$ is defined which maps $\mathbb{R}^p_+(\pi_\tau(\mathfrak{h}))$

into $\mathbb{R}^n_+(\mathfrak{h})$. Define $\beta_\tau = \log \pi_\tau(\mathfrak{h}) \in \mathbb{R}^p$. Given a function $g : \mathbb{R}^n_+(\mathfrak{h}) \to \mathbb{R} \cup \{-\infty\}$

define

$$h_\tau : \mathbb{R}^p(\beta_\tau) \to \mathbb{R} \cup \{-\infty\} \quad \text{by} \quad h_\tau(x) = g(\tau_0(e^x)). \tag{3.2}$$

LEMMA 3.1. If g is logarithmic convex, then h_τ is convex.

PROOF. Take x and y in $\mathbb{R}^p(\beta_\tau)$. Take $\lambda \in \mathbb{R}(0,1)$. Define $\mu = 1 - \lambda$. Then

$$h_\tau(\lambda x + \mu y) = g(\tau_0(e^{\lambda x} e^{\mu y})) = g(\tau_0(e^x)^\lambda \tau_0(e^y)^\mu)$$

$$\leq \lambda g(\tau_0(e^x)) + \mu g(\tau_0(e^y)) = \lambda h_\tau(x) + \mu h_\tau(y) \qquad \text{q.e.d.}$$

Observe, if $\tau = \iota_n$, then $\tau_0(\mathbb{R}^n(\beta_\tau)) = \overset{+}{\mathbb{R}^n}(\mathfrak{h})$ is the interior of $\mathbb{R}^n_+(\mathfrak{h})$. In

particular $g(e^x)$ is convex and g is continuous on $\overset{+}{\mathbb{R}^n}(\mathfrak{h})$, if g is logarithmic convex.

LEMMA 3.2. If h_τ is convex for each $\tau \in \mathfrak{T}(p,n)$ and $p \in \mathbb{N}[1,n]$ then (3.1) holds

for all $x \in \mathbb{R}^n_+(\mathfrak{h})$ and $y \in \mathbb{R}^n_+(\mathfrak{h})$ with $\sigma_x = \sigma_y$ and for all $\lambda \in \mathbb{R}(0,1)$.

PROOF. Define $z = x^\lambda y^{1-\lambda}$. Then $\sigma_z = \sigma_x = \sigma_y = \tau$ where the jth coordinate of the vectors z, x or y is positive if and only if $j \in \text{Im } \tau$. Hence $\tilde{x}$ and $\tilde{y}$ exist in $\mathbb{R}^p(\beta_\tau)$ such that $x = \tau_0(e^{\tilde{x}})$ and $y = \tau_0(e^{\tilde{y}})$. Define $\mu = 1 - \lambda$. Then

$$g(x^\lambda y^\mu) = g(\tau_0(e^{\tilde{x}})^\lambda \tau_0(e^{\tilde{y}})^\mu) = g(\tau_0(e^{\lambda\tilde{x}+\mu\tilde{y}})) = h_\tau(\lambda\tilde{x} + \mu\tilde{y})$$

$$\leq \lambda h_\tau(\tilde{x}) + \mu h_\tau(\tilde{y}) = \lambda g(\tau_0(e^{\tilde{x}})) + \mu g(\tau_0(e^{\tilde{y}})) = \lambda g(x) + \mu g(y) \qquad \text{q.e.d.}$$

If $\sigma_x \neq \sigma_y$, the conclusion of Lemma 3.2 may be incorrect: Take $n = 2$ and $b = (2,2)$. Define $g(x,y) = 1$ if $x = 0$ or $y = 0$ and $g(x,y) = 0$ if $0 < x < 2$ and $0 < y < 2$. Each function h_τ is constant and therefore convex. If $0 < \lambda < 1$, then $(0,0)^\lambda(1,1)^{1-\lambda} = (0,0)$ and

$$g((0,0)^\lambda(1,1)^{1-\lambda}) = g(0,0) = 1 > \lambda = \lambda g(0,0) + (1 - \lambda)g(1,1).$$

However we have the following result:

LEMMA 3.3. If g is increasing on $\mathbb{R}^n_+(b)$ and if h_τ is convex on $\mathbb{R}^p(\beta_\tau)$ for all $\tau \in \mathcal{C}(p,n)$ and $p \in \mathbb{N}[1,n]$, then g is logarithmic convex.

PROOF. If $x^\lambda y^{1-\lambda} = 0$, then (3.1) is trivial since $g(0) \leq g(x)$ and $g(0) \leq g(y)$. Define $\mu = 1 - \lambda$ and $z = x^\lambda y^\mu$. Then $\text{Im } \sigma_z = \text{Im } \sigma_x \wedge \text{Im } \sigma_y$. Put $\tilde{x}_j = 0 = \tilde{y}_j$ if $j \notin \text{Im } \sigma_z$ and $\tilde{x}_j = x_j > 0$ and $\tilde{y}_j = y_j > 0$ if $j \in \text{Im } \sigma_z$. Then $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n) \in \mathbb{R}^n_+(b)$ and $\tilde{y} = (\tilde{y}_1,\ldots,\tilde{y}_n) \in \mathbb{R}^n_+(b)$ with $\sigma_{\tilde{x}} = \sigma_{\tilde{y}} = \sigma_z$ and $\tilde{x} \leq x$ and $\tilde{y} \leq y$. Also we have $z = \tilde{x}^\lambda \tilde{y}^\mu$. Lemma 3.2 implies

$$g(x^\lambda y^\mu) = g(\tilde{x}^\lambda \tilde{y}^\mu) \leq \lambda g(\tilde{x}) + \mu g(\tilde{y}) \leq \lambda g(x) + \mu g(y) \qquad \text{q.e.d.}$$

PROPOSITION 3.4. Take $b \in \overset{+}{\mathbb{R}^n}$. An <u>upper semi-continuous</u> function $g : \mathbb{R}^n_+(b) \to \mathbb{R} \cup \{-\infty\}$ increases and is logarithmic convex if and only if the function $u : \mathbb{D}(b) \to \mathbb{R} \cup \{-\infty\}$ defined by $u(z) = g(|z|)$ is plurisubharmonic.

PROOF. If u is plurisubharmonic, a theorem of Vladimirov [10] p. 88 (see also

Ronkin [1] Theorem 2.1.2) implies that g increases and that all h_τ are convex. Hence

g increases and is logarithmic convex. If g increases and is logarithmic convex,

then g increases and all h_τ are convex. By Vladimirov's theorem u is

plurisubharmonic, q.e.d.

Take $\mathfrak{h} \in \mathbb{R}^n_+$. Let u be a plurisubharmonic function on $\mathbb{D}(\mathfrak{h})$. For $\mathbf{r} \in \mathbb{R}^n_+(\mathfrak{h})$ define

$$M(\mathbf{r},u) = \text{Max}\{u(\mathbf{z}) \mid \mathbf{z} \in \mathbb{D}\langle\mathbf{r}\rangle\} \tag{3.3}$$

$$\mathfrak{M}(\mathbf{r},u) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} u(re^{i\phi}) \, d\phi_1 \cdots d\phi_n. \tag{3.4}$$

Then $M(|\mathbf{z}|;u)$ and $\mathfrak{M}(|\mathbf{z}|,u)$ are plurisubharmonic functions of $\mathbf{z}$ on $\mathbb{D}(\mathfrak{h})$ (Ronkin [1]

pages 75 and 84). Therefore $M(\mathbf{r},u)$ and $\mathfrak{M}(\mathbf{r},u)$ increase and are logarithmic convex

functions of $\mathbf{r}$ in $\mathbb{R}^n_+(\mathfrak{h})$. If $u \not\equiv -\infty$, then $M(\mathbf{r},u)$ and $\mathfrak{M}(\mathbf{r},u)$ are real numbers for all

$\mathbf{r} \in \mathbb{R}^n_+(\mathfrak{h})$. If some coordinates of $\mathbf{r}$ are zero, the same remains true if $u \circ \sigma_{\mathbf{r}_0} \not\equiv -\infty$.

PROPOSITION 3.5. Take $\mathfrak{h} \in \mathbb{R}^n_+$. Let u : $\mathbb{D}(\mathfrak{h}) \to \mathbb{R}$ be a pluriharmonic function.

Then $\mathfrak{M}(\mathbf{r},u) = u(0)$ for all $\mathbf{r} \in \mathbb{R}^n_+(\mathfrak{h})$.

PROOF. Since u and -u are plurisubharmonic we have

$$u(0) \le \mathfrak{M}(\mathbf{r},u) = -\mathfrak{M}(\mathbf{r},-u) \le -(-u(0)) = u(0) \qquad \text{q.e.d.}$$

A function g : $\mathbb{D}(\mathfrak{h}) \to \mathbb{R} \cup \{-\infty,+\infty\}$ is said to be quasipluriharmonic if there are

plurisubharmonic functions $u \not\equiv -\infty$ and $v \not\equiv -\infty$ such that g = u - v. Strictly speaking

g is defined except for a pluripolar set

$$I_g \subseteq \{\mathbf{z} \mid u(\mathbf{z}) = -\infty = v(\mathbf{z})\}$$

called the indeterminacy. For $0 < \mathbf{r} < \mathfrak{h}$, the integral average

$$\mathfrak{M}(\mathbf{r},g) = \mathfrak{M}(\mathbf{r},u) - \mathfrak{M}(\mathbf{r},v) \tag{3.5}$$

is well defined.

Let $\mathfrak{S}(\mathfrak{h})$ be the real vector space of all quasipluriharmonic functions on $\mathbb{D}(\mathfrak{h})$.
Then the set $\mathfrak{P}(\mathfrak{h})$ of all pluriharmonic functions on $\mathbb{D}(\mathfrak{h})$ is a linear subspace of
$\mathfrak{S}(\mathfrak{h})$. Let $\mathbb{Q}(\mathfrak{h}) = \mathfrak{S}(\mathfrak{h})/\mathfrak{P}(\mathfrak{h})$ be the quotient vector space and let $\rho : \mathfrak{S}(\mathfrak{h}) \to \mathbb{Q}(\mathfrak{h})$ be
the residual map. An element of $\mathbb{Q}(\mathfrak{h})$ is called a __quasipluriharmonic class__ on $\mathbb{D}(\mathfrak{h})$.
Take $h \in \mathbb{Q}(\mathfrak{h})$. Then $h = \rho(g)$ with $g \in \mathfrak{S}(\mathfrak{h})$. If $h = \rho(\tilde{g})$ with $\tilde{g} \in \mathfrak{S}(\mathfrak{h})$ then
$\tilde{g} = g + v$ where $v \in \mathfrak{P}(\mathfrak{h})$. Take $\mathfrak{r} \in \mathbb{R}^n_+(\mathfrak{h})$ and $\mathfrak{q} \in \mathbb{R}^n_+(\mathfrak{h})$. Then

$$\mathfrak{M}(\mathfrak{r},\tilde{g}) = \mathfrak{M}(\mathfrak{r},g) + v(0)$$

$$\mathfrak{M}(\mathfrak{q},\tilde{g}) = \mathfrak{M}(\mathfrak{q},g) + v(0).$$

Subtraction implies

$$\mathfrak{M}(\mathfrak{r},g) - \mathfrak{M}(\mathfrak{q},g) = \mathfrak{M}(\mathfrak{r},\tilde{g}) - \mathfrak{M}(\mathfrak{q},\tilde{g}).$$

Therefore the __valence function__ of h is well defined independent of the choice of the
representative g by

$$\mathfrak{N}(h,\mathfrak{r},\mathfrak{q}) = \mathfrak{M}(\mathfrak{r},g) - \mathfrak{M}(\mathfrak{q},g) \in \mathbb{R} \tag{3.6}$$

for $0 < \mathfrak{r} < \mathfrak{h}$ and $0 < \mathfrak{q} < \mathfrak{h}$. For fixed $\mathfrak{r}$ and $\mathfrak{q}$, the function $\mathfrak{N}(h,\mathfrak{r},\mathfrak{q})$ is linear
in h. If $\mathfrak{r}$, $\mathfrak{q}$, $\mathfrak{p}$ belong to $\mathbb{R}^n_+(\mathfrak{h})$, then

$$\mathfrak{N}(h,\mathfrak{r},\mathfrak{q}) + \mathfrak{N}(h,\mathfrak{q},\mathfrak{p}) = \mathfrak{N}(h,\mathfrak{r},\mathfrak{p}) \tag{3.7}$$

$$\mathfrak{N}(h,\mathfrak{r},\mathfrak{q}) = -\mathfrak{N}(h,\mathfrak{q},\mathfrak{r}). \tag{3.8}$$

The set $\mathfrak{S}_+(\mathfrak{h})$ of all plurisubharmonic functions $u \not\equiv -\infty$ on $\mathbb{D}(\mathfrak{h})$ is contained in
$\mathfrak{S}(\mathfrak{h})$ and $\mathfrak{S}_+(\mathfrak{h})$ is closed under addition and multiplication with non-negative numbers.
Define $\mathbb{Q}_+(\mathfrak{h}) = \rho(\mathfrak{S}_+(\mathfrak{h}))$ in $\mathbb{Q}(\mathfrak{h})$. Each $g \in \mathbb{Q}_+(\mathfrak{h})$ is called a __plurisubharmonic class__.
If $u \in \mathfrak{S}_+(\mathfrak{h})$ such that $\rho(u) = g$ and if $0 < \mathfrak{r} < \mathfrak{h}$ and $0 < \mathfrak{q} < \mathfrak{h}$, then

$$\mathfrak{N}(g,\mathfrak{r},\mathfrak{q}) = \mathfrak{M}(\mathfrak{r},u) - \mathfrak{M}(\mathfrak{q},u). \tag{3.9}$$

The definition extends to all $\mathfrak{q} \in \mathbb{R}^n_+(\mathfrak{h})$ with $u \circ \sigma_{\mathfrak{q},0} \not\equiv -\infty$, and the extension does not
depend on the choice of u. The function $\mathfrak{N}(g,\mathfrak{r},\mathfrak{q})$ increases in $\mathfrak{r}$ and decreases in $\mathfrak{q}$.

If $0 \leq \mathfrak{q} \leq \mathfrak{r} < \mathfrak{h}$, then

$$\mathfrak{N}(g,\mathfrak{r},\mathfrak{q}) \geq 0. \qquad (3.10)$$

If $\mathfrak{q}$ is fixed and $\mathfrak{r}$ converges to $\mathfrak{h}$, then $\mathfrak{N}(g,\mathfrak{r},\mathfrak{q})$ measures the growth of g.

Now, we will give examples of valence functions of divisors, meromorphic maps and meromorphic functions.

Take $\mathfrak{h} \in \overset{+}{\mathbb{R}}{}^{u}$. Let $\mathcal{O}(\mathfrak{h})$ be the integral domain of all holomorphic functions on $\mathbb{D}(\mathfrak{h})$. Let $\mathcal{K}(\mathfrak{h})$ be the field of meromorphic functions on $\mathbb{D}(\mathfrak{h})$. Then $\mathcal{K}(\mathfrak{h})$ is the field of quotients of $\mathcal{O}(\mathfrak{h})$. Define $\mathcal{K}_{*}(\mathfrak{h}) = \mathcal{K}(\mathfrak{h}) - \{0\}$ and let

$$\mathcal{O}^{*}(\mathfrak{h}) = \{f \in \mathcal{O}(\mathfrak{h}) \mid f(z) \neq 0 \text{ for all } z \in \mathbb{D}(\mathfrak{h})\} \qquad (3.11)$$

be the multiplicative group of units in $\mathcal{O}(\mathfrak{h})$. The quotient group $\mathfrak{D}(\mathfrak{h}) = \mathcal{K}_{*}(\mathfrak{h})/\mathcal{O}^{*}(\mathfrak{h})$ is written additively and is called the <u>module of divisor</u> on $\mathbb{D}(\mathfrak{h})$. Let $\mu : \mathcal{K}_{*}(\mathfrak{h}) \to \mathfrak{D}(\mathfrak{h})$ be the residual map. Define $\mathcal{O}_{*}(\mathfrak{h}) = \mathcal{O}(\mathfrak{h}) - \{0\}$. Then

$\mathfrak{D}^{+}(\mathfrak{h}) = \mu(\mathcal{O}_{*}(\mathfrak{h}))$ is the additively closed subset of <u>non-negative</u> <u>divisors</u>. We write the variable as an index, $\mu_{f} = \mu(f)$ for all $f \in \mathcal{K}_{*}(\mathfrak{h})$. If μ_{f} is non-negative, we write $\mu_{f} \geq 0$. If $f \in \mathcal{K}_{*}(\mathfrak{h})$, then $f = g/h$ where $g \neq 0$ and $h \neq 0$ are holomorphic functions on $\mathbb{D}(\mathfrak{h})$. Therefore $\log|f| = \log|g| - \log|h| \in \mathcal{S}(\mathfrak{h})$ and the map $f \mapsto \log|f|$ is a homomorphism. If $f \in \mathcal{O}^{*}(\mathfrak{h})$, then $\log|f| \in \mathcal{P}(\mathfrak{h})$. Hence if $\nu = \mu_{f} \in \mathfrak{D}(\mathfrak{h})$, the <u>valence</u> <u>function</u> of the divisor ν

$$N_{\nu}(\mathfrak{r},\mathfrak{q}) = \mathfrak{N}(\rho(\log|f|),\mathfrak{r},\mathfrak{q}) \qquad (3.12)$$

is well defined independent of the representative f with $\nu = \mu_{f}$. Obviously, if $0 < \mathfrak{r} < \mathfrak{h}$ and $0 < \mathfrak{q} < \mathfrak{h}$, the definition (3.12) is also known as the <u>Jensen</u> <u>formula</u>

$$N_{\mu_{f}}(\mathfrak{r},\mathfrak{q}) = \int_{\mathbb{D}\langle \mathfrak{r}\rangle} \log|f|\Omega_{n} - \int_{\mathbb{D}\langle \mathfrak{q}\rangle} \log|f|\Omega_{n}. \qquad (3.13)$$

The map $\nu \to N_{\nu}(\mathfrak{r},\mathfrak{q})$ is an additive homomorphism. If $\mathfrak{r}$, $\mathfrak{p}$, $\mathfrak{q}$ belong to $\overset{+}{\mathbb{R}}{}^{n}(\mathfrak{h})$, then

$$N_{\nu}(\mathfrak{r},\mathfrak{q}) + N_{\nu}(\mathfrak{q},\mathfrak{p}) = N_{\nu}(\mathfrak{r},\mathfrak{p}) = -N_{\nu}(\mathfrak{p},\mathfrak{r}). \qquad (3.14)$$

If $\nu \in \mathfrak{D}^+(\mathfrak{h})$, then $N_\nu(\mathfrak{r},\mathfrak{q})$ increases and is logarithmic convex in $\mathfrak{r}$ with

$$N_\nu(\mathfrak{r},\mathfrak{q}) \geq 0 \quad \text{if } 0 \leq \mathfrak{q} \leq \mathfrak{r} < \mathfrak{h}. \tag{3.15}$$

Naturally, if we consider divisors on complex manifolds and spaces the definition of divisors has to be localized.

Let V be a complex vector space of dimension $m + 1 > 1$. Define $V_* = V - \{0\}$. Then $\mathbb{C}_*$ operates on V_* by multiplication. The quotient space $\mathbb{P}(V) = V_*/\mathbb{C}_*$ is a connected, compact complex manifold of dimension m called the <u>complex projective</u> <u>space of</u> V. The residual map $\mathbb{P} : V_* \to \mathbb{P}(V)$ is holomorphic. If $A \subseteq V$, define $\mathbb{P}(A) = \mathbb{P}(A \cap V_*) = \{\mathbb{P}(z) \mid 0 \neq z \in A\}$. Let $\mathcal{O}(\mathfrak{h},V)$ be the set of all holomorphic vector functions $\mathfrak{v} : \mathbb{D}(\mathfrak{h}) \to V$. Define $\mathcal{O}_*(\mathfrak{h},V) = \mathcal{O}(\mathfrak{h},V) - \{0\}$. Two holomorphic vector functions $\mathfrak{v} \in \mathcal{O}_*(\mathfrak{h},V)$ and $\mathfrak{w} \in \mathcal{O}_*(\mathfrak{h},V)$ are called equivalent $\mathfrak{v} \sim \mathfrak{w}$, if $\mathfrak{v} \wedge \mathfrak{w} \equiv 0$. This defines an equivalence relation on $\mathcal{O}_*(\mathfrak{h},V)$. An equivalence class f is said to be a <u>meromorphic</u> <u>map</u> from $\mathbb{D}(\mathfrak{h})$ into $\mathbb{P}(V)$ and each $\mathfrak{v}$ in f is said to be a <u>representation of</u> f. The representation $\mathfrak{v}$ is said to be <u>reduced</u> if for each representation $\mathfrak{w}$ of f there exists a holomorphic function g such that $\mathfrak{w} = g\mathfrak{v}$. Each meromorphic map has a reduced representation and if $\mathfrak{v}$ and $\tilde{\mathfrak{v}}$ are reduced representations, there are holomorphic functions h and $\tilde{h}$ on $\mathbb{D}(\mathfrak{h})$ such that $\mathfrak{v} = \tilde{h}\tilde{\mathfrak{v}}$ and $\tilde{\mathfrak{v}} = h\mathfrak{v}$. Hence $h\tilde{h} = 1$ and $h \in \mathcal{O}^*(\mathfrak{h})$ has no zeros. Moreover $\mathfrak{w} = \tilde{g}\tilde{\mathfrak{v}} = gh\mathfrak{v}$. Hence $\mu_{\tilde{g}} = \mu_g$. Therefore the non-negative divisor $\mu_{\mathfrak{w}} = \mu_g$ is well defined independent of the choice of the reduced representation $\mathfrak{v}$. Moreover $\mathfrak{w}$ is reduced if and only if $\mu_{\mathfrak{w}} = 0$.

Let $\mathfrak{v}$ and $\tilde{\mathfrak{v}}$ be reduced representations of the meromorphic map f, then the <u>indeterminacy</u>

$$I_f = \{z \in \mathbb{D}(\mathfrak{h}) \mid \mathfrak{v}(z) = 0\} = \{z \in \mathbb{D}(\mathfrak{h}) \mid \tilde{\mathfrak{v}}(z) = 0\} \tag{3.16}$$

is well defined, analytic, with dim $I_f \leq n - 2$. If $z \in \mathbb{D}(\mathfrak{h}) - I_f$, then

$$f(z) = \mathbb{P}(\mathfrak{v}(z)) = \mathbb{P}(\tilde{\mathfrak{v}}(z)) \in \mathbb{P}(V) \tag{3.17}$$

is well defined and the map $f : \mathbb{D}(\mathfrak{h}) - I_f \to \mathbb{P}(V)$ is holomorphic. If $\mathfrak{w}$ is any

representation of f and if $\mathfrak{z} \in \mathbb{D}(\mathfrak{h})$ with $\mathfrak{w}(\mathfrak{z}) \neq 0$, then $f(\mathfrak{z}) = \mathbb{P}(\mathfrak{w}(\mathfrak{z}))$.

Let W be a $(p+1)$-dimensional linear subspace of V. The projective space $\mathbb{P}(W)$

is a <u>projective plane</u> <u>of</u> <u>dimension</u> p in $\mathbb{P}(V)$. If $p = m - 1$, then $\mathbb{P}(W)$ is <u>called a</u>

<u>hyperplane</u>. Let V* be the dual vector space of V. Take a $\in \mathbb{P}(V*)$. Then $a = \mathbb{P}(\alpha)$

where $\alpha \in V_*^*$ is a linear map $\alpha : V \to \mathbb{C}$. Then $E[a] = \mathbb{P}(\ker \alpha)$ is a hyperplane in

$\mathbb{P}(V)$. The map $a \to E[a]$ is a bijective parameterization of all hyperplanes in $\mathbb{P}(V)$.

If $a \in \mathbb{P}(V*)$ and if $f(\mathbb{D}(\mathfrak{h}) - I_f) \not\subseteq E[a]$, then f is said to be <u>linearly</u>

<u>non-degenerate</u> <u>for</u> a. If $\mathfrak{v}$ is a reduced representation of f and if $\alpha \in V_*^*$ then

$\alpha \circ \mathfrak{v} \neq 0$. The divisor $\mu_f^a = \mu_{\alpha \circ \mathfrak{v}}$ depends on a and f only and is called the

a-<u>divisor</u> <u>of</u> f. The valence function of f for a is defined by

$$N_f(\mathfrak{r}, \mathfrak{q}; a) = N_{\mu_f^a}(\mathfrak{r}, \mathfrak{q}). \qquad (3.18)$$

If $\mathfrak{w}$ is a representation of f, there is a holomorphic function $g \neq 0$ with $\mathfrak{w} = g\mathfrak{v}$.
Hence $\alpha \circ \mathfrak{w} = g\alpha \circ \mathfrak{v}$. By definition $\mu_{\mathfrak{w}} = \mu_g$. Hence

$$\mu_{\alpha \circ \mathfrak{w}} = \mu_{\mathfrak{w}} + \mu_f^a. \qquad (3.19)$$

Take a positive hermitian form (1) : $V \times V \to \mathbb{C}$, called a <u>hermitian product</u> on V.
The associated <u>norm</u> is defined by $\|\mathfrak{z}\| = \sqrt{(\mathfrak{z}|\mathfrak{z})}$. If $x \subset V$ and $\mathfrak{z} \in V$, we have the
Schwarz inequality

$$|(\mathfrak{z}|x)| \leq \|\mathfrak{z}\| \, \|x\|. \qquad (3.20)$$

If $\alpha \in V*$ and $\beta \in V*$, vectors $\mathfrak{a} \in V$ and $\mathfrak{h} \in V$ exist uniquely such that

$$\alpha(\mathfrak{z}) = (\mathfrak{z}|\mathfrak{a}) \quad \beta(\mathfrak{z}) = (\mathfrak{z}|\mathfrak{h}) \quad \text{for all } \mathfrak{z} \in V. \qquad (3.21)$$

A dual hermitian product on V* is defined by

$$(\alpha|\beta) = (\mathfrak{a}|\mathfrak{h}). \qquad (3.22)$$

If $\alpha = \beta$, then $\|\alpha\| = \|\mathfrak{a}\|$. Therefore (3.20) and (3.21) imply the Schwarz inequality

$$|\alpha(\mathfrak{z})| \leq \|\alpha\| \ \|\mathfrak{z}\|. \tag{3.23}$$

If $z \in \mathbb{P}(V)$ and $a \in \mathbb{P}(V^*)$, then $z = \mathbb{P}(\mathfrak{z})$ and $a = \mathbb{P}(\alpha)$ with $\mathfrak{z} \in V_*$ and $\alpha \in V_*^*$. The projective distance from z to $E[a]$ is defined by

$$0 \leq \|z,a\| = \frac{|\alpha(\mathfrak{z})|}{\|\alpha\| \ \|\mathfrak{z}\|} \leq 1 \tag{3.24}$$

independent of the choice of the representatives $\mathfrak{z}$ and α.

The compensation function of f for a is defined for $r \in \mathbb{R}^n_+(\mathfrak{h})$ by

$$0 \leq m_f(r,a) = \mathfrak{M}\left(\log \frac{1}{\|f,a\|}, \ r\right) = \int_{\mathbb{D}\langle r\rangle} \log \frac{1}{\|f,a\|} \ \Omega_n < \infty. \tag{3.25}$$

The definition extends to all $r \in \mathbb{R}^n_+(\mathfrak{h})$ with $\sigma_{r_0}(\mathbb{D}(\pi_{\sigma_r}(r))) \nsubseteq I_f$.

Let $\mathfrak{v}$ be a reduced representation of f. Then $\log\|\mathfrak{v}\|$ is a plurisubharmonic function on $\mathbb{D}(\mathfrak{h})$. If $\tilde{\mathfrak{v}}$ is another reduced representation of f a holomorphic function h without zeros exists such that $\tilde{\mathfrak{v}} = h\mathfrak{v}$. Then

$$\log\|\tilde{\mathfrak{v}}\| = \log\|\mathfrak{v}\| + \log|h|$$

where $\log|h|$ is pluriharmonic. Hence $\log\|\tilde{\mathfrak{v}}\|$ and $\log\|\mathfrak{v}\|$ define the same class

$$\rho(\log\|\tilde{\mathfrak{v}}\|) = \rho(\log\|\mathfrak{v}\|).$$

The characteristic function of f is defined as the valence function of this class

$$T_f(r,\mathfrak{q}) = \mathfrak{N}(\rho(\log\|\mathfrak{v}\|),r,\mathfrak{q}) \tag{3.26}$$

which is

$$T_f(r,\mathfrak{q}) = \mathfrak{M}(r,\log\|\mathfrak{v}\|) - (\mathfrak{q},\log\|\mathfrak{v}\|) \tag{3.27}$$

$$T_f(r,\mathfrak{q}) = \int_{\mathbb{D}\langle r\rangle} \log\|\mathfrak{v}\|\Omega_n - \int_{\mathbb{D}\langle \mathfrak{q}\rangle} \log\|\mathfrak{v}\|\Omega_n. \tag{3.28}$$

The definition extends to all $\mathfrak{r} \in \mathbb{R}_+^n(\mathfrak{h})$ and $\mathfrak{q} \in \mathbb{R}_+^n(\mathfrak{h})$ with $\sigma_{\mathfrak{r}_0}(\mathbb{D}(\pi_{\sigma_\mathfrak{r}}(\mathfrak{r}))) \not\subseteq I_f$ and

$\sigma_{\mathfrak{q}_0}(\mathbb{D}(\pi_{\sigma_\mathfrak{q}}(\mathfrak{q}))) \not\subseteq I_f$. If $\mathfrak{r}, \mathfrak{p}, \mathfrak{q}$ belong to $\mathbb{R}_+^n(\mathfrak{h})$, then

$$T_f(\mathfrak{r},\mathfrak{q}) + T_f(\mathfrak{q},\mathfrak{p}) = T_f(\mathfrak{r},\mathfrak{p}) = -T_f(\mathfrak{p},\mathfrak{r}) \qquad (3.29)$$

$$T_f(\mathfrak{r},\mathfrak{q}) \geq 0 \quad \text{if } 0 < \mathfrak{q} \leq \mathfrak{r} < \mathfrak{h}. \qquad (3.30)$$

The function $T_f(\mathfrak{r},\mathfrak{q})$ is logarithmic convex and increasing in $\mathfrak{r}$ and decreasing in $\mathfrak{q}$

and continuous where $0 < \mathfrak{q} \leq \mathfrak{r} < \mathfrak{h}$.

Let $\mathfrak{w}$ be a representation of f. Take a reduced representation $\mathfrak{v}$ of f. A

holomorphic function $g \not\equiv 0$ exists uniquely such that $\mathfrak{w} = g\mathfrak{v}$. By definition $\mu_\mathfrak{w} = \mu_g$.

Then

$$\int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log\|\mathfrak{w}\|\Omega_n = \int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log\|\mathfrak{v}\|\Omega_n + \int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log|g|\Omega_n.$$

Therefore (3.28) and (3.13) imply

$$T_f(\mathfrak{r},\mathfrak{q}) = \int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log\|\mathfrak{w}\|\Omega_n - \int_{\mathbb{D}\langle\mathfrak{q}\rangle} \log\|\mathfrak{w}\|\Omega_n - N_{\mu_\mathfrak{w}}(\mathfrak{r},\mathfrak{q}) \qquad (3.31)$$

which generalizes (3.28).

Let $\mathfrak{v}$ be a reduced representation of f. Take $a \in \mathbb{P}(V)$. Assume that f is not

linearly degenerate for a. Take $\alpha \in V_*^*$ with $\mathbb{P}(\alpha) = a$. Take $0 < \mathfrak{r} < \mathfrak{h}$ and $0 < \mathfrak{q} < \mathfrak{h}$.

Then

$$T_f(\mathfrak{r},\mathfrak{q}) = \int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log\|\mathfrak{v}\|\Omega_n - \int_{\mathbb{D}\langle\mathfrak{q}\rangle} \log\|\mathfrak{v}\|\Omega_n$$

$$N_f(\mathfrak{r},\mathfrak{q},a) = \int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log|\alpha \circ \mathfrak{v}|\Omega_n - \int_{\mathbb{D}\langle\mathfrak{q}\rangle} \log|\alpha \circ \mathfrak{v}|\Omega_n$$

$$0 = \int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log\|\alpha\|\Omega_n - \int_{\mathbb{D}\langle\mathfrak{q}\rangle} \log\|\alpha\|\Omega_n$$

$$
m_f(\mathfrak{r},a) = \int\limits_{\mathbb{D}\langle\mathfrak{r}\rangle} \log \frac{\|\mathfrak{v}\| \; \|\alpha\|}{|\alpha \circ \mathfrak{v}|} \, \Omega_n
$$

$$
m_f(\mathfrak{q},a) = \int\limits_{\mathbb{D}\langle\mathfrak{q}\rangle} \log \frac{\|\mathfrak{v}\| \; \|\alpha\|}{|\alpha \circ \mathfrak{v}|} \, \Omega_n .
$$

Addition and subtraction imply the <u>First Main Theorem</u>

$$
T_f(\mathfrak{r},\mathfrak{q}) = N_f(\mathfrak{r},\mathfrak{q},a) + m_f(\mathfrak{r},a) - m_f(\mathfrak{q},a). \tag{3.32}
$$

Define $\tau_V : V \to \mathbb{R}$ by $\tau_V(\mathfrak{z}) = \|\mathfrak{z}\|^2$. One and only one positive form $\omega > 0$ of

bidegree $(1,1)$ exists on $\mathbb{P}(V)$ such that $\mathbb{P}^*(\omega) = dd^c \log \tau_V$. Here $d = \partial + \bar\partial$ and

$d^c = (i/4\pi)(\bar\partial - \partial)$. Here ω is the exterior form of the <u>Fubini-Study Kaehler metric</u>

on $\mathbb{P}(V)$. We have

$$
\int\limits_{\mathbb{P}(V)} \omega^m = 1. \tag{3.33}
$$

Denote the corresponding form on $\mathbb{P}(V^*)$ also by ω. Then

$$
\int\limits_{a \in \mathbb{P}(V^*)} \log \frac{1}{\|z,a\|} \, \omega^m(a) = \frac{1}{2} \sum_{\mu=1}^{m} \frac{1}{\mu} . \tag{3.34}
$$

An exchange of integration implies

$$
\int\limits_{a \in \mathbb{P}(V^*)} m_f(\mathfrak{r},a) \omega^m(a) = \frac{1}{2} \sum_{\mu=1}^{m} \frac{1}{\mu} . \tag{3.35}
$$

Integration of the First Main Theorem yields

$$
T_f(\mathfrak{r},\mathfrak{q}) = \int\limits_{a \in \mathbb{P}(V^*)} N_f(\mathfrak{r},\mathfrak{q},a) \omega^n(a). \tag{3.36}
$$

For $V = \mathbb{C}^{m+1}$ write $\mathbb{P}_m = \mathbb{P}(V)$ and $\mathbb{P}_m^* = \mathbb{P}(V^*)$. On $\mathbb{C}^{m+1}$ define

$(\mathfrak{z}|\mathfrak{w}) = z_1\bar w_1 + \ldots + z_m \bar w_m$. Identify the compactified plane $\mathbb{C} \cup \{\infty\}$ with $\mathbb{P}_1$ by

setting

$$\mathbb{P}(z,w) = \frac{z}{w} \qquad\qquad \mathbb{P}(z,1) = z \qquad\qquad \mathbb{P}(w,0) = \mathbb{P}(1,0) = \infty \qquad (3.37)$$

if $z \in \mathbb{C}$ and $0 \neq w \in \mathbb{C}$. If $\alpha \in (\mathbb{C}^2)^*$, define $\alpha(1,0) = \alpha_1$ and $\alpha(0,1) = \alpha_2$ and

identify $\mathbb{P}_1^*$ with $\mathbb{C} \cup \{\infty\}$ by setting

$$\mathbb{P}(\alpha) = -\frac{\alpha_2}{\alpha_1} \quad \text{if } \alpha_1 \neq 0 \quad \text{and} \quad \mathbb{P}(\alpha) = \infty \quad \text{if } \alpha_1 = 0 \neq \alpha_2.$$

If $z \in \mathbb{C}$ and $a \in \mathbb{C}$, then $(z,1) \in \mathbb{C}^2$ with $\mathbb{P}(z,1) = z$. Also $\alpha \in (\mathbb{C}^2)^*$ is defined by

setting $\alpha(z_1,z_2) = z_1 - az_2$. Then $\mathbb{P}(\alpha) = a$ and $\alpha(z,1) = z - a$. Define α_∞ by

$\alpha_\infty(1,0) = 0$ and $\alpha_\infty(0,1) = 1$. Then

$$\|z,a\| = \frac{|\alpha(z,1)|}{\|\alpha\| \; \|(z,1)\|} = \frac{|z - a|}{\sqrt{1 + |a|^2} \sqrt{|z|^2 + 1}} \qquad (3.38)$$

$$\|\infty,a\| = \frac{|\alpha(1,0)|}{\|\alpha\| \; \|(1,0)\|} = \frac{1}{\sqrt{1 + |a|^2}} \qquad (3.39)$$

$$\|z,\infty\| = \frac{|\alpha_\infty(z,1)|}{\|\alpha_\infty\| \; \|(z,1)\|} = \frac{1}{\sqrt{1 + |z|^2}} \qquad (3.40)$$

$$\|\infty,\infty\| = \frac{|\alpha_\infty(1,0)|}{\|\alpha_\infty\| \; \|(1,0)\|} = 0. \qquad (3.41)$$

Hence the projective distance $\| \; , \; \|$ on $\mathbb{P}_1$ is the chordale distance on the Riemann

sphere of diameter 1 in $\mathbb{R}^3$. On $\mathbb{C}$ the Fubini-Study form is given by

$$\omega = \frac{i}{2\pi} \; \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} \; . \qquad (3.42)$$

A meromorphic function f on $\mathbb{D}(\mathfrak{h})$ is the quotient of two holomorphic functions g

and $h \neq 0$, such that $hf = g$ and $\mathfrak{v} = (g,h) : \mathbb{D}(\mathfrak{h}) \to \mathbb{C}^2$ defines a meromorphic map from

$\mathbb{D}(\mathfrak{h})$ into $\mathbb{P}_1$ which is identified with f. All meromorphic maps from $\mathbb{D}(\mathfrak{h})$ into $\mathbb{P}_1$ are

obtained this way except the map identical to ∞. The representation $\mathfrak{v}$ is reduced if

and only if g and h are coprime at every point of $\mathbb{D}(\mathfrak{h})$. For $a \in \mathbb{C} \subseteq P_1^*$ we have

$$\mu_f^a = \mu_{g-ah} \qquad\qquad \mu_f^\infty = \mu_h \qquad\qquad (3.43)$$

$$T_f(\mathfrak{r},\mathfrak{q}) = \int\limits_{\mathbb{D}\langle\mathfrak{r}\rangle} \log\sqrt{|g|^2 + |h|^2}\ \Omega_n - \int\limits_{\mathbb{D}\langle\mathfrak{q}\rangle} \log\sqrt{|g|^2 + |h|^2}\ \Omega_n \qquad (3.44)$$

$$m_f(\mathfrak{r},a) = \int\limits_{\mathbb{D}\langle\mathfrak{r}\rangle} \log \frac{\sqrt{1+|a|^2}\sqrt{1+|f|^2}}{|f-a|}\ \Omega_n \qquad\qquad (3.45)$$

$$m_f(\mathfrak{r},\infty) = \int\limits_{\mathbb{D}\langle\mathfrak{r}\rangle} \log\sqrt{1+|f|^2}\ \Omega_n . \qquad\qquad (3.46)$$

Our definition of the value distribution functions corresponds to the Ahlfors-Shimizu definition and is guided by the intrinsic nature of the algebraic and geometric structures which are involved in the given mathematical situation. The classical definitions do not account properly of these features. We would destroy the intrinsic coherence of the theory if we would insist that $\mathfrak{q} = 0$ and that $m_f(\mathfrak{r},\infty)$ be defined as the integral average of $\log^+|f|$.

Our definitions recognize the absence of a natural one parameter exhaustion of the polydisc and take advantage of the fact that every holomorphic line bundle on a polydisc is trivial. If the polydisc is replaced by another complex manifold, the definitions have to be localized and holomorphic line bundles become unavoidable, also an appropriate exhaustion has to be chosen.

At present our definitions do not reflect the possible holomorphic slicings of a polydisc and do not present the valence function as an integrated counting function or the characteristic as an integrated spherical image. These properties will be studied in the next two sections.

4.   THE COUNTING FUNCTION.

Take $\mathfrak{h} \in \mathbb{R}^n_+$ and $\mathfrak{a} \in \mathbb{D}(\mathfrak{h})$. Let $g \not\equiv 0$ be a holomorphic function on $\mathbb{D}(\mathfrak{h})$. Then there exists uniquely a non-negative integer $\mu_g(\mathfrak{a})$ called the zero multiplicity of g

<u>at</u> $\mathbf{a}$ such that for each integer $j \geq \mu_g(\mathbf{a})$ there is a homogeneous polynomial P_j of

degree j such that

$$g(\mathbf{z}) = \sum_{j=\mu_g(\mathbf{a})}^{\infty} P_j(\mathbf{z} - \mathbf{a}) \tag{4.1}$$

for all $\mathbf{z}$ in a neighborhood of $\mathbf{a}$ in $\mathbb{D}(\mathbf{h})$ and such that $P_{\mu_g(\mathbf{a})} \not\equiv 0$. Obviously

$\mu_g(\mathbf{a}) = 0$ if and only if $g(\mathbf{a}) \neq 0$. If $h \not\equiv 0$ is a holomorphic function on $\mathbb{D}(\mathbf{h})$ then

$$\mu_{gh}(\mathbf{a}) = \mu_g(\mathbf{a}) + \mu_h(\mathbf{a}) \tag{4.2}$$

$$\mu_{g+h}(\mathbf{a}) \geq \text{Min}(\mu_g(\mathbf{a}), \mu_h(\mathbf{a})). \tag{4.3}$$

Therefore the map $\mathcal{O}_*(\mathbf{h}) \to \mathbb{Z}_+$ defined by $g \to \mu_g(\mathbf{a})$ is a valuation of the ring $\mathcal{O}(\mathbf{h})$.

Moreover $\mu_{gh}(\mathbf{a}) = \mu_g(\mathbf{a})$ if and only if $h(\mathbf{a}) \neq 0$.

Let ν be a divisor on $\mathbb{D}(\mathbf{h})$. Then $\nu = \mu_f$ where $f \not\equiv 0$ is a meromorphic function

on $\mathbb{D}(\mathbf{h})$. Let g and h be holomorphic functions on $\mathbb{D}(\mathbf{h})$ which are coprime at every

point of $\mathbb{D}(\mathbf{h})$ such that $hf = g$. Then (g,h) is a reduced representation of the

meromorphic map f. Let $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ be another choice. A holomorphic function

$u \in \mathcal{O}^*(\mathbf{h})$ on $\mathbb{D}(\mathbf{h})$ without zeros exists such that $\tilde{f} = uf$. Then (ug,h) is another

reduced representation of $\tilde{f}$. Hence $v \in \mathcal{O}^*(\mathbf{h})$ exists such that $(\tilde{g},\tilde{h}) = (vug, vh)$.

Consequently

$$\nu(\mathbf{a}) = \mu_{\tilde{g}}(\mathbf{a}) - \mu_{\tilde{h}}(\mathbf{a}) = \mu_g(\mathbf{a}) - \mu_h(\mathbf{a}) \in \mathbb{Z} \tag{4.4}$$

is well defined independent of the choice of f, g, and h and is called <u>the</u>

<u>multiplicity</u> <u>of</u> ν <u>at</u> $\mathbf{a}$. The map $\mathfrak{D}(\mathbf{h}) \to \mathbb{Z}$ defined by $\nu \to \nu(\mathbf{a})$ is a homomorphism:

If $\nu_j \in \mathfrak{D}(\mathbf{h})$ and $p \in \mathbb{Z}$, then

$$(p\nu_1 + \nu_2)(\mathbf{a}) = p\nu_1(\mathbf{a}) + \nu_2(\mathbf{a}) \qquad\qquad 0(\mathbf{a}) = 0. \tag{4.5}$$

The divisor ν is non-negative if and only if $\nu(\mathbf{a}) \geq 0$ for all $\mathbf{a} \in \mathbb{D}(\mathbf{h})$. The divisor

ν is the zero divisor if and only if $\nu(\mathbf{a}) = 0$ for all $\mathbf{a} \in \mathbb{D}(\mathbf{h})$. Therefore we can

identify a divisor ν with its multiplicity function $\mathbf{a} \to \nu(\mathbf{a})$ and in fact this is one

way of defining divisors on complex manifolds.

Let f be a meromorphic function on $\mathbb{D}(\mathfrak{h})$. Take $a \in \mathbb{P}_1$. If $f \not\equiv a$, the a-divisor

μ_f^a is defined. Then $\mu_f^a(\mathfrak{x})$ is called the a-multiplicity of f at $\mathfrak{x}$. If $f \not\equiv 0$, the

divisor $\mu_f = \mu_f^0 - \mu_f^\infty$ is defined and $\mu_f(\mathfrak{x})$ is said to be the multiplicity of f at $\mathfrak{x}$.

The function $f \not\equiv 0$ is holomorphic if and only if $\mu_f(\mathfrak{x}) \geq 0$ for all $\mathfrak{x} \in \mathbb{D}(\mathfrak{h})$, and f is

holomorphic and without zeros if and only if $\mu_f(\mathfrak{x}) = 0$ for all $\mathfrak{x} \in \mathbb{D}(\mathfrak{h})$. Also

$\mu_{1/f}(\mathfrak{x}) = -\mu_f(\mathfrak{x})$. If $f_1 \not\equiv 0$ and $f_2 \not\equiv 0$ are meromorphic functions on $\mathbb{D}(\mathfrak{h})$ then

$$\mu_{f_1 f_2}(\mathfrak{x}) = \mu_{f_1}(\mathfrak{x}) + \mu_{f_2}(\mathfrak{x}). \qquad (4.6)$$

The map $\mathcal{K}_*(\mathfrak{h}) \to \mathbb{Z}$ defined by $f \to \mu_f(\mathfrak{x})$ is a homomorphism.

Let ν be a divisor on $\mathbb{D}(\mathfrak{h})$. The closure in $\mathbb{D}(\mathfrak{h})$ of the set of all $\mathfrak{x} \in \mathbb{D}(\mathfrak{h})$ with

$\nu(\mathfrak{x}) \neq 0$ is called the support of ν and denoted by $\mathrm{supp}\,\nu$. If $\nu = 0$ then $\mathrm{supp}\,\nu = \emptyset$.

If $\nu \neq 0$, then the support of ν is an analytic subset of pure dimension $n - 1$ of

$\mathbb{D}(\mathfrak{h})$. Let $\mathcal{R}(\mathrm{supp}\,\nu)$ be the set of regular points of $\mathrm{supp}\,\nu$. Then the function ν is

constant on the connectivity components of $\mathcal{R}(\mathrm{supp}\,\nu)$. If $\nu \geq 0$, then

$\mathrm{supp}\,\nu = \{\mathfrak{x} \in \mathbb{D}(\mathfrak{h}) \mid \nu(\mathfrak{x}) > 0\}$.

Let f be a meromorphic function on $\mathbb{D}(\mathfrak{h})$ with indeterminacy I_f. Take $a \in \mathbb{P}_1$.

Assume that $f \not\equiv a$. Then

$$\mathrm{supp}\,\mu_f^a = I_f \cup \{\mathfrak{x} \in \mathbb{D}(\mathfrak{h}) - I_f \mid f(\mathfrak{x}) = a\}. \qquad (4.7)$$

If $a = 0$, then $\mathrm{supp}\,\mu_f = \mathrm{supp}\,\mu_f^0 \cup \mathrm{supp}\,\mu_f^\infty$.

Take $p \in \mathbb{N}$ and $\mathfrak{r} \in \mathbb{R}^p_+$. Let $\phi : \mathbb{D}(\mathfrak{r}) \to \mathbb{D}(\mathfrak{h})$ be a holomorphic map. Take a

divisor ν on $\mathbb{D}(\mathfrak{h})$ such that $\phi(\mathbb{D}(\mathfrak{r})) \not\subseteq \mathrm{supp}\,\nu$. Let f be a meromorphic function on

$\mathbb{D}(\mathfrak{h})$ such that $\nu = \mu_f$. Then $f \circ \phi \not\equiv 0$ is a meromorphic function on $\mathbb{D}(\mathfrak{r})$. The

pullback divisor

$$\phi^*(\nu) = \mu_{f \circ \phi} \qquad (4.8)$$

is well defined and independent of the choice of f. If $\nu \geq 0$, then $\phi^*(\nu) \geq 0$. If

$p_j \in \mathbb{Z}$ and ν_j are divisors on $\mathbb{D}(\mathfrak{h})$ with $\phi(\mathbb{D}(\mathfrak{r})) \not\subseteq \mathrm{supp}\,\nu_j$ for $j = 1, 2$, then

$\phi(\mathbb{D}(r)) \nsubseteq \text{supp}(p_1 \nu_1 + p_2 \nu_2)$ and

$$\phi^*(p_1 \nu_1 + p_2 \nu_2) = p_1 \phi^*(\nu_1) + p_2 \phi^*(\nu_2). \qquad (4.9)$$

For $0 \neq z \in \mathbb{C}^n$ an injective linear map $\ell_z : \mathbb{C} \to \mathbb{C}^n$ is defined by $\ell_z(u) = uz$ for all $u \in \mathbb{C}$. If $0 \neq z \in \mathbb{D}\langle r \rangle$ and $0 \leq r < \mathfrak{h}$, a largest number $t_0(r) > 1$ exists such that $\ell_z(\mathbb{D}(t_0(r))) \subseteq \mathbb{D}(\mathfrak{h})$. Let ν be a divisor on $\mathbb{D}(\mathfrak{h})$. Take $z \in \mathbb{D}\langle r \rangle$, then ν is said to be <u>restrictable to</u> z, if and only if $\ell_z(\mathbb{D}(t_0(r))) \nsubseteq \text{supp } \nu$. Let $R_\nu(r)$ be the set of all $z \in \mathbb{D}\langle r \rangle$ such that ν is not restrictable to z. Then $R_\nu(r)$ is a thin real analytic subset of $\mathbb{D}\langle r \rangle$. For $z \in \mathbb{D}\langle r \rangle - R_\nu(r)$, the pullback divisor $\nu[z] = \ell_z^*(\nu)$ is defined. If $u \in \mathbb{C}(t_0(r))$, abbreviate $\nu[z,u] = \nu[z](u)$. For $z \in \mathbb{D}\langle r \rangle - R_\nu(r)$ and $0 \leq t < t_0(r)$ define <u>the counting function of</u> ν <u>in the direction</u> z as the finite sum

$$n_\nu[z,t] = \sum_{u \in \mathbb{D}[t]} \nu[z,u]. \qquad (4.10)$$

The function $t \to n_\nu[z,t]$ is semi-continuous from the right and is of bounded variation on each compact subinterval of $\mathbb{R}[0,t_0)$. There is a thin analytic cone C_ν with vertex 0 in $\mathbb{C}^n$ such that

$$\lim_{0 < t \to 0} n_\nu[z,t] = n_\nu[0,t] = \nu(0) \qquad (4.11)$$

for all $0 \neq z \in \mathbb{D}(\mathfrak{h}) - C_\nu$. Here $\mathbb{D}\langle r \rangle \cap C_\nu$ is a thin real analytic subset of $\mathbb{D}\langle r \rangle$.

If $0 \neq \lambda \in \mathbb{D}(t_0(r))$ and $z \in \mathbb{D}\langle r \rangle - R_\nu(r)$, then $\lambda z \in \mathbb{D}\langle |\lambda| r \rangle - R_\nu(|\lambda| r)$ and

$$n_\nu[\lambda z, t] = n_\nu[z, |\lambda| t]. \qquad (4.12)$$

If $\nu \geq 0$, the function $t \to n_\nu[z,t]$ is non-negative and increases.

If ν_j are divisors on $\mathbb{D}(\mathfrak{h})$ restrictable to $z \in \mathbb{D}\langle r \rangle$, and if $p_j \in \mathbb{Z}$ for $j = 1, 2$, then $p_1 \nu_1 + p_2 \nu_2$ is restrictable to z and

$$n_{p_1 \nu_1 + p_2 \nu_2}[z,t] = p_1 n_{\nu_1}[z,t] + p_2 n_{\nu_2}[z,t]. \tag{4.13}$$

Let $f \not\equiv 0$ be a meromorphic function on $\mathbb{D}(\mathfrak{h})$ such that $\nu = \mu_f$. Take

$z \in \mathbb{D}\langle r \rangle - R_\nu(r)$. Then $\nu[z] = \mu_{f \circ \ell_z}$. For $0 < q < r < t_0(r)$ the Jensen formula and

the definition of the valence function for $\nu[z]$ imply

$$N_{\nu[z]}(r,q) = \int_{\mathbb{D}\langle r \rangle} \log|f \circ \ell_z| \Omega_1 - \int_{\mathbb{D}\langle q \rangle} \log|f \circ \ell_z| \Omega_1 = \int_q^r n_\nu[z,t] \frac{dt}{t} . \tag{4.14}$$

The function $z \to N_{\nu[z]}(r,q)$ is continuous on $\mathbb{D}\langle r \rangle - C_\nu$, hence measurable on $\mathbb{D}\langle r \rangle$.

Also

$$\frac{N_{\nu[z]}(\theta r, r)}{\log \theta} \to n_\nu[z,r] \quad \text{for } 1 < \theta \to 1 \tag{4.15}$$

implies that $n_\nu[z,r]$ is a measurable function of z on $\mathbb{D}(r)$. By (2.6) $N_{\nu[z]}(\theta r, r)$ is

integrable over $\mathbb{D}\langle r \rangle$ if $r < \theta r < t_0(r)$. If $\nu \geq 0$ then

$$n_\nu[z,r] \leq \frac{N_{\nu[z]}(\theta r, r)}{\log \theta} \quad \text{for } 1 < \theta < \frac{t_0(r)}{r} . \tag{4.16}$$

Hence $n_\nu[z,r]$ is integrable over $\mathbb{D}\langle r \rangle$. Since every divisor is the difference of two

non-negative divisors, $n_\nu[z,r]$ is integrable over $\mathbb{D}(\mathfrak{h})$ for all $\nu \in \mathfrak{D}(\mathfrak{h})$. The

<u>counting function</u> $n_\nu(r)$ of ν is defined by

$$n_\nu(r) = \int_{z \in \mathbb{D}\langle r \rangle} n_\nu[z,1] \Omega_n(z). \tag{4.17}$$

Then

$$n_\nu(tr) = \int_{z \in \mathbb{D}\langle r \rangle} n_\nu[z,t] \Omega_n(z). \tag{4.18}$$

If $0 < q < r < t_0(r)$, (2.6) implies

$$N_\nu(r\mathfrak{r},q\mathfrak{r}) = \int_{\mathfrak{z}\in\mathbb{D}\langle\mathfrak{r}\rangle} N_{\nu[\mathfrak{z}]}(r,q)\Omega_n(\mathfrak{z}) \tag{4.19}$$

$$= \int_{\mathfrak{z}\in\mathbb{D}\langle\mathfrak{r}\rangle} \int_q^r n_\nu[\mathfrak{z},t]\,\frac{dt}{t}\,\Omega_n(\mathfrak{z}) = \int_q^r n_\nu(t\mathfrak{r})\,\frac{dt}{t}\,.$$

If $p_j \in \mathbb{Z}$ and $\nu_j \in \mathfrak{D}(\mathfrak{h})$ for $j = 1, 2$, then

$$n_{p_1\nu_1+p_2\nu_2}(\mathfrak{r}) = p_1 n_{\nu_1}(\mathfrak{r}) + p_2 n_{\nu_2}(\mathfrak{r}). \tag{4.20}$$

If $\nu \geq 0$, then $n_\nu(\mathfrak{r}) \geq 0$ and $n_\nu(t\mathfrak{r})$ increases in t.

PROPOSITION 4.1. Take $\mathfrak{h} \in \overset{+}{\mathbb{R}}{}^n$. Let ν be a divisor on $\mathbb{D}(\mathfrak{h})$ such that

supp $\nu \cap \mathbb{D}\langle t\mathfrak{h}\rangle = \emptyset$ for all $t \in \mathbb{R}[0,1)$. Then ν is the zero divisor.

PROOF. W.l.o.g. we can assume that $\nu \geq 0$. Then $0 \notin$ supp ν. If $0 < r < 1$,

then $\nu[r\mathfrak{h}] = 0$. Hence $n_\nu[r\mathfrak{h},t] = 0$ for $0 \leq t \leq 1$. By (4.17) we have $n_\nu(r\mathfrak{h}) = 0$ for

all $0 < r < 1$. Since $0 \notin$ supp ν, we are permitted to take $q = 0$ in (4.19) and

obtain $N_\nu(r\mathfrak{h},0) = 0$ for all $0 \leq r < 1$.

Assume that there exists $\mathfrak{z}_0 \in$ supp ν such that $r = |\mathfrak{z}_0| > 0$. Then $\mathfrak{r} < \mathfrak{h}$. Take

$1 < p < q$ such that $q\mathfrak{r} < \mathfrak{h}$ and such that $u\mathfrak{z}_0 \notin$ supp ν if $p \leq |u| \leq q$. An open

connected neighborhood U of $\mathfrak{z}_0$ in $\mathbb{D}\langle\mathfrak{r}\rangle$ exists such that $u\mathfrak{z} \notin$ supp ν for all $u \in \mathbb{C}$

with $p \leq |u| \leq q$ and $\mathfrak{z} \in U$. Rouche's theorem implies

$$n_\nu[\mathfrak{z},t] = n_\nu[\mathfrak{z}_0,t] \geq 1$$

for all $\mathfrak{z} \in U$ and $t \in \mathbb{R}[p,q]$. Hence $n_\nu(t\mathfrak{r}) > 0$ for $p \leq t \leq q$, which implies

$N_\nu(q\mathfrak{r},0) > 0$ by (4.19). A number $s \in \mathbb{R}(0,1)$ exists such that $q\mathfrak{r} < s\mathfrak{h}$. Then

$$0 < N_\nu(q\mathfrak{r},0) \leq N_\nu(s\mathfrak{h},0) = 0$$

which is a contradiction. Hence $\mathfrak{z} \in$ supp ν at most if a coordinate of $\mathfrak{z}$ is zero.

Since $0 \notin$ supp ν we conclude supp $\nu = \emptyset$; q.e.d.

By the same procedures as in Lemma 10.4 and Lemma 10.5 of [2], the following

result can be proved:

PROPOSITION 4.2. Take $\mathfrak{h} \in \mathbb{R}^n_+$ and $\mathfrak{r} \in \mathbb{R}^n_+(\mathfrak{h})$. Let $\nu \not\equiv 0$ be a divisor on $\mathbb{D}(\mathfrak{h})$.

Then $A = \text{supp } \nu$ is an analytic subset of pure dimension $n - 1$ in $\mathbb{D}(\mathfrak{h})$. Define

$A_* = A - \{0\}$. Let A^0 be the set of all $\mathfrak{z} \in A_*$ such that the restriction

$\mathbb{P} : A_* \to \mathbb{P}_{n-1}$ is locally biholomorphic. Then A^0 is either empty or a complex

manifold of dimension $n - 1$. The complement $A_* - A^0$ is analytic in A_*. Also

$A^0 \cap \Delta(\mathfrak{r})$ is either empty or an $(n - 1)$-dimensional real analytic manifold which

can be oriented such that ξ_n defines a positive measure on $A^0 \cap \Delta(\mathfrak{r})$. We have

$$n_\nu(\mathfrak{r}) = \int_{A^0 \cap \Delta(\mathfrak{r})} \nu \xi_n + \nu(0). \qquad (4.21)$$

Take $\mathfrak{h} \in \mathbb{R}^n_+$ and let ν be a divisor on $\mathbb{D}(\mathfrak{h})$. Take $p \in \mathbb{N}[1,n)$ and define

$\tilde{p} = n - p$. Take $\tau \in \mathfrak{C}(p,n)$. Then $\tilde{\tau} \in \mathfrak{C}(\tilde{p},n)$. Take any $\mathfrak{z} \in \mathbb{D}(\pi_{\tilde{\tau}}(\mathfrak{h}))$. Then

$\tau_{\mathfrak{z}} : \mathbb{D}(\pi_\tau(\mathfrak{h})) \to \mathbb{D}(\mathfrak{h})$ is an injective holomorphic map. We say that ν is

restrictable to $\mathfrak{z}$ by τ if

$$\tau_{\mathfrak{z}}(\mathbb{D}(\pi_\tau(\mathfrak{h}))) \not\subseteq \text{supp } \nu. \qquad (4.22)$$

Let $R_\nu(\tau)$ be the set of all $\mathfrak{z} \in \mathbb{D}(\pi_{\tilde{\tau}}(\mathfrak{h}))$ such that ν is not restrictable to $\mathfrak{z}$ by τ.

If $\mathfrak{z} \in \mathbb{D}(\pi_{\tilde{\tau}}(\mathfrak{h})) - R_\nu(\tau)$, the pullback divisor

$$\nu[\tau,\mathfrak{z}] = \tau_{\mathfrak{z}}^*(\nu) \qquad (4.23)$$

exists.

PROPOSITION 4.3. Take $\mathfrak{r} \in \mathbb{R}^n_+(\mathfrak{h})$ and $\mathfrak{q} \in \mathbb{R}^n_+(\mathfrak{h})$. Assume that $\mathfrak{p} = \pi_{\tilde{\tau}}(\mathfrak{r}) = \pi_{\tilde{\tau}}(\mathfrak{q})$.
Then

$$N_\nu(\mathfrak{r},\mathfrak{q}) = \int_{\mathfrak{z} \in \mathbb{D}\langle \mathfrak{p} \rangle} N_{\nu[\tau,\mathfrak{z}]}(\pi_\tau(\mathfrak{r}),\pi_\tau(\mathfrak{q}))\Omega_{\tilde{p}}(\mathfrak{z}). \qquad (4.24)$$

PROOF. A meromorphic function $f \not\equiv 0$ exists on $\mathbb{D}(\mathfrak{h})$ such that $\mu_f = \nu$. By (2.22)

we have

$$N_\nu(\mathfrak{r},\mathfrak{q}) = \int_{\mathbb{D}\langle \mathfrak{r}\rangle} \log|f|\Omega_n - \int_{\mathbb{D}\langle \mathfrak{q}\rangle} \log|f|\Omega_n = \int_{\mathfrak{z}\in\mathbb{D}\langle \mathfrak{p}\rangle}\left(\int_{\mathbb{D}\langle \pi_\tau(\mathfrak{r})\rangle} \log|f\circ\tau_\mathfrak{z}|\Omega_p\right)\Omega_{\widetilde{p}}(\mathfrak{z})$$

$$- \int_{\mathfrak{z}\in\mathbb{D}\langle \mathfrak{p}\rangle}\left(\int_{\mathbb{D}\langle \pi_\tau(\mathfrak{q})\rangle} \log|f\circ\tau_\mathfrak{z}|\Omega_p\right)\Omega_{\widetilde{p}}(\mathfrak{z}) = \int_{\mathfrak{z}\in\mathbb{D}\langle \mathfrak{p}\rangle} N_{\nu[\tau,\mathfrak{z}]}(\pi_\tau(\mathfrak{r}),\pi_\tau(\mathfrak{q}))\Omega_{\widetilde{p}}(\mathfrak{z})$$

If $\pi_\tau(\mathfrak{q}) = q\pi_\tau(\mathfrak{r})$ with $0 < q < 1$, then (4.19) implies

$$N_{\nu[\tau,\mathfrak{z}]}(\pi_\tau(\mathfrak{r}),q\pi_\tau(\mathfrak{r})) = \int_q^1 n_{\nu[\tau,\mathfrak{z}]}(t\pi_\tau(\mathfrak{r}))\,\frac{dt}{t}\ . \tag{4.25}$$

For $\mathfrak{x}\in\pi_\tau(\mathfrak{h})$ define

$$n_{\nu,\tau}(\mathfrak{p},\mathfrak{x}) = \int_{\mathfrak{z}\in\mathbb{D}\langle \mathfrak{p}\rangle} n_{\nu[\tau,\mathfrak{z}]}(\mathfrak{x})\Omega_{\widetilde{p}}. \tag{4.26}$$

We have

$$N_\nu(\mathfrak{r},\mathfrak{q}) = \int_q^1 n_{\nu,\tau}(\mathfrak{p},t\pi_\tau(\mathfrak{r}))\,\frac{dt}{t} \tag{4.27}$$

provided $\pi_{\widetilde{\tau}}(\mathfrak{r}) = \pi_{\widetilde{\tau}}(\mathfrak{q}) = \mathfrak{p}$ and $\pi_\tau(\mathfrak{q}) = q\pi_\tau(\mathfrak{r})$ with $0 < q < 1$.

If $\widetilde{\tau}_0(\mathbb{D}(\pi_{\widetilde{\tau}}(\mathfrak{h}))) \not\subseteq \operatorname{supp}\nu$, we are permitted to use $q = 0$

$$N_\nu(\mathfrak{r},\mathfrak{q}) = \int_0^1 n_{\nu,\tau}(\mathfrak{p},t\pi_\tau(\mathfrak{r}))\,\frac{dt}{t}\ . \tag{4.28}$$

If we abbreviate $\mu = \widetilde{\tau}_0^*(\nu)$, then we have $N_\nu(\mathfrak{r},\mathfrak{q}) = N_\nu(\mathfrak{r},0) - N_\mu(\mathfrak{p},0)$ which leads us

to a generalization of formula 4.2.3 by Ronkin [1]

$$N_\nu(\mathfrak{r},0) - N_\mu(\mathfrak{p},0) = \int_0^1 n_{\nu,\tau}(\mathfrak{p},t\pi_\tau(\mathfrak{r}))\,\frac{dt}{t}\ . \tag{4.29}$$

Take $p = 1$ and identify $\tau : \{1\} \to \mathbb{N}[1,n]$ with $\tau(1) \in \mathbb{N}[1,n]$. Then $\pi_\tau(\mathfrak{r}) = r_\tau$

is the τ^{st} coordinate and $q_\tau = qr_\tau$. Hence

$$\int_q^1 n_{\nu,\tau}(\mathfrak{p},t\pi_\tau(\mathfrak{r}))\,\frac{dt}{t} = \int_{q_\tau}^{r_\tau} n_{\nu,\tau}(\mathfrak{p},t)\,\frac{dt}{t}\ . \tag{4.30}$$

If $\mathfrak{r} = (r_1,\ldots,r_n)$ and $\mathfrak{q} = (r_1,\ldots,r_{\tau-1},q_\tau,r_{\tau+1},\ldots,r_n)$, then

$$N_\nu(\mathfrak{r},\mathfrak{q}) = \int_{q_\tau}^{r_\tau} n_{\nu,\tau}(\mathfrak{p},t) \frac{dt}{t} \tag{4.31}$$

where $\mathfrak{p} = (r_1,\ldots,r_{\tau-1},r_{\tau+1},\ldots,r_n)$ and

$$n_{\nu,\tau}(\mathfrak{p},t) = \int_{\mathfrak{z}\in\mathbb{D}\langle\mathfrak{p}\rangle} n_{\nu[\tau,\mathfrak{z}]}(t)\Omega_{n-1}(\mathfrak{z}) \tag{4.32}$$

$$n_{\nu[\tau,\mathfrak{z}]}(t) = \sum_{u\in\mathbb{D}[t]} \nu[\tau,\mathfrak{z},u]. \tag{4.33}$$

If $0 < \mathfrak{q} < \mathfrak{r} < \mathfrak{h}$ are arbitrarily picked, we can represent $N_\nu(\mathfrak{r},\mathfrak{q})$ as an integral

sum. Define

$$\mathfrak{r}_\tau = (q_1,\ldots,q_\tau,r_{\tau+1},\ldots,r_n) \tag{4.34}$$

$$\mathfrak{p}_\tau = (q_1,\ldots,q_{\tau-1},r_{\tau+1},\ldots,r_n). \tag{4.35}$$

Then $\mathfrak{r}_0 = \mathfrak{r}$ and $\mathfrak{r}_n = \mathfrak{q}$. Also $\pi_{\tilde\tau}(\mathfrak{r}_\tau) = \mathfrak{p}_\tau = \pi_{\tilde\tau}(\mathfrak{r}_{\tau-1})$. Hence

$$N_\nu(\mathfrak{r}_{\tau-1},\mathfrak{r}_\tau) = \int_{q_\tau}^{r_\tau} n_{\nu,\tau}(\mathfrak{p}_\tau,t) \frac{dt}{t}. \tag{4.36}$$

By addition

$$N_\nu(\mathfrak{r},\mathfrak{q}) = \sum_{\tau=1}^{n} \int_{q_\tau}^{r_\tau} n_{\nu,\tau}(\mathfrak{p}_\tau,t) \frac{dt}{t}. \tag{4.37}$$

5. THE SPHERICAL IMAGE AND THE CHARACTERISTIC.

First let us consider the 1-dimensional case. Take $b \in \mathbb{R}^+$. Let V be a complex

vector space of dimension $m + 1$ with a hermitian product. Let ω be the associated

Fubini-Study form on $\mathbb{P}(V)$. Take a meromorphic map $f : \mathbb{D}(b) \to \mathbb{P}(V)$. Since $\mathbb{D}(b)$

has dimension 1, the map f is holomorphic. The spherical image of f is defined by

$$A_f(t) = \int_{\mathbb{D}(t)} f^*(\omega) \quad \forall\ t \in \mathbb{R}[0,b). \tag{5.1}$$

The function A_f increases and is non-negative. Also $A_f(t) > 0$ if and only if f is

not constant and $0 < t < b$. If $0 \le q \le r < b$, the characteristic is given by

$$T_f(r,q) = \int_q^r A_f(t) \frac{dt}{t} . \tag{5.2}$$

The identity (5.2) is easily derived from the definition (3.28) in applying Stokes

Theorem twice.

Take $\mathfrak{h} \in \mathbb{R}^{n}_{+}$. Let $f : \mathbb{D}(\mathfrak{h}) \to \mathbb{P}(V)$ be a meromorphic map. Let I_f be the

indeterminacy of f. Let $\mathfrak{u} : \mathbb{D}(\mathfrak{h}) \to V$ be a reduced representation. If M is a

connected complex manifold and if $\phi : M \to \mathbb{D}(\mathfrak{h})$ is a holomorphic map such that

$\phi(M) \not\subseteq I_f$, then $f \circ \phi$ is a meromorphic function on M, and $\mathfrak{u} \circ \phi$ is a representation

which may not be reduced. In order to counterbalance this effect, we have to

introduce an indeterminacy multiplicity $\iota_f(\mathfrak{a})$ for each $\mathfrak{a} \in \mathbb{D}(\mathfrak{h})$. There is an integer

$\nu \ge 0$ and for each integer $j \ge \nu$ a homogeneous vector polynomial $\mathfrak{u}_j : \mathbb{C}^n \to \mathbb{C}$ of

degree j such that $\mathfrak{u}_\nu \not\equiv 0$ and such that

$$\mathfrak{u}(\mathfrak{z}) = \sum_{j=\nu}^{\infty} \mathfrak{u}_j(\mathfrak{z} - \mathfrak{a}) \tag{5.3}$$

for all $\mathfrak{z}$ in an open neighborhood U of $\mathfrak{a}$ in $\mathbb{D}(\mathfrak{h})$. Define $\iota_f(\mathfrak{a}) = \nu$. Obviously

$\iota_f(\mathfrak{a}) > 0$ if and only if $\mathfrak{a} \in I_f$. If $G \ne \emptyset$ is an open, connected subset of $\mathbb{C}$, if

$\phi : G \to \mathbb{D}(\mathfrak{h})$ is holomorphic such that $\phi(G) \not\subseteq I_f$, then $\mathfrak{u} \circ \phi \not\equiv 0$ and $\mu_{\mathfrak{u} \circ \phi}(x) \ge \iota_f(\phi(x))$.

Take $\mathfrak{r} \in \mathbb{R}^{n}_{+}(\mathfrak{h})$ and $\mathfrak{z} \in \mathbb{D}\langle \mathfrak{r} \rangle$. The meromorphic map f is said to be restrictable

to $\mathfrak{z}$ if and only if $\ell_{\mathfrak{z}}(\mathbb{D}(t_0(\mathfrak{r}))) \not\subseteq I_f$. The set $R_f(\mathfrak{r})$ of all $\mathfrak{z} \in \mathbb{D}\langle \mathfrak{r} \rangle$ such that f is

not restrictable to $\mathfrak{z}$, is a thin real analytic subset of $\mathbb{D}\langle \mathfrak{r} \rangle$. For $\mathfrak{z} \in \mathbb{D}\langle \mathfrak{r} \rangle - R_f(\mathfrak{r})$,
a meromorphic map

$$f[\mathfrak{z}] = f \circ \ell_{\mathfrak{z}} : \mathbb{D}(t_0(\mathfrak{r})) \to \mathbb{P}(V) \tag{5.4}$$

is defined. For $0 \le t < t_0(\mathfrak{r})$ the spherical image of f in the direction $\mathfrak{z}$ is

defined by

$$A_f[z,t] = A_{f[z]}(t) + \iota_f(0) = \int_{\mathbb{D}(t)} f[z]*(\omega) + \iota_f(0) \geq 0. \tag{5.5}$$

If $0 \neq ut \in \mathbb{D}(t_0(r))$, then $uz \in \mathbb{D}\langle |u|r\rangle - R_f(|u|r)$ and

$$A_f[uz,t] = A_f[z,|u|t]. \tag{5.6}$$

The function $t \rightarrow A_f[z,t]$ increases. For $0 < q < r < t_0(r)$ define the <u>characteristic</u>

<u>of f in the direction</u> z by

$$T_f[z,r,q] = T_{f[z]}(r,q) + \iota_f(0) \log \frac{r}{q} = \int_q^r A_f[z,t] \frac{dt}{t} \geq 0. \tag{5.7}$$

Let $\mathfrak{v} : \mathbb{D}(\mathfrak{h}) \rightarrow V$ be a reduced representation of f. For $z \in \mathbb{D}\langle r\rangle - R_f(r)$ the

vector function $\mathfrak{v} \circ \ell_z \not\equiv 0$ is a representation of $f[z]$ on $\mathbb{D}(t_0(r))$. Let $\mu_{\mathfrak{v} \circ \ell_z}$ be the

divisor of $\mathfrak{v} \circ \ell_z$ on $\mathbb{C}$. Define the divisor $\mu_{\mathfrak{v}}[z] \geq 0$ on $\mathbb{D}(t_0(r))$ by

$$\mu_{\mathfrak{v}}[z,u] = \begin{cases} \mu_{\mathfrak{v} \circ \ell_z}(u) \geq 0 & \text{if } 0 \neq u \in \mathbb{D}(t_0(r)) \\[2mm] \mu_{\mathfrak{v} \circ \ell_z}(0) - \iota_f(0) \geq 0 & \text{if } u = 0. \end{cases} \tag{5.8}$$

If $0 \neq u \in \mathbb{D}(t_0(r))$, then $\mu_{\mathfrak{v}}[z,u] > 0$ if and only if $\ell_z(u) = uz \in I_f$. If $u = 0$,

then $\mu_{\mathfrak{v}}[z,u] > 0$ if in (5.3) with $\mathfrak{a} = 0$ we have $\mathfrak{v}_\nu(z) = 0$. Since $\dim I_f \leq n - 2$ and

$\mathfrak{v}_\nu \not\equiv 0$, we see that $\mu_{\mathfrak{v}} \equiv 0$ for almost all $z \in \mathbb{D}\langle r\rangle - R_f(r)$. By (3.31) and (5.7)

we obtain

$$T_f[z,r,q] = \int_{\mathbb{D}\langle r\rangle} \log\|\mathfrak{v} \circ \ell_z\|\Omega_1 - \int_{\mathbb{D}\langle q\rangle} \log\|\mathfrak{v} \circ \ell_z\|\Omega_1 - N_{\mu_{\mathfrak{v}}[z]}(r,q). \tag{5.9}$$

Since $N_{\mu_{\mathfrak{v}}[z]} = 0$ for almost all z in $\mathbb{D}\langle r\rangle$, the function $T_f[z,r,q]$ is integrable over

$\mathbb{D}\langle r\rangle$ and (2.6), (3.27) and (5.9) imply

$$T_f(rr,qq) = \int_{z \in \mathbb{D}\langle r\rangle} T_f[z,r,q]\Omega_n(z). \tag{5.10}$$

If $z \in \mathbb{D}\langle r \rangle - R_f(r)$ and if $0 < t < \theta t < t_0(r)$, then

$$A_f[z,t] \leq \frac{T_f[z,\theta t,t]}{\log \theta} \to A_f[z,t] \quad \text{for } \theta \to 1. \tag{5.11}$$

Therefore $A_f[z,t]$ is integrable over $\mathbb{D}\langle r \rangle$. The _spherical_ _image_ _of_ f is defined by

$$A_f(r) = \int_{z \in \mathbb{D}\langle r \rangle} A_f[z,1]\Omega_n. \tag{5.12}$$

The identity (5.6) implies

$$A_f(tr) = \int_{z \in \mathbb{D}\langle r \rangle} A_f[z,t]\Omega_n \quad \text{for } 0 \leq t < t_0(r). \tag{5.13}$$

From (5.7), (5.10) and (5.13) we obtain

$$T_f(rr,qr) = \int_q^r A_f(tr) \frac{dt}{t}. \tag{5.14}$$

If $0 < q < r < \infty$ and $x \in \mathbb{R}^+$ define

$$L(x,r,q) = \log^+ \frac{r}{x} - \log^+ \frac{q}{x} = \begin{cases} 0 & \text{if } x \geq r \\[2mm] \log \dfrac{r}{x} & \text{if } q \leq x \leq r \\[2mm] \log \dfrac{r}{q} & \text{if } 0 < x \leq q. \end{cases} \tag{5.15}$$

PROPOSITION 5.1. If $r \in \mathbb{R}^n_+(\mathfrak{h})$ and $0 < q < r < t_0(r)$, then

$$A_f(r) = \int_{\Delta(r)} f*(\omega) \wedge \xi_n + \imath_f(0) \tag{5.16}$$

$$T_f(rr,qr) = \int_{z \in \Delta(r)} L\left(\frac{\|z\|}{\|r\|},r,q\right) f*(\omega) \wedge \xi_n + \imath_f(0) \log \frac{r}{q}. \tag{5.17}$$

PROOF. If $z \in \mathbb{D}\langle r \rangle - R_f(r)$, then

$$f[z]*(\omega) = (f \circ \ell_z)*(\omega) = \ell_z^*(f*(\omega)).$$

Therefore Lemma 2.1 implies

$$A_f(r) = \int_{z \in \mathbb{D}\langle r \rangle} \left(\int_{\mathbb{D}} f[z]*(\omega) + \iota_f(0) \right) \Omega_n(z)$$

$$= \int_{z \in \mathbb{D}\langle r \rangle} \int_{\mathbb{D}} \ell_z^*(f*(\omega))\Omega_n(z) + \iota_f(0) = \int_{\Delta(r)} f*(\omega) \wedge \xi_n + \iota_f(0).$$

If $z \in \Delta(t_0(r)r)$, then $|z| = u(z)r$ with $0 < u(z) < t_0(r)$. Moreover

$\|z\| = u(z)\|r\|$. If $0 < t < t_0(r)$ then $z \in \Delta(tr)$ if and only if $0 < u(z) < t$.

Define $\chi(x,t) = 1$ if $q \leq x \leq t \leq r$ and $\chi(x,t) = 0$ if $q \leq t < x \leq r$. Then

$$T_f(rr,qr) = \int_q^r \left(\int_{\Delta(tr)} f*(\omega) \wedge \xi_n + \iota_f(0) \right) \frac{dt}{t}$$

$$= \int_q^r \left(\int_{\Delta(rr)-\Delta(qr)} \chi(u(z),t) f*(\omega) \wedge \xi_n \right) \frac{dt}{t} + A_f(qr) \log \frac{r}{q}$$

$$= \int_{\Delta(rr)-\Delta(qr)} \int_q^r \chi(u(z),t) \frac{dt}{t} f*(\omega) \wedge \xi_n + A_f(qr) \log \frac{r}{q}$$

$$= \int_{\Delta(rr)-\Delta(qr)} \log \frac{r}{u(z)} f*(\omega) \wedge \xi_n + \int_{\Delta(qr)} \log \frac{r}{q} f*(\omega) \wedge \xi_n + \iota_f(0) \log \frac{r}{q}$$

$$= \int_{\Delta(rr)} L(u(z),r,q) f*(\omega) \wedge \xi_n + \iota_f(0) \log \frac{r}{q} \qquad \text{q.e.d.}$$

Take $a \in \mathbb{P}(V*)$ and assume that f is not linearly degenerate for a. Take

$r \in \mathbb{R}^n_+(\mathfrak{h})$ and $z \in \mathbb{D}\langle r \rangle - R_f(r)$. If μ_f^a is restrictable to z, then

$$m_{f[z]}(r,a) = \int_{\mathbb{D}\langle r \rangle} \log \frac{1}{\|f \circ \ell_z, a\|} \Omega_1 \qquad (5.18)$$

is defined for $0 < r < t_0(r)$. If $0 \neq u \in \mathbb{C}$ with $|u|t < t_0(r)$, then

$$m_{f[uz]}(r,a) = m_{f[z]}(|u|r,a). \qquad (5.19)$$

Also (2.6) implies

$$m_f(r,a) = \int_{\mathbb{D}\langle r \rangle} \log \frac{1}{\|f,a\|} \Omega_n = \int_{z \in \mathbb{D}\langle r \rangle} \int_{\mathbb{D}\langle 1 \rangle} \log \frac{1}{\|f \circ \ell_z, a\|} \Omega_1 \Omega_n(z)$$

$$m_f(r,a) = \int\limits_{\mathbf{z} \in \mathbb{D}\langle r\rangle} m_{f[\mathbf{z}]}(1,a)\Omega_n(\mathbf{z}) \qquad (5.20)$$

$$m_f(rr,a) = \int\limits_{\mathbf{z} \in \mathbb{D}\langle r\rangle} m_{f[\mathbf{z}]}(r,a)\Omega_n(\mathbf{z}). \qquad (5.21)$$

Now, we will slice the characteristic parallel to the coordinate planes. Here we encounter the difficulty, that the pullback of a reduced representation of a meromorphic map may not be a reduced representation anymore. We will show that the problem does not occur for "almost all slices". Some preparations are needed. For the technical apparatus we refer to Andreotti-Stoll [11] §1.

LEMMA 5.2. Take $\mathbf{h} \in \overset{+}{\mathbb{R}}^n$ and $r \in \overset{+}{\mathbb{R}}^n(\mathbf{h})$. Let A be an almost thin subset of $\mathbb{D}(\mathbf{h})$. Then $A \cap \mathbb{D}\langle r\rangle$ has measure zero on $\mathbb{D}\langle r\rangle$.

PROOF. By definition there is a sequence $\{A_\nu\}_{\nu \in \mathbb{N}}$ of thin subsets of $\mathbb{D}(\mathbf{h})$ such that $A = \overset{\infty}{\underset{\nu=1}{\cup}} A_\nu$. For each $p \in A_\nu$ there exists an open neighborhood $U_\nu(p)$ of p in $\mathbb{D}(\mathbf{h})$ and analytic subset $B_\nu(p)$ of $U_\nu(p)$ such that $A_\nu \cap U_\nu(p) \subseteq B_\nu(p)$ and such that $\dim B_\nu(p) < n$. Hence $B_\nu(p) \cap \mathbb{D}\langle r\rangle$ has measure zero on $\mathbb{D}\langle r\rangle$. Consequently, $A_\nu \cap U_\nu(p) \cap \mathbb{D}\langle r\rangle$ has measure zero on $\mathbb{D}\langle r\rangle$. A countable union of these neighborhoods $U_\nu(p)$ covers A_ν. Hence $A_\nu \cap \mathbb{D}\langle r\rangle$ has measure zero on $\mathbb{D}\langle r\rangle$ for each $\nu \in \mathbb{N}$. Therefore $A \cap \mathbb{D}\langle r\rangle$ has measure zero on $\mathbb{D}\langle r\rangle$; q.e.d.

Take $\mathbf{h} \in \overset{+}{\mathbb{R}}^n$. Let $f : \mathbb{D}(\mathbf{h}) \to \mathbb{P}(V)$ be a meromorphic map. Let $\mathbf{u} : \mathbb{D}(\mathbf{h}) \to V$ be a reduced representation of f. Take $p \in \mathbb{N}[1,n]$. Then $\tilde{p} = n - p \in \mathbb{N}[1,n]$. Take $\tau \in \mathfrak{T}(p,n)$. Then $\tilde{\tau} \in \mathfrak{T}(\tilde{p},n)$. We have the surjective projection

$$\pi_{\tilde{\tau}} : \mathbb{D}(\mathbf{h}) \to \mathbb{D}(\pi_{\tilde{\tau}}(\mathbf{h})). \qquad (5.22)$$

For $\mathbf{z} \in \mathbb{D}(\pi_{\tilde{\tau}}(\mathbf{h}))$ we have an injective holomorphic map

such that
$$\tau_{\mathbf{z}} : \mathbb{D}(\pi_\tau(\mathbf{h})) \to \mathbb{D}(\mathbf{h}) \qquad (5.23)$$

$$\tau_{\mathbf{z}} : \mathbb{D}(\pi_\tau(\mathbf{h})) \to \pi_{\tilde{\tau}}^{-1}(\mathbf{z}) \qquad (5.24)$$

is biholomorphic. Define

$$R_f(\tau) = \{ z \in \mathbb{D}(\pi_{\widetilde{\tau}}(\mathfrak{h})) \mid \tau_z^{-1}(I_f) = \mathbb{D}(\pi_\tau(\mathfrak{h})) \} \tag{5.25}$$

$$S_f(\tau) = \{ z \in \mathbb{D}(\pi_{\widetilde{\tau}}(\mathfrak{h})) \mid \dim \tau_z^{-1}(I_f) \geq p - 1 \}. \tag{5.26}$$

Then $R_f(\tau) \subseteq S_f(\tau)$. If $z \in \mathbb{D}(\pi_{\widetilde{\tau}}(\mathfrak{h})) - R_f(\tau)$, the meromorphic map f restricts to a

meromorphic map

$$f[\tau, z] = f \circ \tau_z : \mathbb{D}(\pi_\tau(\mathfrak{h})) \to \mathbb{P}(V). \tag{5.27}$$

If $z \in \mathbb{D}(\pi_{\widetilde{\tau}}(\mathfrak{h})) - S_f(\tau)$, the reduced representation $\mathfrak{v} : \mathbb{D}(\mathfrak{h}) \to V$ restricts to a

reduced representation $\mathfrak{v} \circ \tau_z : \mathbb{D}(\pi_\tau(\mathfrak{h})) \to V$ of $f[\tau, z]$. By (5.24) we have

$$S_f(\tau) = \{ z \in \mathbb{D}(\pi_{\widetilde{\tau}}(\mathfrak{h})) \mid \dim I_f \cap \pi_{\widetilde{\tau}}^{-1}(z) \geq p - 1 \}.$$

LEMMA 5.3. $S_f(\tau)$ is almost thin in $\mathbb{D}(\pi_{\widetilde{\tau}}(\mathfrak{h}))$.

PROOF. By [1] Theorem 1.14 $E = \{ x \in I_f \mid \text{rank}_x \pi_{\widetilde{\tau}} | I_f \leq \widetilde{p} - 1 \}$ is analytic in I_f

and by [11] Lemma 1.30 $E' = \pi_{\widetilde{\tau}}(E)$ is almost thin. Take $z \in S_f(\tau)$. Then

$x \in I_f \cap \pi_{\widetilde{\tau}}^{-1}(z)$ exists such that $\dim_x I_f \cap \pi_{\widetilde{\tau}}^{-1}(z) \geq p - 1$. An open, connected

neighborhood U of x in $\mathbb{D}(\mathfrak{h})$ exists such that $I_f \cap U = X_1 \cup \dots \cup X_r$ where each X_j is

a branch of $I_f \cap U$ with $x \in X_j$ for $j = 1, \dots, r$ and where X_j is locally irreducible

at x. A branch N of $I_f \cap \pi_{\widetilde{\tau}}^{-1}(z) \cap U$ exists with $x \in N$ and $\dim_x N \geq p - 1$. Then

$N \subseteq X_j$ for some j. By [11] Lemma 1.7 we obtain

$$\text{rank}_x \pi_{\widetilde{\tau}} | I_f \leq \text{rank}_x \pi_{\widetilde{\tau}} | X_j = \dim X_j - \dim_x X_j \cap \pi_{\widetilde{\tau}}^{-1}(z)$$

$$\leq \dim I_f - \dim N \leq n - 2 - (p - 1) = \widetilde{p} - 1.$$

Therefore $x \in E$ and $z = \pi_{\widetilde{\tau}}(x) \in E'$. Hence $S_f(\tau) \subseteq E'$ is almost thin; q.e.d.

THEOREM 5.4. Take $r \in \mathbb{R}^n_+(\mathfrak{h})$ and $\mathfrak{q} \in \mathbb{R}^n_+(\mathfrak{h})$. Assume that $\mathfrak{p} = \pi_{\widetilde{\tau}}(r) = \pi_{\widetilde{\tau}}(\mathfrak{q})$. Then

$$T_f(r, \mathfrak{q}) = \int_{z \in \mathbb{D}\langle \mathfrak{p} \rangle} T_{f[\tau, z]}(\pi_\tau(r), \pi_\tau(\mathfrak{q})) \Omega_{\widetilde{p}}(z). \tag{5.28}$$

PROOF. Let $\mathfrak{v} : \mathbb{D}(\mathfrak{h}) \to V$ be a reduced representation of f. Take $\mathfrak{z} \in \mathbb{D}\langle\mathfrak{p}\rangle - S_f(\tau)$. Then $f[\tau,\mathfrak{z}]$ exists and $\mathfrak{v} \circ \tau_{\mathfrak{z}}$ is a reduced representation of f.

Therefore

$$T_{f[\tau,\mathfrak{z}]}(\pi_\tau(\mathfrak{r}),\pi_\tau(\mathfrak{q})) = \int_{\mathbb{D}\langle\pi_\tau(\mathfrak{r})\rangle} \log\|\mathfrak{v} \circ \tau_{\mathfrak{z}}\|\Omega_p - \int_{\mathbb{D}\langle\pi_\tau(\mathfrak{q})\rangle} \log\|\mathfrak{v} \circ \tau_{\mathfrak{z}}\|\Omega_p .$$

Now (2.22) implies

$$\int_{\mathbb{D}\langle\mathfrak{p}\rangle} T_{f[\tau,\mathfrak{z}]}(\pi_\tau(\mathfrak{r}),\pi_\tau(\mathfrak{q}))\Omega_{\tilde{p}} = \int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log\|\mathfrak{v}\|\Omega_n - \int_{\mathbb{D}\langle\mathfrak{q}\rangle} \log\|\mathfrak{v}\|\Omega_n = T_f(\mathfrak{r},\mathfrak{q}) \quad \text{q.e.d.}$$

If $\pi_\tau(\mathfrak{q}) = q\pi_\tau(\mathfrak{r})$ with $0 < q < 1$, then (5.14) implies

$$T_{f[\tau,\mathfrak{z}]}(\pi_\tau(\mathfrak{r}),q\pi_\tau(\mathfrak{r})) = \int_q^1 A_{f[\tau,\mathfrak{z}]}(t\pi_\tau(\mathfrak{r}))\frac{dt}{t} . \tag{5.29}$$

Define

$$A_{f,\tau}(\mathfrak{p},t\pi_\tau(\mathfrak{r})) = \int_{\mathfrak{z}\in\mathbb{D}\langle\mathfrak{p}\rangle} A_{f[\tau,\mathfrak{z}]}(t\pi_\tau(\mathfrak{r}))\frac{dt}{t} . \tag{5.30}$$

Then we have

$$T_f(\mathfrak{r},\mathfrak{q}) = \int_q^1 A_{f,\tau}(\mathfrak{p},t\pi_\tau(\mathfrak{r}))\frac{dt}{t} \tag{5.31}$$

provided $\pi_{\tilde{\tau}}(\mathfrak{r}) = \mathfrak{p} = \pi_{\tilde{\tau}}(\mathfrak{q})$ and $\pi_\tau(\mathfrak{q}) = q\pi_\tau(\mathfrak{r})$ with $0 < q < 1$.

Take $p = 1$ and identify $\tau \in \mathfrak{T}(1,n)$ with $\tau(1) \in \mathbb{N}[1,n]$. Then

$$\int_1^q A_{f,\tau}(\mathfrak{p},t\pi_\tau(\mathfrak{r}))\frac{dt}{t} = \int_{q_\tau}^{r_\tau} A_{f,\tau}(\mathfrak{p},t)\frac{dt}{t} . \tag{5.32}$$

If $\mathfrak{r} = (r_1,\ldots,r_n)$ and $\mathfrak{q} = (r_1,\ldots,r_{\tau-1},q_\tau,r_{\tau+1},\ldots,r_n)$, then

$$T_f(\mathfrak{r},\mathfrak{q}) = \int_{q_\tau}^{r_\tau} A_{f,\tau}(\mathfrak{p},t)\frac{dt}{t} \tag{5.33}$$

where $\mathfrak{p} = (r_1,\ldots,r_{\tau-1},r_{\tau+1},\ldots,r_n)$ and

$$A_{f\tau}(\mathfrak{p},t) = \int_{\mathfrak{z}\in\mathbb{D}\langle\mathfrak{p}\rangle} A_{f[\tau,\mathfrak{z}]}(t)\frac{dt}{t} \tag{5.34}$$

$$A_{f[\tau,z]}(t) = \int_{\mathbb{D}[t]} f[\tau,z]*(\omega).$$ (5.35)

Take $0 < \mathfrak{q} < \mathfrak{r} < \mathfrak{h}$ and define $\mathfrak{r}_\tau$ and $\mathfrak{p}_\tau$ as in (4.34) and (4.35). We obtain

$$T_f(\mathfrak{r},\mathfrak{q}) = \sum_{\tau=1}^{n} \int_{q_\tau}^{r_\tau} A_{f,\tau}(\mathfrak{p}_\tau,t) \frac{dt}{t}.$$ (5.36)

Take $a \in \mathbb{P}(V^*)$ and assume that f is not linearly degenerate for a. Take $p \in \mathbb{N}[1,n]$. Then $\tilde{p} = n - p \in \mathbb{N}[1,n]$. Take $\tau \in \mathfrak{T}(p,n)$. Then $\tilde{\tau} \in \mathfrak{T}(\tilde{p},n)$. Take $\mathfrak{h} \in \overset{+}{\mathbb{R}}{}^n$ and $\mathfrak{r} \in \overset{+}{\mathbb{R}}{}^n(\mathfrak{h})$. Take $z \in \mathbb{D}(\pi_{\tilde{\tau}}(\mathfrak{h})) - S_f(\tau)$ such that μ_f^a is restrictable to z by τ. Then

$$m_{f[\tau,z]}(\mathfrak{p},a) = \int_{\mathbb{D}\langle\mathfrak{p}\rangle} \log \frac{1}{\|f \circ \tau_z, a\|} \Omega_p$$ (5.37)

is defined for all $\mathfrak{p} \in \overset{+}{\mathbb{R}}{}^p(\pi_\tau(\mathfrak{h}))$. Then (2.22) implies

$$m_f(\mathfrak{r},a) = \int_{z \in \mathbb{D}\langle\pi_{\tilde{\tau}}(\mathfrak{r})\rangle} m_{f[\tau,z]}(\pi_\tau(\mathfrak{r}),a)\Omega_{\tilde{p}}(z).$$ (5.38)

6. THE LEMMA OF THE LOGARITHMIC DERIVATIVE.

Take $0 < r < R < b \leq +\infty$. Let $f \not\equiv 0$ be a meromorphic function on $\mathbb{D}(b)$. By [9] Lemma 2.1 (see also Hayman [12], Lemma 2.3) we have

$$\int_{\mathbb{D}\langle r\rangle} \log^+|f'/f|\Omega_1 \leq \log^+ m_f(R,0) + \log^+ m_f(R,\infty) + 3 \log^+ n_f(R,0)$$ (6.1)

$$+ 3 \log^+ n_f(R,\infty) + \log^+ \frac{R}{(R-r)^2} + \log^+ \frac{1}{r} + \log^+ \frac{r}{R-r} + 8 \log 2.$$

We keep the notation of the last three sections. Take $\mathfrak{h} \in \overset{+}{\mathbb{R}}{}^n$ and $\mathfrak{r}$ and $\mathfrak{R}$ in $\overset{+}{\mathbb{R}}{}^n(\mathfrak{h})$. Take $\tau \in \mathbb{N}[1,n]$ and assume that

$$0 < r = \pi_\tau(\mathfrak{r}) < \pi_\tau(\mathfrak{R}) = R < b \qquad \qquad \pi_{\tilde{\tau}}(\mathfrak{r}) = \mathfrak{p} = \pi_{\tilde{\tau}}(\mathfrak{R}).$$ (6.2)

Let $f \not\equiv 0$ be a meromorphic function on $\mathbb{D}(\mathfrak{h})$. Abbreviate

$$\nu = \mu_f^0 \qquad\qquad \eta = \mu_f^\infty \qquad\qquad f_\tau' = \frac{\partial f}{\partial z_\tau} \ . \tag{6.3}$$

THEOREM 6.1. Under these assumptions we have

$$\int_{\mathbb{D}\langle \mathbf{r}\rangle} \log^+|f_\tau'/f|\Omega_n \le \log^+ m_f(\mathfrak{R},0) + \log^+ m_f(\mathfrak{R},\infty) + 3\log^+ n_{\nu,\tau}(\mathfrak{p},R) \tag{6.4}$$

$$+ 3\log^+ n_{\mu,\tau}(\mathfrak{p},R) + \log^+\frac{R}{(R-r)^2} + \log^+\frac{1}{r} + \log^+\frac{r}{R-r} + 16\log 2.$$

PROOF. For $\mathbf{z} \in \mathbb{D}\langle\mathfrak{p}\rangle - R_f(\tau)$ we have $f[\tau,\mathbf{z}] = f \circ \tau_{\mathbf{z}}$. The chain rule implies

$f[\tau,\mathbf{z}]' = f_\tau' \circ \tau_{\mathbf{z}}$. Now (2.22), (4.26) and (5.38) imply

$$\int_{\mathbb{D}\langle \mathbf{r}\rangle} \log^+|f_\tau'/f|\Omega_n$$

$$= \int_{\mathbb{D}\langle\mathfrak{p}\rangle} \left(\int_{\mathbb{D}\langle \mathbf{r}\rangle} \log^+|f[\tau,\mathbf{z}]'/f[\tau,\mathbf{z}]|\Omega_1 \right)\Omega_{n-1}(\mathbf{z})$$

$$\le \int_{\mathbb{D}\langle\mathfrak{p}\rangle} \log^+ m_{f[\tau,\mathbf{z}]}(R,0)\Omega_{n-1} + \int_{\mathbb{D}\langle\mathfrak{p}\rangle} \log^+ m_{f[\tau,\mathbf{z}]}(R,\infty)\Omega_{n-1} + 3\int_{\mathbb{D}\langle\mathfrak{p}\rangle} \log^+ n_{\nu[\tau,\mathbf{z}]}(R)\Omega_{n-1}$$

$$+ 3\int_{\mathbb{D}\langle\mathfrak{p}\rangle} \log^+ n_{\mu[\tau,\mathbf{z}]}(R)\Omega_{n-1} + \log^+\frac{R}{(R-r)^2} + \log^+\frac{1}{r} + \log^+\frac{r}{R-r} + 8\log 2$$

$$\le \log^+ \int_{\mathbb{D}\langle\mathfrak{p}\rangle} m_{f[\tau,\mathbf{z}]}(R,0)\Omega_{n-1} + \log^+ \int_{\mathbb{D}\langle\mathfrak{p}\rangle} m_{f[\tau,\mathbf{z}]}(R,\infty)\Omega_{n-1} + 3\log^+ \int_{\mathbb{D}\langle\mathfrak{p}\rangle} n_{\nu[\tau,\mathbf{z}]}(R)\Omega_{n-1}$$

$$+ 3\log^+ \int_{\mathbb{D}\langle\mathfrak{p}\rangle} n_{\mu[\tau,\mathbf{z}]}(R)\Omega_{n-1} + \log^+\frac{R}{(R-r)^2} + \log^+\frac{1}{r} + \log^+\frac{r}{R-r} + 16\log 2$$

$$= \log^+ m_f(\mathfrak{R},0) + \log^+ m_f(\mathfrak{R},\infty) + 3\log^+ n_{\nu,\tau}(\mathfrak{p},R) + 3\log^+ n_{\mu,\tau}(\mathfrak{p},R)$$

$$+ \log^+\frac{R}{(R-r)^2} + \log^+\frac{1}{r} + \log^+\frac{r}{R-r} + 16\log 2 \qquad\qquad \text{q.e.d.}$$

THEOREM 6.2. If $0 < \mathfrak{q} \le \mathbf{r}$, we obtain under the same assumptions

$$\int_{\mathbb{D}\langle \mathbf{r}\rangle} \log^+|f_\tau'/f|\Omega_n \le 8\log^+ T_f(\mathfrak{R},\mathfrak{q}) + 4\log^+ m_f(\mathfrak{q},0) + 4\log^+ m_f(\mathfrak{q},\infty) \tag{6.5}$$

$$+ 9\log\frac{2R}{R-r} + 2\log^+\frac{1}{r} + 24\log 2.$$

PROOF. Define $\Theta = (R + r)/2R < 1$. Then $r < \Theta R < R$ and $\Theta R - r = (R - r)/2$. Define

$$\hat{r} = (r_1, \ldots, r_{\tau-1}, \Theta R, r_{\tau+1}, \ldots, r_n).$$

Then $r \leq \hat{r} \leq \Re$ and $\pi_\tau(r) = r < \pi_\tau(\hat{r}) = \Theta R < \pi_\tau(\Re) = R$ and $\pi_{\widetilde{\tau}}(r) = \pi_{\widetilde{\tau}}(\hat{r}) = \pi_{\widetilde{\tau}}(\Re) = \mathfrak{p}$. Since $N_\nu(\hat{r}, \mathfrak{q}) \geq 0$, the First Main Theorem implies

$$m_f(\hat{r}, 0) \leq T_f(\hat{r}, \mathfrak{q}) + m_f(\mathfrak{q}, 0) \leq T_f(\Re, \mathfrak{q}) + m_f(\mathfrak{q}, 0)$$

$$\log^+ m_f(\hat{r}, 0) \leq \log^+ T_f(\Re, \mathfrak{q}) + \log^+ m_f(\mathfrak{q}, 0) + \log 2 \qquad (6.6)$$

$$\log^+ m_f(\hat{r}, \infty) \leq \log^+ T_f(\Re, \mathfrak{q}) + \log^+ m_f(\mathfrak{q}, \infty) + \log 2. \qquad (6.7)$$

By (4.31) we have

$$N_\nu(\Re, \mathfrak{q}) = N_\nu(\Re, r) + N_\nu(r, \mathfrak{q}) \geq N_\nu(\Re, r) = \int_r^R n_{\nu, \tau}(\mathfrak{p}, t) \, \frac{dt}{t} \geq \int_{\Theta R}^R n_{\nu, \tau}(\mathfrak{p}, t) \, \frac{dt}{t}$$

$$\geq n_{\nu, \tau}(\mathfrak{p}, \Theta R) \, \frac{R - \Theta R}{R} = n_{\nu, \tau}(\mathfrak{p}, \Theta R) \, \frac{R - r}{2R}$$

$$\log^+ n_{\nu, \tau}(\mathfrak{p}, \Theta R) \leq \log^+ N_\nu(\Re, \mathfrak{q}) + \log^+ \frac{2R}{R - r} \leq \log^+(T_f(\Re, \mathfrak{q}) + m_f(\mathfrak{q}, 0)) + \log \frac{2R}{R - r}$$

or

$$\log^+ n_{\nu, \tau}(\mathfrak{p}, \Theta R) \leq \log^+ T_f(\Re, \mathfrak{q}) + \log^+ m_f(\mathfrak{q}, 0) + \log \frac{2R}{R - r} + \log 2 \qquad (6.8)$$

$$\log^+ n_{\mu, \tau}(\mathfrak{p}, \Theta R) \leq \log^+ T_f(\Re, \mathfrak{q}) + \log^+ m_f(\mathfrak{q}, \infty) + \log \frac{2R}{R - r} + \log 2. \qquad (6.9)$$

Also we have

$$\log^+ \frac{\Theta R}{(\Theta R - r)^2} \leq \log^+ \frac{\Theta R}{\Theta R - r} + \log^+ \frac{R}{\Theta R - r} + \log^+ \frac{1}{R} \qquad (6.10)$$

$$= \log^+ \frac{R + r}{R - r} + \log^+ \frac{2R}{R - r} + \log^+ \frac{1}{R} \leq 2 \log \frac{2R}{R - r} + \log^+ \frac{1}{r}$$

$$\log^+ \frac{r}{R - r} \leq \log \frac{2R}{R - r} \, . \qquad (6.11)$$

Now, Theorem 6.1 for r and $\hat{r}$ and (6.6)-(6.11) imply (6.5), q.e.d.

THEOREM 6.3. Take $\mathfrak{h} \in \overset{+}{\mathbb{R}}^n$. Take $\mathfrak{r} \in \overset{+}{\mathbb{R}}^n(\mathfrak{h})$ and $\mathfrak{q} \in \overset{+}{\mathbb{R}}^n(\mathfrak{h})$ with $0 < \mathfrak{q} \le \mathfrak{r} < \mathfrak{h}$.

Take $1 < \Theta \in \mathbb{R}$ and $0 < q \in \mathbb{R}$ such that $0 < q \cdot \mathfrak{h} \le \mathfrak{q} \le \mathfrak{r} < \Theta r < \mathfrak{h}$. Let $f \ddagger 0$ be a

meromorphic function on $\mathbb{D}(\mathfrak{h})$. Take $\tau \in \mathbb{N}[1,n]$. Define $f'_\tau = \dfrac{\partial f}{\partial z_\tau}$. Then

$$\int_{\mathbb{D}\langle \mathfrak{r} \rangle} \log^+ |f'_\tau/f| \, \Omega_n \le 8 \log^+ T_f(\Theta r, \mathfrak{q}) + 4 \log^+ m_f(\mathfrak{q}, 0) + 4 \log^+ m_f(\mathfrak{q}, \infty) \qquad (6.12)$$

$$+ \, 9 \log \frac{2\Theta}{\Theta - 1} + 2 \log^+ \frac{1}{qb_\tau} + 24 \log 2.$$

PROOF. Theorem 6.2 can be applied with $r = r_\tau$ and $R = \Theta r_\tau$ and

$$\mathfrak{R} = (r_1, \ldots, r_{\tau-1}, R, r_{\tau+1}, \ldots, r_n).$$

Then $\mathfrak{R} \le \Theta r$ and $T_f(\mathfrak{R}, \mathfrak{q}) \le T_f(\Theta r, \mathfrak{q})$. Also $2R/(R - r) = 2\Theta/(\Theta - 1)$ and $1/r \le 1/qb_\tau$.

Consequently (6.5) implies (6.12); q.e.d.

Theorem 6.3 is a preliminary version of the Lemma of the logarithmic derivative.

For $\mathfrak{r}$ to approach $\mathfrak{h}$, we have to eliminate Θ. If we restrict ourself to the approach

$r\mathfrak{h} \to \mathfrak{h}$ with $1 > r \to 1$ this could be accomplished along classical lines and would

result in the usual "exceptional intervals". Rubel and Henson want to consider the

approach $\mathfrak{h} > \mathfrak{r} \to \mathfrak{h}$. In order to satisfy both cases, we introduce a more general

method resulting in "exceptional sets" of dimensions between 1 and n.

Take $\mathfrak{h} \in \overset{+}{\mathbb{R}}^n$ and $p \in \mathbb{Z}[0,n)$. Define $\delta : \overset{+}{\mathbb{R}}^n(\mathfrak{h}) \to \mathbb{R}(0,1]$ by

$$\delta(x) = \text{Min}\{1 - (x_j/b_j) \mid j = 1, \ldots, n\} \qquad (6.13)$$

for $x \in \overset{+}{\mathbb{R}}^n(\mathfrak{h})$. Let B be a real, pure p-dimensional, oriented submanifold of class C^k

in $\overset{+}{\mathbb{R}}^n$ with $k \ge 1$. Denote by μ_B the differential geometric measure on B. It equals

the p-dimensional Hausdorff measure. For $x \in B$, let T_x be the real tangent space and

N_x the normal space of B at x. Then we have the orthogonal decomposition

$\mathbb{R}^n = T_x \oplus N_x$. Let $\rho_x : \mathbb{R}^n \to N_x$ be the projection. The manifold B is said to be an

approach base if the following conditions are satisfied.

(a)   The manifold B has finite volume $\mu_B(B) < \infty$.

(b)   If $x \in B$, then $0 \le x < \mathfrak{h}$, that is, $B \subseteq \mathbb{R}^n_+(\mathfrak{h})$.

(c)   If $x \in \overline{B}$ and $t \in \mathbb{R}$ and $x + t(\mathfrak{h} - x) \in \overline{B}$, then $t = 0$.

(d)   If $p > 0$, there is a number $\gamma > 0$ such that for each $x \in B$ we have

$$\| \rho_x(\mathfrak{h} - x) \| \ge \gamma. \tag{6.14}$$

(e)   If $p > 0$, then

$$\int_B \log(1/\delta) \, d\mu_B < \infty. \tag{6.15}$$

The closure $\overline{B}$ is compact and contained in $\mathbb{R}^n[\mathfrak{h}]$. If $\mathfrak{h} \in \overline{B}$, then $\mathfrak{h} = \mathfrak{h} + (\mathfrak{h} - \mathfrak{h}) \in \overline{B}$

which contradicts (c). Hence $\mathfrak{h} \notin \overline{B}$.

    Define the <u>associated</u> <u>approach</u> <u>cone</u> by

$$M = \{ x + t(\mathfrak{h} - x) \mid x \in B \text{ and } t \in \mathbb{R}(0,1) \}. \tag{6.16}$$

If $x \in B$, then $x < \mathfrak{h}$ and $x < x + t(\mathfrak{h} - x) < \mathfrak{h}$ for $0 < t < 1$. Hence $M \subseteq \mathbb{R}^n_+(\mathfrak{h})$.
    EXAMPLE 1. Take

$$B = \left\{ x \in \mathbb{R}^n_+(\mathfrak{h}) \mid \sum_{j=1}^n x_j/b_j = 1 \right\}$$

$$M = \left\{ x \in \mathbb{R}^n_+(\mathfrak{h}) \mid \sum_{j=1}^n x_j/b_j > 1 \right\}.$$

Simple calculations show that the conditions (a)-(e) are satisfied. Hence B is an

approach base and M the associated approach cone. Observe that $M \cup \{\mathfrak{h}\}$ is an open

neighborhood of $\mathfrak{h}$ in $\mathbb{R}^n_+(\mathfrak{h}) \cup \{\mathfrak{h}\}$.

    EXAMPLE 2. Take $B = \{0\}$. Then $M = \{t\mathfrak{h} \mid 0 < t < 1\}$.

    While the geometric meaning of the conditions (a)-(d) is obvious, the

condition (e) is more difficult to analyze. For this purpose we offer the following

considerations:

    LEMMA 6.4. Let B be a real, pure p-dimensional, oriented submanifold of class

C^k in $\mathbb{R}^n$ with $k \geq 1$, with $p \geq 1$ and with $\mu_B(B) < \infty$. For each $\lambda \in \mathbb{R}(0,1]$ define

$$B(\lambda) = B - \mathbb{R}^n_+((1 - \lambda)\mathfrak{h}).$$

Assume there are constants $c > 0$ and $q > 0$ such that $\mu_B(B(\lambda)) \leq c\lambda^q$ for all

$\lambda \in \mathbb{R}(0,1]$. Then (6.15) is satisfied.

PROOF. Observe that $B(1) = B$ and $B(\lambda) = \{x \in B \mid \delta(x) \leq \lambda\}$. For $m \in \mathbb{N}$ we have

$$A(m) = B(1/m) - B(1/(m + 1)) = \{x \in B \mid 1/(m + 1) < \delta(x) \leq 1/m\}$$

$$d_m = \mu_B(B(1/m)) \leq cm^{-q}$$

$$a_m = \mu_B(A(m)) = d_m - d_{m+1}.$$

This yields the estimate

$$\int_B \log(1/\delta) \, d\mu_B \leq \sum_{m=1}^{\infty} \int_{A_m} \log(m + 1) \, d\mu_B = \sum_{m=1}^{\infty} a_m \log(m + 1)$$

$$= \lim_{r \to \infty} \left(\sum_{m=1}^{r-1} d_m \log(m + 1) - \sum_{m=1}^{r-1} d_{m+1} \log(m + 1) \right)$$

$$= \lim_{r \to \infty} \left(\sum_{m=1}^{r} d_m \log(1 + 1/m) - d_r \log(r + 1) \right)$$

$$\leq c \sum_{m=1}^{\infty} m^{-q} \log(1 + 1/m) \leq c \sum_{m=1}^{\infty} m^{-1-q} < \infty \qquad \text{q.e.d.}$$

A map $\mathfrak{y} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is defined by $\mathfrak{y}(t,x) = x + t(\mathfrak{h} - x)$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Obviously $\mathfrak{y}(0,x) = x$ and $\mathfrak{y}(1,x) = \mathfrak{h}$. If $0 < s < t < 1$ then $x < \mathfrak{y}(t,x) < \mathfrak{h}$. By definition $M = \mathfrak{y}(\mathbb{R}(0,1) \times B)$.

LEMMA 6.5. The map $\mathfrak{y} : (\mathbb{R} - \{1\}) \times \overline{B} \to \mathbb{R}^n$ is injective.

PROOF. Assume that x and $\tilde{x}$ exist in $\overline{B}$ and $1 \neq t \in \mathbb{R}$ and $1 \neq \tilde{t} \in \mathbb{R}$ such that $\mathfrak{y}(t,x) = \mathfrak{y}(\tilde{t},\tilde{x})$. Then $t(\mathfrak{h} - x) = \tilde{x} - x + \tilde{t}(\mathfrak{h} - \tilde{x})$. Hence

$$x + \frac{t - \tilde{t}}{1 - \tilde{t}} (\mathfrak{h} - x) = x + \frac{1}{1 - \tilde{t}} (\tilde{x} - x + \tilde{t}(\mathfrak{h} - \tilde{x}) - \tilde{t}(\mathfrak{h} - x)) = \tilde{x} \in \overline{B}.$$

Hence $t = \tilde{t}$ and $x = \tilde{x}$; q.e.d.

LEMMA 6.6. The map $\mathfrak{y} : \mathbb{R}(0,1) \times B \to M$ is a homeomorphism.

PROOF. By definition the map is surjective, hence by Lemma 6.5 the map $\mathfrak{y} : \mathbb{R}(0,1) \times B \to M$ is bijective. Trivially the map is continuous. It remains to be shown that $\mathfrak{y}$ is open. Let U be an open subset of $\mathbb{R}(0,1) \times B$. Take $(t,\mathbf{x}) \in U$. Assume that $\mathfrak{y}(t,\mathbf{x})$ is not an interior point of $\mathfrak{y}(U)$ in M. A sequence $\{(t_\nu, \mathbf{x}_\nu)\}_{\nu \in \mathbb{N}}$ exists in $\mathbb{R}(0,1) \times B - U$ such that $\mathfrak{y}(t_\nu, \mathbf{x}_\nu) \to \mathfrak{y}(t,\mathbf{x})$ for $\nu \to \infty$. By going to a subsequence we can assume that $t_\nu \to \tilde{t}$ and $\mathbf{x}_\nu \to \tilde{\mathbf{x}}$ for $\nu \to \infty$ with $(\tilde{t}, \tilde{\mathbf{x}}) \in \mathbb{R}[0,1] \times \bar{B} - U$. Then $\mathfrak{y}(t,\mathbf{x}) = \mathfrak{y}(\tilde{t}, \tilde{\mathbf{x}})$. If $\tilde{t} = 1$, then $\mathfrak{h} = \mathfrak{y}(1,\tilde{\mathbf{x}}) = \mathfrak{y}(t,\mathbf{x}) \in M$ which is wrong. Therefore $\tilde{t} \neq 1$ and $(\tilde{t}, \tilde{\mathbf{x}}) = (t,\mathbf{x}) \in U$ which is impossible. Therefore $\mathfrak{y}(U)$ is open in M; q.e.d.

Let $V \neq \emptyset$ be an open, connected subset of $\mathbb{R}^p$. Let $U \neq \emptyset$ be an open subset of B and let $\mathfrak{g} : V \to U$ be an orientation preserving diffeomorphism of class C^1. Let $\mathfrak{g}_{v_j}$ be the partial derivatives of $\mathfrak{g}$. Define $\mathfrak{w} : \mathbb{R} \times V \to \mathbb{R}^n$ by $\mathfrak{w}(t,\mathfrak{v}) = \mathfrak{y}(t, \mathfrak{g}(\mathfrak{v}))$ for $t \in \mathbb{R}$ and $\mathfrak{v} \in V$. Then

$$\mathfrak{w}_t(t,\mathfrak{v}) = \mathfrak{h} - \mathfrak{g}(\mathfrak{v}) \qquad\qquad \mathfrak{w}_{v_j}(t,\mathfrak{v}) = (1-t)\mathfrak{g}_{v_j}(\mathfrak{v}). \qquad (6.17)$$

$$\mathfrak{w}_t(t,\mathfrak{v}) \wedge \mathfrak{w}_{v_1}(t,\mathfrak{v}) \wedge \ldots \wedge \mathfrak{w}_{v_p}(t,\mathfrak{v}) = (1-t)^p(\mathfrak{h} - \mathfrak{g}(\mathfrak{v})) \wedge \mathfrak{g}_{v_1}(\mathfrak{v}) \wedge \ldots \wedge \mathfrak{g}_{v_p}(\mathfrak{v}). \qquad (6.18)$$

The vectors $\mathfrak{g}_{v_1}(\mathfrak{v}), \ldots, \mathfrak{g}_{v_p}(\mathfrak{v})$ span $T_{\mathfrak{g}(\mathfrak{v})}$. By (d), we have $\mathfrak{h} - \mathfrak{g}(\mathfrak{v}) \notin T_{\mathfrak{g}(\mathfrak{v})}$. Therefore

$$(\mathfrak{h} - \mathfrak{g}(\mathfrak{v})) \wedge \mathfrak{g}_{v_1}(\mathfrak{v}) \wedge \ldots \wedge \mathfrak{g}_{v_p}(\mathfrak{v}) \neq 0. \qquad (6.19)$$

In fact, if $\mathbf{x} = \mathfrak{g}(\mathfrak{v})$, then

$$\Delta(\mathbf{x}) = \frac{\| (\mathfrak{h} - \mathfrak{g}(\mathfrak{v})) \wedge \mathfrak{g}_{v_1}(\mathfrak{v}) \wedge \ldots \wedge \mathfrak{g}_{v_p}(\mathfrak{v}) \|}{\| \mathfrak{g}_{v_1}(\mathfrak{v}) \wedge \ldots \wedge \mathfrak{g}_{v_p}(\mathfrak{v}) \|} \qquad (6.20)$$

depends only on $\mathbf{x}$ and not on the parameterization $\mathfrak{g}$. Hence $\Delta : B \to \mathbb{R}$ is globally defined.

LEMMA 6.7. If $x \in B$, then

$$0 < \gamma \leq \Delta(x) = \| \rho_x(\mathfrak{h} - x) \| \leq \| \mathfrak{h} \|.$$ (6.21)

PROOF. We have $\mathfrak{h} - x = \rho_x(\mathfrak{h} - x) + \sum_{j=1}^{p} a_j \mathfrak{g}_{v_j}(\mathfrak{v})$. Hence

$$\| (\mathfrak{h} - x) \wedge \mathfrak{g}_{v_1}(\mathfrak{v}) \wedge \ldots \wedge \mathfrak{g}_{v_p}(\mathfrak{v}) \| = \| \rho_x(\mathfrak{h} - x) \| \ \| \mathfrak{g}_{v_1}(\mathfrak{v}) \wedge \ldots \wedge \mathfrak{g}_{v_p}(\mathfrak{v}) \|$$

or $\Delta(x) = \| \rho_x(\mathfrak{h} - x) \|$. By (6.14) we have $\Delta(x) \geq \gamma$. Also

$$\Delta(x)^2 = \| \rho_x(\mathfrak{h} - x) \|^2 \leq \| \mathfrak{h} - x \|^2 = \sum_{j=1}^{n} (b_j - x_j)^2 \leq \sum_{j=1}^{n} b_j^2 = \| \mathfrak{h} \|^2$$

which implies $\Delta(x) \leq \| \mathfrak{h} \|$, q.e.d.

By (6.18) and (6.19) the map $\mathfrak{y} : \mathbb{R}(0,1) \times B \to \mathbb{R}^n$ is smooth. In conjunction with Lemma 6.6 we have proved:

THEOREM 6.8. M is an embedded, oriented, differentiable manifold of pure dimension $p + 1$ and of class C^1 such that $\mathfrak{y} : \mathbb{R}(0,1) \times B \to M$ is an orientation preserving diffeomorphism of class C^1.

Let μ_M and μ_B be the differential geometric measures on M and B respectively. In the situation (6.17)-(6.20) we have

$$d\mu_B = \| \mathfrak{g}_{v_1} \wedge \ldots \wedge \mathfrak{g}_{v_p} \| \ dv_1 \wedge \ldots \wedge dv_p$$

$$d\mu_M = (1 - t)^p \| (\mathfrak{h} - \mathfrak{g}) \wedge \mathfrak{g}_{v_1} \wedge \ldots \wedge \mathfrak{g}_{v_p} \| \ dt \wedge dv_1 \wedge \ldots \wedge dv_p.$$

Hence globally

$$d\mu_M = (1 - t)^p \Delta(x) \ dt \wedge d\mu_B.$$ (6.22)

Now, Lemma 6.7 implies

$$0 \leq \frac{\gamma}{p + 1} \mu_B(B) \leq \mu_M(M) \leq \frac{\| \mathfrak{h} \|}{p + 1} \mu_B(B) < \infty.$$ (6.23)

Here we are able to give another example where the conditions (a)-(d) imply condition (e).

PROPOSITION 6.9. Take $\mathfrak{h} \in \mathbb{R}^n_+$. Let A be a pure $(n-1)$-dimensional, oriented submanifold of class C^k in $\mathbb{R}^n$ with $k \geq 1$. Assume that $B = A \cap \mathbb{R}^n_+(\mathfrak{h})$ is relative compact in A. Then $\mu_B(B) < \infty$. Assume that there exists a number $\gamma > 0$ such that (6.14) holds. Then (6.15) is satisfied. If also condition (c) holds, then B is an approach base.

PROOF. Define $B_j = \{\mathbf{x} \in B \mid \delta(\mathbf{x}) = 1 - (x_j/b_j)\}$ for $j = 1, \ldots, n$. Then $B = B_1 \cup \ldots \cup B_n$. Each $\overline{B}_j$ is a compact subset of A. Take $\mathfrak{a} \in \overline{B}_j$ with $a_j = b_j$. Then there exists an orientation preserving diffeomorphism $\mathfrak{g} : V \rightarrow U$ of class C^1 where U is an open neighborhood of $\mathfrak{a}$ in A and where V is an open neighborhood of 0 in $\mathbb{R}^{n-1}$ and where $\mathfrak{g}(0) = \mathfrak{a}$. Let ∇ be the gradient. Abbreviate

$$G = \| \mathfrak{g}_{v_1} \wedge \ldots \wedge \mathfrak{g}_{v_{n-1}} \| > 0$$

$$G_i = (-1)^{i-1} \det(\nabla g_1, \ldots, \nabla g_{i-1}, \nabla g_{i+1}, \ldots, \nabla g_n).$$

Then (6.20) reads

$$(\Delta \circ \mathfrak{g})G = \left| \sum_{i=1}^{n} (b_i - g_i)G_i \right|.$$

Assume that $\nabla g_j(0) = 0$. Since $\Delta(\mathbf{x}) \geq \gamma > 0$ for all $\mathbf{x} \in B$, we have $\Delta(\mathfrak{a}) \geq \gamma > 0$ by continuity. Also $\nabla g_j(0) = 0$ implies $G_i(0) = 0$ for all $i \neq j$. Therefore

$$0 < \Delta(\mathfrak{a})G(0) = (b_j - g_j(0))G_j(0) = (b_j - a_j)G_j(0) = 0.$$

This contradiction shows that $\nabla g_j(0) \neq 0$. Therefore $d(b_j - x_j) \neq 0$ at $\mathfrak{a}$. Consequently $\log(b_j - x_j)$ is locally integrable at $\mathfrak{a} \in \overline{B}_j$ if $a_j = b_j$. If $a_j \neq b_j$, then $a_j < b_j$ and $\log(b_j - x_j)$ is trivially locally integrable at $\mathfrak{a}$. Since $\overline{B}_j$ is compact, $\log(b_j - x_j)$ is integrable over B_j. Therefore

$$\int_B \log(1/\delta) \, d\mu_B \leq \sum_{j=1}^{n} \int_{B_j} \log \frac{b_j}{b_j - x_j} \, d\mu_B < \infty, \qquad\qquad \text{q.e.d.}$$

If E is a measurable subset of M, then the set

$$E_{\mathbf{x}} = \{ t \in \mathbb{R}(0,1) \mid \mathbf{x} + t(\mathfrak{h} - \mathbf{x}) \in E \}$$

is measurable for almost all $\mathbf{x} \in B$. Let $g : E \to \mathbb{R}$ be integrable over E. Fubini's

theorem implies

$$\int_E g \, d\mu_M = \int_{\mathbf{x} \in B} \left(\int_{E(\mathbf{x})} g(\mathbf{y}(t,\mathbf{x}))(1-t)^P \, dt \right) \Delta(\mathbf{x}) \, d\mu_B(\mathbf{x}). \tag{6.24}$$

Define

$$\Lambda_M(E) = \int_{\mathbf{r} \in E} \frac{d\mu_M(\mathbf{r})}{\|\mathbf{h} - \mathbf{r}\|^{P+1}} = \int_{\mathbf{x} \in B} \left(\int_{E(\mathbf{x})} \frac{dt}{1-t} \right) \frac{\Delta(\mathbf{x}) \, d\mu_B(\mathbf{x})}{\|\mathbf{h} - \mathbf{x}\|^{P+1}}. \tag{6.25}$$

Then $0 \leq \Lambda_M(E) \leq \infty$.

LEMMA 6.10. Define $\gamma_0 = \text{Min}\{\|\mathbf{h} - \mathbf{x}\| \mid \mathbf{x} \in \overline{B}\}$. Then $\gamma_0 > 0$. Define

$$\gamma_1 = (1/\gamma_0)^P \qquad\qquad \gamma_2 = \gamma \|\mathbf{h}\|^{-P-1}.$$

Let E be a measurable set on M. Then

$$\gamma_2 \int_{\mathbf{x} \in B} \left(\int_{E(\mathbf{x})} \frac{dt}{1-t} \right) d\mu_B(\mathbf{x}) \leq \Lambda_M(E) \leq \gamma_1 \int_{\mathbf{x} \in B} \left(\int_{E(\mathbf{x})} \frac{dt}{1-t} \right) d\mu_B(\mathbf{x}). \tag{6.26}$$

PROOF. Recall that $0 < \gamma \leq \Delta(\mathbf{x}) \leq \|\mathbf{h} - \mathbf{x}\| \leq \|\mathbf{h}\|$. Hence

$$\gamma_2 = \frac{\gamma}{\|\mathbf{h}\|^{P+1}} \leq \frac{\Delta(\mathbf{x})}{\|\mathbf{h} - \mathbf{x}\|^{P+1}} \leq \frac{1}{\|\mathbf{h} - \mathbf{x}\|^P} \leq \left(\frac{1}{\gamma_0} \right)^P = \gamma_1$$

which with (6.25) implies (6.26); q.e.d.

For $0 < r \leq s \leq 1$ define

$$M[r,s] = \mathbf{y}(\mathbb{R}[r,s] \times B) \qquad\qquad M(r,s) = \mathbf{y}(\mathbb{R}(r,s) \times B)$$

$$M(r,s] = \mathbf{y}(\mathbb{R}(r,s] \times B) \qquad\qquad M[r,s) = \mathbf{y}(\mathbb{R}[r,s) \times B).$$

Then $M = M(0,1)$ and $M \cup \{\mathbf{h}\} = M(0,1]$. If $t_j \in \mathbb{R}(0,1)$ for $j \in \mathbb{N}$ and if $t_j \to 1$ for

$j \to \infty$, then $\{M(t_j,1]\}_{j \in \mathbb{N}}$ is a base of open neighborhoods of $\mathbf{h}$ in $M \cup \{\mathbf{h}\}$. If

$0 < r \leq s < 1$, then (6.26) implies

$$\gamma_2 \mu_B(B) \, \log \frac{1-r}{1-s} \leq \Lambda_M(M[r,s]) \leq \gamma_1 \mu_B(B) \, \log \frac{1-r}{1-s} \tag{6.27}$$

$$\Lambda_M(M[r,1)) = \infty \qquad\qquad \Lambda_M(M(0,s]) < \infty. \tag{6.28}$$

LEMMA 6.11. Define

$$Q = \{ \mathfrak{y}(t,\mathbf{x}) \mid \mathbf{x} \in B \text{ and } 0 < t < 1 - \delta(\mathbf{x}) \}. \qquad (6.29)$$

Then $\Lambda_M(Q) < \infty$. If $\mathbf{x} \in B$, if $1 - \delta(\mathbf{x}) \le t < 1$ and if $\mathbf{r} = \mathfrak{y}(t,\mathbf{x})$, then $\mathbf{r} \in M - Q$ and

$$\| \mathfrak{h} - \mathbf{r} \| \le \delta(\mathbf{x}) \| \mathfrak{h} - \mathbf{x} \|. \qquad (6.30)$$

PROOF. According to (6.26) we have

$$\Lambda_M(Q) \le \gamma_1 \int_{\mathbf{x} \in B} \int_0^{1-\delta(\mathbf{x})} \frac{dt}{1-t} \, d\mu_B(\mathbf{x}) = \gamma_1 \int_B \log(1/\delta) \, d\mu_B < \infty.$$

Also $\mathbf{r} = \mathbf{x} + t(\mathfrak{h} - \mathbf{x}) \in M - Q$ if and only if $0 \le 1 - \delta(\mathbf{x}) \le t < 1$. If so, then $\mathfrak{h} - \mathbf{r} = (1 - t)(\mathfrak{h} - \mathbf{x})$ where $(1 - t) \le \delta(\mathbf{x})$. Hence

$$\| \mathfrak{h} - \mathbf{r} \| = (1 - t) \| \mathfrak{h} - \mathbf{x} \| \le \delta(\mathbf{x}) \| \mathfrak{h} - \mathbf{x} \| \qquad \text{q.e.d.}$$

Let $g : M \to \mathbb{R}$ and $h : M \to \mathbb{R}$ be functions. We say that $g(\mathbf{r}) \le h(\mathbf{r})$ for <u>most</u> $\mathbf{r}$ in M and write $g(\mathbf{r}) \overset{\le}{\cdot} h(\mathbf{r})$ on M if there exists a measurable subset E of M with $\Lambda_M(E) < \infty$ such that $g(\mathbf{r}) \le h(\mathbf{r})$ for all $\mathbf{r} \in M - E$.

LEMMA 6.12. Let g, h, k and ℓ be real valued functions on M. Then

(a) If $g \overset{\le}{\cdot} h$ and $h \overset{\le}{\cdot} k$ on M, then $g \overset{\le}{\cdot} k$ on M.

(b) If $g \overset{\le}{\cdot} h$ and $0 \overset{\le}{\cdot} k$ on M, then $gk \overset{\le}{\cdot} hk$ on M.

(c) If $g \overset{\le}{\cdot} h$ and $k \overset{\le}{\cdot} \ell$ on M, then $g + k \overset{\le}{\cdot} h + \ell$.

(d) If $t_0 \in \mathbb{R}(0,1)$ and $g(\mathbf{r}) \le h(\mathbf{r})$ for all $\mathbf{r} \in M[t_0,1)$, then $g(\mathbf{r}) \overset{\le}{\cdot} h(\mathbf{r})$.

(e) If $g(\mathbf{r}) \to \infty$ for $\mathbf{r} \to \mathfrak{h}$ with $\mathbf{r} \in M$ and if $a \in \mathbb{R}$ and $0 < \varepsilon \in \mathbb{R}$ then $a \overset{\le}{\cdot} \varepsilon g(\mathbf{r})$.

(f) If $g(\mathbf{r}) \overset{\le}{\cdot} h(\mathbf{r})$, then $\underset{M \ni \mathbf{r} \to \mathfrak{h}}{\lim \inf} \, g(\mathbf{r}) \le \underset{M \ni \mathbf{r} \to \mathfrak{h}}{\lim \sup} \, h(\mathbf{r})$.

PROOF. (a) to (e) are trivial. In the case (f), there exists a measurable set E in M with $\Lambda_M(E) < \infty$ such that $g(\mathbf{r}) \le h(\mathbf{r})$ for all $\mathbf{r} \in M - E$. Take a sequence $\{t_j\}_{j \in \mathbb{N}}$ with $0 < t_j < 1$ such that $t_j \to 1$ for $j \to \infty$. By (6.28), $\mathbf{r}_j \in M[t_j,1) - E$ exists. Then $\mathbf{r}_j \to \mathfrak{h}$ for $j \to \infty$ and $g(\mathbf{r}_j) \le h(\mathbf{r}_j)$. Therefore

$$\underset{M \ni \mathbf{r} \to \mathfrak{h}}{\lim \inf} \, g(\mathbf{r}) \le \underset{j \to \infty}{\lim \inf} \, g(\mathbf{r}_j) \le \underset{j \to \infty}{\lim \inf} \, h(\mathbf{r}_j) \le \underset{M \ni \mathbf{r} \to \mathfrak{h}}{\lim \sup} \, h(\mathbf{r}) \qquad \text{q.e.d.}$$

LEMMA 6.13. Let $g : \mathbb{R}^n_+(\mathbf{h}) \to \mathbb{R}$ be an increasing function. Define

$$S = \sup\{g(\mathbf{r}) \mid \mathbf{r} \in \mathbb{R}^n_+(\mathbf{h})\} \leq \infty. \quad \text{Then}$$

$$g(\mathbf{r}) \to S \quad \text{for } \mathbf{r} \to \mathbf{h} \text{ with } \mathbf{r} \in M.$$

PROOF. The function $g(r\mathbf{h})$ of $r \in \mathbb{R}(0,1)$ increases. Therefore $g(r\mathbf{h}) \to s$ for $r \to \infty$. Obviously $s \leq S$. Take $T < S$. A point $\mathbf{r}_0$ with $0 \leq \mathbf{r}_0 < \mathbf{h}$ exists such that $T < g(\mathbf{r}_0)$. A number $r_0 \in \mathbb{R}(0,1)$ exists such that $\mathbf{r}_0 < r_0\mathbf{h}$. Then $T < g(\mathbf{r}_0) \leq g(r_0\mathbf{h}) \leq s$. Hence $S \leq s$. We find that $s = S$ and $g(r\mathbf{h}) \to S$ for $r \to 1$ if $0 < r < 1$.

Again take $T < S$. A number $r_0 = r_0(T) \in \mathbb{R}(0,1)$ exists such that $g(r_0\mathbf{h}) > T$. Take $\mathbf{r} \in M[r_0,1)$. Then $\mathbf{r} = \mathbf{x} + t(\mathbf{h} - \mathbf{x})$ with $r_0 \leq t < 1$ and $\mathbf{x} \in B$. Therefore $\mathbf{r} = (1 - t)\mathbf{x} + t\mathbf{h} \geq t\mathbf{h} \geq r_0\mathbf{h}$ which implies $T < g(r_0\mathbf{h}) \leq g(\mathbf{r}) \leq S$ for all $\mathbf{r} \in M[r_0,1)$. Hence $g(\mathbf{r}) \to S$ for $\mathbf{r} \to \mathbf{h}$ if $\mathbf{r} \in M$; q.e.d.

In order to eliminate Θ we need the following well-known Lemma.

LEMMA 6.14. (Hayman [12] Lemma 2.4 p. 38) Take $t_0 \in \mathbb{R}[0,1)$. Let $T : \mathbb{R}[t_0,1) \to \mathbb{R}$ be a continuous, increasing function. Assume that there is a positive number $c > 0$ such that $T(t) \geq c$ for all $t \in \mathbb{R}[t_0,1)$. Define

$$E = \left\{ t \in \mathbb{R}[t_0,1) \mid T\left(t + \frac{(1 - t)c}{eT(t)}\right) > 2T(t) \right\}. \quad (6.31)$$

Then E is open in $\mathbb{R}[t_0,1)$ with $\int_E (1/1 - t)\, dt \leq 2$.

PROPOSITION 6.15. Define Q by (6.29). Let $T : M \to \mathbb{R}$ be a continuous, increasing function. Assume that there are constants $c > 0$ and $s \in \mathbb{R}(0,1)$ such that $T(\mathbf{r}) \geq c$ for $\mathbf{r} \in M[s,1)$. Define $T_0 = \text{Max}(c,T)$ as a function on M. A continuous function $\mathbf{p} : M \to \mathbb{R}^n$ is defined by

$$\mathbf{p}(\mathbf{r}) = \mathbf{y}\left(t + \frac{(1 - t)c}{eT_0(\mathbf{r})},\, \mathbf{x}\right) \quad (6.32)$$

for all $\mathbf{r} \in M$ where $(t,\mathbf{x}) \in \mathbb{R}(0,1) \times B$ is uniquely determined by $\mathbf{r} = \mathbf{y}(t,\mathbf{x})$. Then $\mathbf{r} < \mathbf{p}(\mathbf{r}) < \mathbf{h}$ for $\mathbf{r} \in M$ and $\mathbf{p}(\mathbf{r}) \in \mathbb{R}^n_+(\mathbf{h})$. Let $\Theta(\mathbf{r})$ be the largest number such that

$\Theta(\mathbf{r})\mathbf{r} \leq \mathfrak{p}(\mathbf{r})$ for $\mathbf{r} \in M$. Then $\Theta(\mathbf{r}) > 1$. Define $c_0 = \log(8\|\mathfrak{h}\|^2/cs)$. Then

$$\log \frac{2\Theta(\mathbf{r})}{\Theta(\mathbf{r}) - 1} \leq \log T(\mathbf{r}) + 2 \log \frac{1}{\|\mathfrak{h} - \mathbf{r}\|} + c_0 \tag{6.33}$$

for all $\mathbf{r} \in M[s,1) - Q$. In particular

$$\log \frac{2\Theta(\mathbf{r})}{\Theta(\mathbf{r}) - 1} \overset{\cdot}{\lessgtr} \log T(\mathbf{r}) + 2 \log \frac{1}{\|\mathfrak{h} - \mathbf{r}\|} + c_0 \tag{6.34}$$

on M. Define $\gamma_0 = \text{Min}\{\|\mathfrak{h} - \mathbf{x}\| \mid \mathbf{x} \in \overline{B}\} > 0$ and

$$\gamma_3 = \mu_B(B)\gamma_0^{-p}\left(2 + \log \frac{1}{1 - s}\right). \tag{6.35}$$

Then there exists an open subset E of M with $\Lambda_M(E) \leq \gamma_3 < \infty$ such that

$$T(\mathfrak{p}(\mathbf{r})) \leq 2T(\mathbf{r}) \quad \text{for all } \mathbf{r} \in M - E. \tag{6.36}$$

In particular $T(\mathfrak{p}(\mathbf{r})) \overset{\cdot}{\lessgtr} 2T(\mathbf{r})$ on M.

   PROOF. Trivially T_0 is continuous and $T_0 \geq c$ on M. Take $\mathbf{r} \in M$. Then

$\mathbf{r} = \mathfrak{y}(t,\mathbf{x})$ with $0 < t < 1$ and $\mathbf{x} \in B$. Hence

$$0 < t < t + \frac{(1 - t)c}{eT(\mathbf{r})} < 1$$

$$0 < \mathbf{r} = \mathfrak{y}(t,\mathbf{x}) < \mathfrak{p}(\mathbf{r}) < \mathfrak{y}(t,\mathbf{x}) < \mathfrak{h}.$$

Consequently, $\Theta(\mathbf{r}) > 1$ and $\mathfrak{p}(\mathbf{r}) \in \overset{+}{\mathbb{R}}{}^n(\mathfrak{h})$. Take $\mathbf{r} \in M[s,1)$. Then $\mathbf{r} = \mathbf{x} + t(\mathfrak{h} - \mathbf{x})$
with $\mathbf{x} \in B$ and $s \leq t < 1$. Also

$$\mathfrak{p}(\mathbf{r}) = \mathbf{x} + \left(t + \frac{(1 - t)c}{eT(\mathbf{r})}\right)(\mathfrak{h} - \mathbf{x}) = \mathbf{r} + \frac{(1 - t)c}{eT(\mathbf{r})}(\mathfrak{h} - \mathbf{x}).$$

For each $j \in \mathbb{N}[1,n]$ we have

$$\Theta(\mathbf{r})r_j \leq r_j + \frac{(1 - t)c}{eT(\mathbf{r})}(b_j - x_j)$$

with equality for at least one j. Since $r_j > 0$ we obtain

$$\Theta(r) = 1 + \frac{(1 - t)c}{eT(r)} \; \underset{1 \leq j \leq n}{\text{Min}} \; \frac{b_j - x_j}{r_j} \; . \tag{6.37}$$

We have $b - r = (1 - t)(b - x)$. Hence $\|b - r\| = (1 - t)\|b - x\|$ and

$$\Theta(r) = 1 + \frac{\|b - r\|}{\|b - x\|} \; \frac{c}{eT(r)} \; \underset{1 \leq j \leq n}{\text{Min}} \; \frac{b_j - x_j}{r_j} \tag{6.38}$$

where $r_j = x_j + t(b_j - x_j) \geq s(b_j - x_j)$. Hence

$$\Theta(r) \leq 1 + \frac{1}{se} = \frac{se + 1}{se} \leq \frac{4}{se} \; . \tag{6.39}$$

Also $r_j = (1 - t)x_j + tb_j < (1 - t)b_j + tb_j = b_j$. Hence

$$\underset{1 \leq j \leq n}{\text{Min}} \; \frac{b_j - x_j}{r_j} \geq \underset{1 \leq j \leq n}{\text{Min}} \; \frac{b_j - x_j}{b_j} = \delta(x)$$

$$\Theta(r) \geq 1 + \frac{\|b - r\|}{\|b - x\|} \; \frac{c}{eT(r)} \; \delta(x) . \tag{6.40}$$

Now assume that $r \in M[s,1) - Q$. Then $1 > t \geq 1 - \delta(x)$ by (6.29). Consequently

$$\delta(x) \geq 1 - t = \frac{\|b - r\|}{\|b - x\|} \; .$$

Since $\|b - x\| \leq \|b\|$, we obtain

$$\Theta(r) \geq 1 + \frac{\|b - r\|^2}{\|b - x\|^2} \; \frac{c}{eT(r)} \geq 1 + \frac{\|b - r\|^2}{\|b\|^2} \; \frac{c}{eT(r)} \; . \tag{6.41}$$

Now (6.39) and (6.41) imply

$$1 \leq \frac{2\Theta(r)}{\Theta(r) - 1} \leq \frac{8\|b\|^2}{sc} \; \frac{T(r)}{\|b - r\|^2}$$

or

$$0 \leq \log \frac{2\Theta(r)}{\Theta(r) - 1} \leq \log T(r) + 2 \log \frac{1}{\|b - r\|} + \log \frac{8\|b\|^2}{sc}$$

for all $r \in M[s,1) - Q$, which proves (6.33). Now (6.28) and Lemma 6.11 prove (6.34).

The open set E is defined by $E = \{r \in M \mid T(\mathfrak{p}(r)) > 2T(r)\} \cup M[0,s)$. Then

$T(\mathfrak{p}(r)) \leq 2T(r)$ for all $r \in M - E$. For $x \in B$ and $t \in \mathbb{R}(0,1)$ define

$T_x(t) = T(\mathfrak{y}(t,x))$. Define

$$E(x) = \left\{ t \in \mathbb{R}[s,1) \mid T_x\left(t + \frac{(1-t)c}{eT_x(t)}\right) > 2T_x(t) \right\} \cup \mathbb{R}[0,s).$$

According to Lemma 6.14 we have

$$\int_{E(x)} \frac{dt}{1-t} \leq 2 + \log\frac{1}{1-s} \,.$$

Therefore Lemma 6.10 implies

$$\Lambda_M(E) \leq \mu_B(B)\gamma_0^{-p}\left(2 + \log\frac{1}{1-s}\right) = \gamma_3 < \infty \qquad\qquad \text{q.e.d.}$$

LEMMA 6.16. Let f be a non-constant meromorphic function on $\mathbb{D}(\mathfrak{h})$. Then

$T_f(r\mathfrak{h},s\mathfrak{h}) > 0$ if $0 < s < r < 1$.

PROOF. If f has a point of indeterminacy at 0, then $\iota_f(0) > 0$ and (5.17)

implies that $T_f(r\mathfrak{h},s\mathfrak{h}) \geq \iota_f(0) \log r/s > 0$. Hence we can assume that f is

holomorphic at 0. If $T_f(r\mathfrak{h},s\mathfrak{h}) = 0$ for some pair $0 < s < r < 1$, identity (5.14)

implies $A_f(t\mathfrak{h}) = 0$ for $s \leq t \leq r$. Then (5.5) and (5.13) show that $A_{f[z]}(t) = 0$ for

almost all $z \in \mathbb{D}\langle\mathfrak{h}\rangle$ and $s \leq t \leq r$. Hence f[z] is constant on $\mathbb{D}(t)$ and therefore on

$\mathbb{D}(1)$ for almost all $z \in \mathbb{D}\langle\mathfrak{h}\rangle$. By continuity f[z] is constant for all $z \in \mathbb{D}\langle\mathfrak{h}\rangle$. Let

$$f(z) = \sum_{\lambda=0}^{\infty} P_\lambda(z)$$

be the development into homogeneous polynomials at 0. Then

$$f(zz) = \sum_{\lambda=0}^{\infty} P_\lambda(z)z^\lambda$$

for all $z \in \mathbb{C}^n$ and $z \in \mathbb{C}$ sufficiently small. Because f(zz) is constant for $z \in \mathbb{D}\langle\mathfrak{h}\rangle$

and $z \in \mathbb{D}(1)$ we have $P_\lambda|\mathbb{D}\langle\mathfrak{h}\rangle \equiv 0$ if $\lambda > 0$. Hence $f \equiv P_0$ is constant.

Contradiction! q.e.d.

LEMMA 6.17. Let f be a non-constant meromorphic function on $\mathbb{D}(\mathfrak{h})$. Take

$\mathfrak{q} \in \mathbb{R}^n_+(\mathfrak{h})$ and $q > 0$ with $q\mathfrak{h} < \mathfrak{q}$. Then there exist numbers $c > 0$ and $s \in \mathbb{R}(0,1)$ such

that $r > \mathfrak{q}$ and $T(r,\mathfrak{q}) \geq c$ for all $r \in M[s,1)$.

PROOF. A number $q_0 \in \mathbb{R}(q,1)$ exists such that $\mathfrak{q} < q_0\mathfrak{h} < \mathfrak{h}$. Take $s \in \mathbb{R}(q_0,1)$.

By Lemma 6.16, $c = T_f(s\mathfrak{h}, q_0\mathfrak{h}) > 0$. Take $\mathfrak{r} \in M[s,1)$. Then

$\mathfrak{r} = \mathfrak{x} + t(\mathfrak{h} - \mathfrak{x}) = (1 - t)\mathfrak{x} + t\mathfrak{h}$ with $\mathfrak{x} \in B$ and $s \le t < 1$. Hence $\mathfrak{r} > s\mathfrak{h} > q_0\mathfrak{h} > \mathfrak{q}$

and

$$T(\mathfrak{r},\mathfrak{q}) \ge T(s\mathfrak{h},\mathfrak{q}) \ge T(s\mathfrak{h},q_0\mathfrak{h}) = c > 0 \qquad\qquad \text{q.e.d.}$$

Now, the Lemma of the logarithmic derivative follows easily, which constitutes

the main result of this paper.

THEOREM 6.18. Take $\mathfrak{h} \in \overset{+}{\mathbb{R}}{}^n$ and $\mathfrak{q} \in \overset{+}{\mathbb{R}}{}^n(\mathfrak{h})$. Let f be a non-constant meromorphic

function on $\mathbb{D}(\mathfrak{h})$. Let M be an approach cone. Take $\tau \in \mathbb{N}[1,n]$. Then

$$\int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log^+\left|\frac{df}{dz_\tau} \Big/ f\right| \Omega^n \overset{\le}{\cdot} 17 \log^+ T_f(\mathfrak{r},\mathfrak{q}) + 19 \log^+ \frac{1}{\|\mathfrak{h} - \mathfrak{r}\|} \qquad (6.42)$$

for $\mathfrak{r} \in M$.

PROOF. Take $q > 0$ with $q\mathfrak{h} < \mathfrak{q}$. Then $c > 0$ and $s \in \mathbb{R}(0,1)$ exist such that

$\mathfrak{r} > \mathfrak{q}$ and $T(\mathfrak{r},\mathfrak{q}) \ge c$ for all $\mathfrak{r} \in M[s,1)$. For $T(\mathfrak{r},\mathfrak{q})$ define $\mathfrak{p} : M \to \mathbb{R}^n$ by (6.32) and

$\Theta(\mathfrak{r})$ as the largest number such that

$$\mathfrak{r} < \Theta(\mathfrak{r})\mathfrak{r} \le \mathfrak{p}(\mathfrak{r}) < \mathfrak{h} \quad \text{for } \mathfrak{r} \in M.$$

Then (6.34) and (6.36) hold for $T(\mathfrak{r}) = T(\mathfrak{r},\mathfrak{q})$. Also we have (6.12). Let c_j be

positive constants. Take $\mathfrak{q} \le \mathfrak{r} < \mathfrak{h}$. Then

$$\int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log^+|f'_\tau/f| \Omega_n \le 8 \log^+ T_f(\Theta(\mathfrak{r})\mathfrak{r},\mathfrak{q}) + 9 \log \frac{2\Theta(\mathfrak{r})}{\Theta(\mathfrak{r}) - 1} + c_1$$

$$\le 8 \log^+ T_f(\mathfrak{p}(\mathfrak{r}),\mathfrak{q}) + 9 \log \frac{2\Theta(\mathfrak{r})}{\Theta(\mathfrak{r}) - 1} + c_1$$

$$\overset{\le}{\cdot} 8 \log^+(2T_f(\mathfrak{r},\mathfrak{q})) + 9 \log T_f(\mathfrak{r},\mathfrak{q}) + 18 \log \frac{1}{\|\mathfrak{h} - \mathfrak{r}\|} + c_2$$

$$\overset{\le}{\cdot} 17 \log^+ T_f(\mathfrak{r},\mathfrak{q}) + 19 \log^+ \frac{1}{\|\mathfrak{h} - \mathfrak{r}\|}$$

where the constant is swallowed by $\log(1/\|\mathfrak{h} - \mathfrak{r}\|)$ (see Lemma 6.12e) q.e.d.

Now, it is possible to derive the defect relation along the lines of Vitter [5]

and Stoll [6].

Naturally, Theorem 6.18 extends immediately to any differentiable operator

$$D = \sum_{j=1}^{n} A_j \frac{\partial}{\partial z_j}$$

where the A_j are bounded continuous functions. If we want Df/f to be meromorphic,

we have to take A_j as bounded holomorphic functions on $\mathbb{D}(\mathfrak{h})$. We even could take A_j

unbounded, if we add a correction term on the right-hand side in (6.42). If the A_j

are bounded holomorphic functions on $\mathbb{D}(\mathfrak{h})$, then Df/f is meromorphic. We have

$$m_{Df/f}(r,\infty) = \int_{\mathbb{D}\langle r\rangle} \log\sqrt{1 + |Df/f|^2}\, \Omega_n \le \int_{\mathbb{D}\langle r\rangle} \log^+ |Df/f|\,\Omega_n + \log 2$$

$$m_{Df/f}(r,\infty) \lesssim 17n \log^+ T_f(r,\mathfrak{q}) + 19n \log^+ \frac{1}{\|\mathfrak{h} - r\|}$$

if the constant c_2 in the proof includes $n \log 2$ and $n \log C$ where C is an upper

bound of the A_j.

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