Internat. J. Math. & Math. Sci.
Vol. 6 No. 3 (1983) 503-510

INFINITE MATRICES AND ABSOLUTE ALMOST CONVERGENCE

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(Received January 26, 1981 and in revised form August 16, 1982)

ABSTRACT. In 1973, Stieglitz [5] introduced a notion of F_B-convergence which provided a wide generalization of the classical idea of almost convergence due to Lorentz [1]. The concept of strong almost convergence was introduced by Maddox [3] who later on generalized this concept analogous to Stieglitz's extension of almost convergence [4]. In the present paper we define absolute F_B-convergence which naturally emerges from the concept of F_B-convergence.

KEY WORDS AND PHRASES. Infinite matrices, almost convergence, strong almost convergence, F_B-convergence, absolute F_B-convergence.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 40C05, 40D05.

1. INTRODUCTION.

Let ℓ_∞, c, and c_0 denote respectively the Banach spaces of bounded, convergent, and null sequences $x = (x_k)$ of complex numbers with norm $||x|| = \sup_k |x_k|$, and let v be the space of sequences of bounded variation, that is,

$$v = \{x: \ ||x|| \equiv \sum_{k=0}^{\infty} |x_k - x_{k-1}| < +\infty, \ x_{-1} = 0\}.$$

Suppose that $B = (B_i)$ is a sequence of infinite complex matrices with $B_i = (b_{np}(i))$. Then $x \in \ell_\infty$ is said to be F_B-convergent [5], to the value Lim Bx, if

$$\lim_{n\to\infty} (B_i x)_n = \lim_{n\to\infty} \sum_{p=0}^{\infty} b_{np}(i) x_p = \text{Lim } Bx,$$

uniformly for $i = 0,1,2,\ldots$.

The space F_B of F_B-convergent sequences depends on the fixed chosen sequence $B = (B_i)$. In case $B = B_0 = (I)$ (unit matrix), the space F_B is same as c and, for

$B = B_1 = (B_i^{(1)})$, it is same as the space f of almost convergent sequences [1], where $B_i^{(1)} = (b_{np}^{(1)}(i))$ with

$$b_{np}^{(1)}(i) = \begin{cases} \frac{1}{n+1}, & i \le p \le i+n \\ 0 & \text{elsewhere} \end{cases}$$

Maddox [4] generalized strong almost convergence by saying that $x_p \to s[F_B]$ if and only if

$$\sum_p b_{np}(i)\ |x_p - s| \to 0\ (n \to \infty, \text{ uniformly in } i) \tag{1.1}$$

assuming that the series in (1.1) converges for each n and i.

In particular, if $B = B_0$, the $[F_B] = c$; if $B = B_i$, then $[F_B] = [f]$, the space of strongly almost convergent sequences [3]. We shall write $e_k = (0,0,\dots,0,1$ (kth entry), $0,\dots)$ and $e = (1,1,1,\dots)$.

Let s be the space of all complex sequences and

$$d_B = \{x \in s:\ \text{Lim } Bx = \lim_{n\to\infty} (B_i x)_n \text{ exists for each } i\}$$

$$F_B = \{x \in (d_B \cap \ell_\infty):\ \lim_{n\to\infty} t_n(i,x) \text{ exists uniformly in } i, \text{ and the limit is independent of } i\},$$

where

$$t_n(i,x) = \begin{bmatrix} \sum_{p=0}^{\infty} b_{np}(i)x_p, & (n \ge 1) \\ \sum_{p=0}^{\infty} \beta_{0p}(i)x_p, & (n = 0) \\ 0, & (n = -1) \end{bmatrix}$$

and

$$\beta_{0p}(i) = \begin{cases} 1 & \text{if } p = i, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\phi_n(i,x) = t_n(i,x) - t_{n-1}(i,x).$$

Therefore, we have

$$\phi_n(i,x) = \begin{bmatrix} \sum_{p=0}^{\infty} [b_{np}(i) - b_{n-1,p}(i)]x_p, & (n \ge 1) \\ \sum_{p=0}^{\infty} \beta_{0p}(i)\ x_p, & (n = 0) \end{bmatrix} \tag{1.2}$$

DEFINITION. Let $B = (B_i)$ be a sequence of infinite matrices with $B_i = (b_{np}(i))$. A sequence $x \in \ell_\infty$ is said to be absolutely F_B-convergent if $\sum_{n=0}^{\infty} |\phi_n(i,x)|$ converges uniformly for $i \geq 0$, and $\lim_{n\to\infty} t_n(i,x)$ which must exist should take the same value for all i. We denote the space of absolute F_B-convergent sequences by $v(B)$.

2. THE MAIN RESULT.

In this note, we denote by $(v,v(B))$ the set of matrices which give new classes of absolute B-conservative matrices and absolute almost B-conservative matrices.

Let A be any infinite complex matrix for which the pth row-sum converges for a given x for all x in some class.

We have

$$A_p\, x = (Ax)_p = \sum_{k=0}^{\infty} a_{pk}\, x_k$$

and

$$(B_i x)_n = \sum_{p=0}^{\infty} b_{np}(i)\, x_p .$$

Therefore,

$$(B_i Ax)_n = \sum_{p=0}^{\infty} b_{np}(i)\, A_p\, x = \sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} a_{pk}\, x_k,$$

and, assuming the interchange of order of summation can be justified (see lemma), we get that

$$(B_i Ax)_n = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} b_{np}(i)\, a_{pk}\, x_k \tag{2.1}$$

Now, by (1.2) and (2.1), we have

$$\phi_n(i,Ax) = t_n(i,Ax) - t_{n-1}(i,Ax)$$

$$= \begin{cases} \sum_{p=0}^{\infty} [b_{np}(i) - b_{n-1,p}(i)]A_p x, & (n \geq 1), \\ \sum_{p=0}^{\infty} \beta_{0p}(i)\, A_p x, & (n = 0), \end{cases}$$

$$= \sum_{k=0}^{\infty} g_{nk}(i)\, x_k, \tag{2.2}$$

where

$$g_{nk}(i) = \begin{cases} \sum_{p=0}^{\infty} [b_{np}(i) - b_{n-1,p}(i)]a_{pk}, & (n \geq 1), \\ \sum_{p=0}^{\infty} \beta 0p\ (i)\ a_{pk}, & (n = 0). \end{cases}$$

THEOREM. Let $B = (B_i)$ be a sequence of infinite matrices with

$$\sup_n \sum_{p=0}^{\infty} |b_{np}(i)| < \infty, \quad \text{for each } i.$$

Let A be an infinite matrix. Then A: $v \to v(B)$ if and only if

(i) $\sup_{p,k} |\sum_{\ell=k}^{\infty} a_{p\ell}| < \infty$,

(ii) there is an N such that for $r,i = 0,1,2,\ldots$

$$\sum_{n=N}^{\infty} |\sum_{k=0}^{r} g_{nk}(i)| \leq K \quad \text{(constant)},$$

(iii) $(a_{pk})_{p\geq 0} \in v(B)$ for each k, and

(iv) $(\sum_{k=0}^{\infty} a_{pk})_{p\geq 0} \in v(B)$.

Let $A \in (v,v(B))$. For each k, let a_{pk} be F_B-convergent with limit α_k. And let $\sum_{k=0}^{\infty} a_{pk}$ be F_B-convergent with limit α. (In each case, limit is taken for $p \geq 0$).

If $x = (x_k) \in v$, then

$$\lim_{n\to\infty} t_n(i,Ax) = \alpha \lim_{k\to\infty} x_k + \sum_{k=0}^{\infty} (x_k - \lim_{k\to\infty} x_k)\ \alpha_k.$$

We use the following lemma in the proof.

LEMMA. If either the necessity part or the sufficiency part of the theorem holds, then, for $x \in v$,

$$\sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} a_{pk}x_k = \sum_{k=0}^{\infty} x_k \sum_{p=0}^{\infty} b_{np}(i)\ a_{pk}.$$

PROOF. If either A: $v \to v(B)$ or the conditions (i)-(iv) of the theorem hold, then by partial summation, for $x \in v$,

$$\sum_{k=0}^{\infty} a_{pk}x_k = \sum_{k=0}^{\infty} d_{pk}\ (x_k - x_{k-1})$$

where $d_{pk} = \sum_{\ell=k}^{\infty} a_{p\ell}$. Since condition (i) holds, d_{pk} is bounded for all p,k. Thus

$$\sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} a_{pk}x_k = \sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} d_{pk} (x_k - x_{k-1})$$
$$= \sum_{k=0}^{\infty} (x_k - x_{k-1}) \sum_{p=0}^{\infty} b_{np}(i) d_{pk},$$

(where the inversion is justified by absolute convergence)

$$= \sum_{k=0}^{\infty} x_k \sum_{p=0}^{\infty} b_{np}(i) a_{pk}$$

since

$$\lim_{k\to\infty} x_k \sum_{p=0}^{\infty} b_{np}(i) d_{pk} = 0.$$

PROOF OF THEOREM. Necessity. Condition (i) follows from the fact that A: $v \to \ell_\infty$. Since $e_k, e \in v$, necessity of (iii) and (iv) is obvious.

It is clear that, for fixed p and j,

$$x \to \sum_{k=0}^{j} a_{pk} x_k$$

is a continuous linear functional on v. We are given that, for all $x \in v$, it tends to a limit as $j \to \infty$ (for fixed p) and hence, by the Banach-Steinhaus Theorem [2], this limit $A_p x$ is also a continuous linear functional on v.

We observe that, although $\sum_{n=0}^{\infty} |\emptyset_n(i,Ax)|$ is uniformly convergent in i, it needs not be uniformly bounded in i. For example, if $\emptyset_o(i,Ax) = i$ and $\emptyset_n(i,Ax) = o$ ($n \geq 1$ and i), then $\sum_{n=o}^{\infty} |\emptyset_n(i,Ax)|$ is uniformly convergent in $i \geq o$ but not uniformly bounded. Now, we can say that uniform convergence bears only on the behaviour of $\emptyset_n(i,Ax)$ for sufficiently large n. Thus, by definition, there is an m such that

$$q_{m,i}(x) = \sum_{n=m}^{\infty} |\emptyset_n(i,Ax)|.$$

For $m \geq o$, $i \geq o$, $q_{m,i}$ is a continuous seminorm on v, and there is an integer N such that $\{q_{N,i}\}_{i\geq 0}$ is pointwise bounded on v. Such an N exists. For suppose not. Then for r = 0,1,2,... ther exists $x_r \in v$ with

$$\sup_{i\geq o} q_{r,i}(x_r) = \infty.$$

By the principle of condensation of singularities [6],

$$\{x \in v: \sup_{i\geq o} q_{r,i}(x) = \infty \quad \text{for} \quad r = 0,1,2,...\}$$

is of second category in v and hence nonempty, i.e., there is $x \in v$ with

$$\sup_{i\geq o} q_{r,i}(x) = \infty \quad \text{for} \quad r = 0,1,2,\ldots$$

But this contradicts the fact that to each $x \in v$ there exists an integer N_x with $\sup_{i\geq o} q_{N_x,i}(x) < \infty$.

Now, by another application of the Banach-Steinhaus Theorem, there exists a constant M such that

$$q_{N,i}(x) \leq M||x||. \tag{2.3}$$

Apply (2.3) with $x = (x_k)$ defined by $x_k = 1$ for $k \leq r$ and o for $k > r$. Hence (ii) must hold.

Sufficiency. Suppose that the conditions (i)-(iv) hold and that $x \in v$. We have defined v(B) as a subspace of ℓ_∞. Thus, in order to show that $Ax \in v(B)$, it is necessary to prove that Ax is bounded. By virtue of condition (i), this follows immediately.

Now, it follows from (iv) and the lemma that

$$\sum_{k=0}^{\infty} g_{nk}(i)$$

converges for all i,n. Hence, if we write

$$h_{nk}(i) = \sum_{\ell=k}^{\infty} g_{n\ell}(i),$$

then $h_{nk}(i)$ is defined, also for fixed i,n,

$$h_{nk}(i) \to 0 \tag{2.4}$$

as $k\to\infty$. Now condition (iv) gives us that

$$\sum_{n=0}^{\infty} |h_{n0}(i)| \tag{2.5}$$

converges uniformly in i, and, for suitable chosen N,

$$\sum_{n=N}^{\infty} |h_{n0}(i)| \tag{2.6}$$

is bounded. By virtue of condition (iii), for fixed k, we get that

$$\sum_{n=0}^{\infty} |g_{nk}(i)|$$

converges uniformly in i. Since

$$h_{nk}(i) = h_{n0}(i) - \sum_{\ell=0}^{k-1} g_{n\ell}(i), \tag{2.7}$$

it follows that, for fixed k,

$$\sum_{n=0}^{\infty} |h_{nk}(i)| \tag{2.8}$$

converges uniformly in i.

Now

$$\begin{aligned}\emptyset(i,Ax) &= \sum_{k=0}^{\infty} g_{nk}(i)\ x_k \\ &= \sum_{k=0}^{\infty} [h_{nk}(i) - h_{n,k+1}(i)]\ x_k \\ &= \sum_{k=0}^{\infty} h_{nk}(i)\ (x_k - x_{k-1}),\end{aligned} \tag{2.9}$$

by (2.4) and the boundedness of x_k.

Condition (ii) and the boundedness of (2.6) show that

$$\sum_{n=N}^{\infty} |h_{nk}(i)| \tag{2.10}$$

is bounded for all k,i. We can make

$$\sum_{k=k_o+1}^{\infty} |x_k - x_{k-1}|$$

arbitrarily small by choosing k_o sufficiently large. It therefore follows that, given $\varepsilon > o$, we can choose k_o so that, for all i,

$$\sum_{n=N}^{\infty} \left| \sum_{k=k_o+1}^{\infty} h_{nk}(i)(x_k - x_{k-1})\right| < \varepsilon. \tag{2.11}$$

By the uniform convergence of (2.8), it follows that, once k_o has been chosen, we can choose n_o so that, for all i,

$$\sum_{n=n_o+1}^{\infty} \left| \sum_{k=o}^{k_o} h_{nk}(i)(x_k - x_{k-1})\right| < \varepsilon.$$

It follows from (2.11) that the same inequality holds when $\sum\limits_{n=N}^{\infty}$ is replaced by $\sum\limits_{n=n_o+1}^{\infty}$; hence, for all i,

$$\sum_{n=n_o+1}^{\infty} \left| \sum_{k=0}^{\infty} h_{nk}(i)(x_k - x_{k-1})\right| < 2\varepsilon. \tag{2.12}$$

Hence,

$$\sum_{n=n_o+1}^{\infty} |\emptyset_n(i,Ax)| < 2\varepsilon.$$

Thus

$$\sum_{n=0}^{\infty} |\emptyset_n(i,Ax)|$$

converges uniformly.

Now, by virtue of (2.9), we have

$$\lim_{n\to\infty} t_n(i,Ax) - t_{N-1}(i,Ax) = \sum_{n=N}^{\infty} \sum_{k=0}^{\infty} h_{nk}(i)\ (x_k - x_{k-1})$$

$$= \sum_{k=0}^{\infty} (x_k - x_{k-1}) \sum_{n=N}^{\infty} h_{nk}(i) \qquad (2.13)$$

the assertion being justified by absolute convergence because of the boundedness of (2.10). By (2.7), we have

$$\sum_{n=N}^{\infty} h_{nk}(i) = \sum_{n=N}^{\infty} h_{no}(i) - \sum_{\ell=0}^{k-1} \sum_{n=N}^{\infty} g_{n\ell}(i)$$

$$= \alpha - \sum_{\ell=0}^{k-1} \alpha_\ell - \sum_{\ell=k}^{\infty} \sum_{p=0}^{\infty} b_{N-1,p}(i)\ a_{p\ell}.$$

Thus,

$$\lim_{n\to\infty} t_n(i,Ax) = \alpha \lim_{k\to\infty} x_k + \sum_{k=0}^{\infty} (x_k - \lim_{k\to\infty} x_k)\ \alpha_k.$$

This completes the proof.

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