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RESEARCH NOTES

A CHARACTERIZATION OF THE DESARGUESIAN PLANES OF ORDER q^2 BY SL(2,q)

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ABSTRACT. The main result is that if the translation complement of a translation plane of order q^2 contains a group isomorphic to SL(2,q) and if the subgroups of order q are elations (shears), then the plane is Desarguesian. This generalizes earlier work of Walker, who assumed that the kernel of the plane contained GF(q).

KEY WORDS AND PHRASES. Translation planes, translation complement, elations.
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THEOREM. Let π be a translation plane of order q^2, where $q = p^r$ and p is a prime. Let $G \cong SL(2,q)$ be a subgroup of the translation complement of π whose elements of order p are elations. Then π is a Desarguesian plane.

This theorem is a special case required in the classification of all translation planes π of order q^2 which admit a collineation group $G \cong SL(2,q)$ [1, 2]. That classification is a generalization of the work of Walker and Schaeffer [3, 4], who assume, in addition, that the kernel of π contains GF(q).

To begin the proof, let W be a vector space of dimension 2r over GF(p). Since

π is a $4r$-dimensional vector space over GF(p), we may represent π as $W \oplus W$ so that the points of π are vectors (x,y), where $x,y \in W$. The components of π (i.e., the lines containing (0,0)) have the form $\{(0,y): y \in W\}$ and $\{(x,xA): x \in W\}$ for various GF(p)—linear transformations $A: W \to W$. We will denote the components by their defining equations $x = 0$ and $y = xA$, respectively. Next, note that each Sylow p-subgroup Q of G is abelian and hence all the elements ($\neq 1$) of Q have the same elation axis. Let S denote the set of all components of π and let N be the subset of elatior axes; thus $|S| = q^2 + 1$ and $|N| = q + 1$.

LEMMA 1. (Hering [5], Ostrom [6]). We may coordinatize π as above such that

$$G = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} : A,B,C,D \in K;\ AD - BC = I \right\}$$

where K is a field of $2r \times 2r$ matrices over GF(p) and $K \cong GF(q)$. Further, the elation axes (that is, the elements of N) have the form $y = xA$ $(A \in K)$ and $x = 0$.

LEMMA 2. There is an element $g \in G$ such that the following conditions are satisfied: (i) $|g| \,\big|\, q+1$; (ii) $|g| \nmid p^t - 1$ for $t < 2r$; and (iii) g fixes a component of π which is not in the set N.

PROOF. The integer s is a <u>p-primitive</u> <u>prime</u> <u>divisor</u> of $q^2 - 1$ if s is a prime, $s \mid q^2 - 1$, and $s \nmid p^t - 1$ for $0 < t < 2r$ (hence $s \mid q + 1$). $q^2 - 1$ has a p-primitive prime divisor s unless $q = 8$ or $q = p$ and $p + 1 = 2^a$ [7]. In the first case, let $|g| = s$ sc that g satisfies conditions (i) and (ii). Then g also satisfies condition (iii) because $|g|$ is a prime and g permutes the $q(q-1)$ components in $S \setminus N$. If $q = 8$, choose g such that $|g| = 9$. Since $|S \setminus N| = 56 \not\equiv 0 \pmod 3$, g must fix one of the elements of $S \setminus N$. Finally, if $q = p$ and $p + 1 = 2^a$, choose h of order 8 in G and let $g = h^2$. Then g^2 has order 2 in $G = SL(2,K)$, so $g^2 = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}$ fixes every component of π. Hence, h has orbits of lengths 1, 2, and 4 in S, and since $4 \nmid p(p-1)$ then h has an orbit of length 1 or 2 on $S \setminus N$. Therefore $g = h^2$ fixes an element of $S \setminus N$.

LEMMA 3. Choose $g \in G$ so that g satisfies the conditions of Lemma 2, and let L(g) be the ring of matrices generated by g over GF(p). Then L(g) is a field $\cong GF(q^2)$ and L(g) contains the subfield

$$\widetilde{K} = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in K \right\}.$$

PROOF. $g \in G \subset GL(2,K)$ by Lemma 1. As a 2×2 matrix over K, g has a minimum polynomial $f(x)$ over K of degree ≤ 2. Since $|g| \nmid q(q-1)$, then the degree of f is 2 and f is irreducible over K. Therefore, g and K generate a field $U \cong GF(q^2)$ which contains $L(g)$ as a subfield. Since $|g| \nmid p^t - 1$ (for $t < 2r$), then $L(g) = U$ and $L(g) \supset \tilde{K}$.

LEMMA 4. Let g of Lemma 2 fix the component $y = xT$ of $S \setminus N$. Then $K[T]$ is a field isomorphic to $GF(q^2)$.

PROOF. $L(g)$ and hence $\tilde{K} = \left\{\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in K\right\}$ fix the component $y = xT$, and thus K centralizes T. T and the elements of K are $2r \times 2r$ matrices which act on a vector space $V = V(2r,p)$ of dimension 2r over $GF(p)$. K makes V into a 2-dimensional vector space and T acts as a K-linear transformation of V. Hence, the minimum polynomial $f(x)$ of T over K has degree ≤ 2. If T has an eigenvalue A in K, then the distinct components $y = xT$ and $y = xA$ of π must intersect, which is impossible. Therefore, T is irreducible over K and $K[T] \cong GF(q^2)$.

We can now complete the proof of the Theorem. Let π^* denote the Desarguesian affine plane of order q^2 coordinatized by the field $L = K[T]$; i.e., the points of π^* are $\{(x,y): x,y \in L\}$ and the components of π^* are $\{y = xC: C \in L\} \cup \{x = 0\}$. Clearly, $GL(2,L)$ acts as a collineation group of π^*. We superimpose π^* on π by identifying the points of π^* and π. Since $K \subset L$ and $T \in L$, the components $y = xA$ of N and $y = xT$ are components both of π^* and π. Since $G = SL(2,K) \subset GL(2,L)$, then G acts both as a collineation group of π^* and of π. Finally, recall that $SL(2,K)$ acts transitively on the $q(q-1)$ components of π^* outside of N (for example, the stabilizer subgroup in $SL(2,K)$ of a component of π^* outside N has order $q+1$). Therefore, the images of $y = xT$ under G constitute $q(q-1)$ components both of π^* and of π; so $\pi^* = \pi$ as required.

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