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SOME REMARKS ON THE SPACE $R^2(E)$

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ABSTRACT. Let E be a compact subset of the complex plane. We denote by R(E) the algebra consisting of the rational functions with poles off E. The closure of R(E) in $L^p(E)$, $1 \leq p < \infty$, is denoted by $R^p(E)$. In this paper we consider the case p = 2. In section 2 we introduce the notion of weak bounded point evaluation of order β and identify the existence of a weak bounded point evaluation of order β, $\beta > 1$, as a necessary and sufficient condition for $R^2(E) \neq L^2(E)$. We also construct a compact set E such that $R^2(E)$ has an isolated bounded point evaluation. In section 3 we examine the smoothness properties of functions in $R^2(E)$ at those points which admit bounded point evaluations.

KEY WORDS AND PHRASES. Rational functions, compact set, L^p-spaces, bounded point evaluation, weak bounded point evaluation, Bessel capacity.

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1. INTRODUCTION.

Let E be a compact subset of the complex plane $\mathbb{C}$. For each p, $1 \leq p < \infty$, let $L^p(E)$ be the linear space of all complex valued functions f for which $|f|^p$ is integrable with the usual norm

$$\left\{\int\int_E |f(z)|^p \, dm(z)\right\}^{1/p},$$ where m denotes the two dimensional

Lebesque measure. Denote by R(E) the subspace of all rational functions having no poles on E and let $R^p(E)$ be the closure of R(E) in $L^p(E)$. A point $z_0 \in E$ is said to be a bounded point evaluation (BPE) for $R^p(E)$, if there is a constant F such that

$$|f(z_0)| \leq F \left\{ \int_E |f(z)|^p \, dm(z) \right\}^{1/p}, \text{ for all } f \in R(E) . \qquad (1.1)$$

In [1] Brennan showed that $R^p(E) = L^p(E)$, $p \neq 2$, if and only if no point of E is a BPE for $R^p(E)$. The theorem is not true for p = 2 (See Fernström [2] or Fernström and Polking [3].) In this paper we show that if the right hand side of (1) is made slightly larger a corresponding theorem is true for p = 2. We also show that this theorem is best possible.

If $z_0 \in E$ is a BPE for $R^p(E)$ there is a function $g \in L^q(E)$, $\frac{1}{p} + \frac{1}{q} = 1$, such that $f(z_0) = \int_E f(z)g(z)dm(z)$ for all $f \in R(E)$. The function g is called a representing function for z_0. Let $B(z,\delta)$ denote the ball with radius δ and centre at z. We say that a set A, $A \subset \mathbb{C}$, has full area density at z if $m(A \cap B(z,\delta))m(B(z,\delta))^{-1}$ tends to one when δ tends to zero.

Suppose now that z_0 is a BPE for $R^p(E)$, $2 < p$, represented by $g \in L^q(E)$ and $(z-z_0)^{-s} \phi(|z-z_0|)^{-1} g \in L^q(E)$, where s is a nonnegative integer and ϕ is a non-decreasing function such that $r\,\phi(r)^{-1} \searrow 0$ when $r \searrow 0$. Then for every $\varepsilon > 0$ there is a set E_0 in E having full area density at z_0 such that for every $f \in R(E)$ and for all $\tau \in E_0$,

$$\left| f(\tau) - f(z_0) - \frac{f'(z_0)}{1!}(\tau - z_0) - \ldots \frac{f^{(s)}(z_0)}{s!}(\tau - z_0)^s \right|$$
$$\leq |\tau - z_0|^s \, \phi(|\tau - z_0|) \left\{ \int_E |f(z)|^p \, dm(z) \right\}^{1/p} .$$

This theorem is due to Wolf [8].

We shall show that the theorem of Wolf is not true for p = 2. We shall also show that a slightly weaker result is true and that this result is best possible. The main tool to show this is to construct a compact set E with exactly one bounded point derivation for $R^2(E)$. A point $z_0 \in E$ is a bounded point derivation (BPD) of order s for $R^p(E)$ if the map $f \to f^{(s)}(z_0)$, $f \in R(E)$, extends from R(E) to a bounded linear functional on $R^p(E)$.

2. BPE'S AND APPROXIMATION IN THE MEAN BY RATIONAL FUNCTIONS.

Denote the Bessel kernel of order one by G where G is defined in terms of its Fourier transform by

$$\hat{G}(z) = (1 + |z|^2)^{-\frac{1}{2}}.$$

For $f \in L^2(\mathbb{C})$ we define the potential

$$u^f(z) = \int G(z-\tau)\, f(\tau)\, dm\,(\tau).$$

The Bessel capacity C_2 for an arbitrary set X, $X \subset \mathbb{C}$, is defined by $C_2(X) = \inf \int |f(z)|^2\, dm(z)$, where the infimum is taken over all $f \in L^2(\mathbb{C})$ such that $f(z) \geq 0$ and $u^f(z) \geq 1$ for all $z \in X$. The set function C_2 is subadditive, increasing, translation invariant and

$$C_2(B(z,\delta)) \approx \left(\log \frac{1}{\delta}\right)^{-1}, \quad \delta \leq \delta_0 < 1.$$

For further details about this capacity see Meyers [5].

The BPD's can be described by the Bessel capacity. Let $A_n(z_0)$ denote the annulus $\left\{z;\ 2^{-n-1} < |z-z_0| \leq 2^{-n}\right\}$. The following theorem is proved in [3]:

Theorem 2.1 Let E be a compact set. Then z is a BPD of order s for $R^2(E)$ if and only if

$$\sum_{n=0}^{\infty} 2^{2n(s+1)}\, C_2(A_n(z) - E) < \infty .$$

Definition Set

$$L_{z_0}(z) = \begin{cases} \log \dfrac{1}{|z-z_0|} & \text{for } |z-z_0| \leq \dfrac{1}{e} \\ 1 & \text{for } |z-z_0| \geq \dfrac{1}{e}. \end{cases}$$

A point $z_0 \in E$ is called a weak bounded point evaluation (w BPE) of order β, $\beta \geq 0$, for $R^2(E)$ if there is a constant F such that

$$|f(z_0)| \leq F \left\{ \int_E |f(z)|^2\, L_{z_0}^{\beta}(z)\, dm(z) \right\}^{\frac{1}{2}}$$

for all $f \in R(E)$.

We are now going to generalize theorem 2.1 in two directions.

Theorem 2.2 Let s be a nonnegative integer and E a compact set. Suppose that z_0 is a BPE for $R^2(E)$ represented by $g \in L^2(E)$ and that ϕ is a positive, nondecreasing function defined on $(0,\infty)$ such that $r\,\phi(r)^{-1}$ is non-decreasing and tends to zero when $r \to 0^+$. Then z_0 is represented by a function $g \in L^2(E)$ such that

$$\frac{g}{(z-z_0)^s\, \phi(|z-z_0|)} \in L^2(E)$$

if and only if

$$\sum_{n=0}^{\infty} 2^{2n(s+1)} \phi(2^{-n})^{-2} C_2(A_n(z_0) - E) < \infty.$$

Theorem 2.3 Let E be a compact set. Then z is a w BPE of order β for $R^2(E)$ if and only if

$$\sum_{n=1}^{\infty} n^{-\beta} 2^{2n} C_2(A_n(z) - E) < \infty .$$

The proofs of theorem 2.2 and theorem 2.3 are almost the same as the proof of theorem 2.1. We omit the proofs. Wolf proved in [8] that the condition $\sum_{n=0}^{\infty} 2^{2n(s+1)} \phi(2^{-n})^{-2} C_2(A_n(z_0 - E)) < \infty$ is necessary in theorem 2.2.

The compact sets E for which $R^2(E) = L^2(E)$ can be described in terms of the Bessel Capacity. The following theorem is proved in Hedberg [4] and Polking [6].

Theorem 2.4 Let E be a compact set. Then the following are equivalent.

(i) $R^2(E) = L^2(E)$.

(ii) $C_2(B(z,\delta)-E) = C_2(B(z,\delta))$ for all balls $B(z,\delta)$.

(iii) $\limsup_{\delta\to 0} \dfrac{C_2(B(z,\delta) - E)}{\delta^2} > 0$ for all z.

If we combine theorem 2.3 and theorem 2.4 we get the following theorem.

Theorem 2.5 Let $\beta > 1$ and E be a compact set. Then $L^2(E) = R^2(E)$ if and only if E admits no w BPE of order β for $R^2(E)$.

Now we shall show that theorem 2.5 is not true for $\beta \leq 1$. We first need the following theorem.

Theorem 2.6 There is a compact set E such that

(i) $C_2(B(0,\tfrac{1}{2}) - E) < C_2(B(0,\tfrac{1}{2}))$

(ii) $\sum_{n=1}^{\infty} n^{-1} 2^{2n} C_2(A_n(z) - E) = \infty$ for all z.

The proof is a modification of a proof in [2] or [3], where a weaker theorem is proved. Since we shall need the construction of E later, we give some details.

Proof. There are constants F_1 and F_2 such that

$$F_1(\log \tfrac{1}{\delta})^{-1} \leq C_2(B(z,\delta)) \leq F_2(\log \tfrac{1}{\delta})^{-1} \quad \text{for all } \delta,\ \delta \leq \delta_0 < 1.$$

Choose α, $\alpha \geq 1$, such that

$$\frac{F_2}{\alpha} \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} < C_2(B(0,\tfrac{1}{2})).$$

Let A_0 be the closed unit square with centre at the origin. Cover A_0 with 4^n squares with side 2^{-n}. Call the squares $A_n^{(i)}$, $i = 1,2,\ldots,4^n$. In every $A_n^{(i)}$ put an open disc $B_n^{(i)}$ such that $B_n^{(i)}$ and $A_n^{(i)}$ have the same centre and the radius of $B_n^{(i)}$ is $\exp(-\alpha 4^n\, n \log^2 n)$. Repeat the construction for all n, $n \geq 2$.

Set

$$E = A_0 - \bigcup_{n=2}^{\infty} \bigcup_{i=1}^{4^n} B_n^{(i)} .$$

The subadditivity of C_2 now gives (i).

In order to prove (ii) it is enough to prove

$$C_2(A_n^{(i)} - E) \geq \frac{F_1}{32\alpha\, 4^n \log n} \quad \text{for all } n,\ n \geq n_0. \tag{2.1}$$

Consider all $B_k^{(i)}$, $n \leq k \leq n^2$, such that $B_k^{(i)} \subset A_n^{(i)}$.

We get 4^ℓ discs with radius $\exp(-\alpha 4^{n+\ell}(n+\ell)\log^2(n+\ell))$, $0 \leq \ell \leq n^2 - n$.

Call the discs

$$D_n^{(r)} , \ r = 1,2,\ldots, \frac{4^{n^2-n+1} - 1}{3} .$$

Thus

$$\frac{F_1}{\alpha 4^n} \sum_{j=n}^{n^2} \frac{1}{j \log^2 j} \leq \sum_r C_2(D_n^{(r)}) \leq \frac{F_2}{\alpha 4^n} \sum_{j=n}^{n^2} \frac{1}{j \log^2 j} .$$

Set $D_n = \bigcup_r D_n^{(r)}$.

Since the distances between the discs are large compared to their radii, it can be shown that

$$C_2(D_n) \geq \frac{1}{8} \sum_r C_2(D_n^{(r)}) \text{ , if n is large.}$$

(See theorem 2' in [2] or theorem 2 in [3] for a proof.)

Thus if n is large,

$$C_2(A_n^{(i)} - E) \geq C_2(D_n) \geq \frac{F_1}{8\alpha 4^n} \sum_{j=n}^{n^2} \frac{1}{j \log^2 j} \geq \frac{F_1}{16\alpha 4^n \log n} ,$$

which is (2.1)

q.e.d.

Theorem 2.7 There is a compact set E such that

(i) $L^2(E) = R^2(E)$

(ii) E has no w BPE of order one for $R^2(E)$.

Proof The theorem follows immediately from theorem 2.3, 2.4, and 2.6.

3. BPE'S AND SMOOTHNESS PROPERTIES OF FUNCTIONS IN $R^2(E)$.

In this section we treat the theorem of Wolf mentioned in the introduction for the case p = 2.

Theorem 3.1 Let ϕ be a positive, nondecreasing function defined on $(0,\infty)$ such that $r\,L_0(r)\,\phi(r)^{-1}$ is nondecreasing and $r\,L_0(r)\,\phi(r)^{-1} \to 0$ when $r \to 0^+$. Suppose that z_0 is a BPE for $R^2(E)$ represented by g and $(z-z_0)^{-s}\,\phi(|z-z_0|)^{-1}\, g \in L^2(E)$, where s is a nonnegative integer. Then for every $\beta > 1$ and $\varepsilon > 0$ there is a set E_0 in E, having full area density at z_0, such that for every $f \in R(E)$ and every $\tau \in E_0$

$$\left| f(\tau) - f(z_0) - \frac{f'(z_0)}{1!}(\tau-z_0) - \dots - \frac{f^{(s)}(z_0)}{s!}(\tau-z_0)^s \right|$$

$$\leq \varepsilon |\tau-z_0|^s\, \phi(|\tau-z_0|) \left\{ \iint_E |f(z)|^2\, L^{\beta}_{z_0}(z) \right\}^{\frac{1}{2}} .$$

The proof of theorem 3.1 is only a minor modification of the proof of theorem 4.1 in [8]. Moreover, there is a proof of theorem 3.1 for $\beta = 2$ in Wolf [7]. We omit the proof.

Remark. Let $z_0 \in \partial E$ (the boundary of E) be both a BPE for $R^2(E)$ and the vertex of a sector contained in Int E. Let L be a line which passes through z_0 and bisects the secotr. Let $\varepsilon > 0$ and let ϕ be as in theorem 2.2. For those $y \in L \cap E$ that are sufficiently near z_0 Wolf showed in [9] that

$$|f(y)-f(z_0)| \leq \varepsilon\, \phi(|y-z_0|) \left\{ \iint | f(z)|^2\, dm(z) \right\}^{\frac{1}{2}} \quad \text{for all } f \in R(E).$$

Our next step is to prove that theorem 3.1 is not true for $\beta = 1$. We first need a theorem, which we think is interesting in itself.

Theorem 3.2 Let s be a nonnegative integer. Then there is a compact set E such that

(i) $\sum_{n=1}^{\infty} n^{-1}\, 2^{2n}\, C_2(A_n(z)-E) = \infty$ if $z \neq 0$.

(ii) $\sum_{n=1}^{\infty} 2^{2n(s+1)}\, C_2(A_n(0)-E) < \infty$.

Proof We shall modify the set constructed in the proof of theorem 2.6. Let $B_j^{(k)}$ denote the same discs as in that proof. Let all $B_j^{(k)}$ which intersect $A_1(0)$ be denoted by $A_{11}, A_{12}, A_{13}, \ldots$ so that their diameters are decreasing.

Choose j_1 so that

$$2^{2(s+1)} \sum_{j>j_1} C_2(A_{1j}) < 2^{-1}$$

and $$\operatorname{diam}(A_{1j_1}) < 2^{-3}.$$

Suppose that we have chosen $j_1, \ldots j_n$. Let all $B_j^{(k)}$ which intersect $A_{n+1}(0)$ and which do not coincide with $A_{11}, \ldots, A_{1j_1}, \ldots, A_{n1}, \ldots, A_{nj_n}$, be denoted by $A_{n+1\ 1}, A_{n+2\ 2}, A_{n+3\ 3}, \ldots$ so that their diameters are decreasing.

Choose j_{n+1} so that

$$2^{2(n+1)(s+1)} \sum_{j>j_{n+1}} C_2(A_{n+1\ j}) < 2^{-(n+1)}$$

and $$\operatorname{diam}(A_{n+1\ j_{n+1}}) < 2^{-(n+3)} .$$

Let A_0 be the closed unit square with centre at the origin. Set $E = A_0 -$ (The union of all $B_j^{(k)}$ such that $B_j^{(k)} \neq A_{nm}$, $1 \leq n < \infty$ and $1 \leq m \leq j_n$).

We have

$$\sum_{n=1}^{\infty} 2^{2n(s+1)} C_2(A_n(0) - E) \leq \sum_{n=1}^{\infty} 2^{-n} < \infty .$$

Let $z \neq 0$. If ℓ is large all $B_j^{(k)}$, $B_j^{(k)} \subset A_\ell(z)$, differ from A_{nm}, $1 \leq n < \infty$ and $1 \leq m \leq j_n$.

Now exactly as in proof of theorem 2.6 it follows

$$\sum_{n=1}^{\infty} n^{-1} 2^{2n} C_2(A_n(z) - E_1) = \infty$$

q.e.d.

Corollary 3.3 There is a compact set E with exactly one BPD of order s for $R^2(E)$.

Proof Just combine theorem 3.2 and 2.1.

Remark The situation for $p \neq 2$ is different. In [1] Brennan showed that if almost all points $z \in E$, E compact, are not BPE for $R^p(E)$, E admits no BPE's for $R^2(E)$.

Theorem 3.4 Let s be a nonnegative integer and ϕ be as in theorem 2.2. Then there is a compact set E such that

(i) z_0 is a BPE for $R^2(E)$.

(ii) There is a representing function g for z_0 that satisfies

$$(z-z_0)^{-s}\ \phi(|z-z_0|)^{-1}\ g \in L^2(E)\ .$$

(iii) For every $\tau \in E$, $\tau \neq z_0$, and every positive integer n there is a function $f \in R(E)$ such that

$$\left| f(\tau)-f(z_0)- \frac{f'(z_0)}{1!}(\tau-z_0) - \dots - \frac{f^{(s)}(z_0)}{s!}(\tau-z_0)^s \right| >$$

$$> n \left\{ \int_E |f(z)|^2\ L_{z_0}(z)dm(z) \right\}^{\frac{1}{2}}\ .$$

Proof Theorem 3.2 gives that there is a compact set E such that

$$\sum_{n=1}^{\infty} n^{-1}\ 2^{2n} C_2(A_n(z) - E) = \infty\ ,\ z \neq z_0$$

$$\sum_{n=1}^{\infty} 2^{2n(s+1)}\ \phi(2^{-n})^{-2}\ C_2(A_n(z_0) - E) < \infty\ .$$

Now theorem 2.1 gives (i) and theorem 2.2 gives (ii). Moreover theorem 2.1 gives that z_0 is a BPD of order s for $R^2(E)$ and theorem 2.3 that τ is not a w BPE of order 1 for $R^2(E)$. This gives (iii).

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