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LOCAL EXPANSIONS AND ACCRETIVE MAPPINGS

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ABSTRACT. Let X and Y be complete metric spaces with Y metrically convex, let $D \subset X$ be open, fix $u_0 \in X$, and let $d(u) = d(u_0,u)$ for all $u \in D$. Let $f : X \longrightarrow 2^Y$ be a closed mapping which maps open subsets of D onto open sets in Y, and suppose f is locally expansive on D in the sense that there exists a continuous nonincreasing function $c : R^+ \longrightarrow R^+$ with $\int^{+\infty} c(s)ds = +\infty$ such that each point $x \in D$ has a neighborhood N for which $\text{dist}(f(u),f(v)) \geq c(\max\{d(u),d(v)\})d(u,v)$ for all $u,v \in N$. Then, given $y \in Y$, it is shown that $y \in f(D)$ iff there exists $x_0 \in D$ such that for $x \in X\setminus D$, $\text{dist}(y,f(x_0)) \leq \text{dist}(u,f(x))$. This result is then applied to the study of existence of zeros of (set-valued) locally strongly accretive and ϕ-accretive mappings in Banach spaces.

KEY WORDS AND PHRASES. Local expansions, accretive mappings, nonexpansive mappings, fixed points, zeros.

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1. INTRODUCTION

This paper may be viewed as a sequel to that of Kirk and Schöneberg [1]. We first prove a general theorem for "local expansions" and we then apply this result in special settings to the study of the existence of zeros of the locally strongly accretive and ϕ-accretive mappings. In the interest of attaining the generality readily offered by our techniques, we formulate our results for set-valued mappings even though some of our assumptions (e.g., continuity, as opposed to semicontinuity) might seem stringent for such mappings. The results themselves, however, represent extensions of those of [1] even in the point-valued case.

Results similar to those obtained here may be found in Ray and Walker [2] and in Torrejón [3]; however, the methods employed are different. Torrejón relies on differential inequalities, while Ray and Walker use the Brezis-Browder order principle to prove a refined version of the Caristi-Ekeland minimization principle, and this in turn is used to obtain, among other things, a Banach space version of the surjectivity part of our Theorem 2.1. On the other hand, Torrejón obtains our Theorem 2.1 under the assumptions that X is a Banach space and Y is a complete and metrically convex metric space. While it is likely that the methods of Ray-Walker and of Torrejón could be modified to attain the generality we obtain, our approach, which is a refinement of the argument of Kirk-Schöneberg [1], seems more direct and more in the spirit of the original work of Browder [4, §4]. In particular, Browder uses an argument (cf. [4, Theorem 4.9]) roughly like the one we use below to show that a local expansion from a complete metric space X to a metric space Y is, under suitable connectedness hypotheses, actually a covering map of X onto Y.

For the most part, we use standard notation. $B(x;r)$ denotes the closed ball centered at a point x of a metric space with radius $r > 0$. We shall use $\mathcal{B}(Y)$ and $\mathcal{C}(Y)$ to denote, respectively, the family of nonempty bounded closed subsets and the family of nonempty compact subsets of a metric space Y, and we assign to these families the usual Hausdorff metric (denoted by H). For a Banach space X, the mapping $J : X \longrightarrow 2^{X^*}$ denotes the usual normalized duality mapping:

$$J(x) = \{j \in X^* : \|j\| = \|x\|,\ \langle x,j\rangle = \|x\|^2\}$$

Also, for a subset A of X, we use $|A|$ to denote $\inf\{\|x\| : x \in A\}$.

Finally, if X and Y are metric spaces, then a set-valued mapping $f : X \longrightarrow 2^X$ is said to be <u>closed</u> if for $\{x_n\}$ in X, the conditions $x_n \longrightarrow x$, $y_n \in T(x_n)$, and $y_n \longrightarrow y$ imply $y \in T(x)$.

2. A THEOREM ON LOCAL EXPANSIONS.

THEOREM 2.1. Let (X,d) be a complete metric space and (Y,d) a rectifiably pathwise connected metric space with intrinsic metric ℓ, let $D \subset X$ be open, fix $u_0 \in X$, and let $d(u) = d(u_0,u)$, $u \in D$. Let $f : X \longrightarrow 2^X$ be a closed mapping which maps open subsets of X onto open sets in Y, and suppose there exists a continuous nonincreasing function $c : [0,\infty) \to (0,\infty)$ with $\int^{+\infty} c(s)ds = +\infty$ such that each point $x \in D$ has a neighborhood N for which

$$\mathrm{dist}(f(u),f(v)) \geq c(\max\{d(u),d(v)\})d(u,v)$$

for all $u,v \in N$. Then, given $y \in Y$, the following are equivalent.

(a) $y \in f(D)$.

(b) There exists $x_0 \in D$ such that for each $x \in X\backslash D$,

$$\inf\{\ell(w,y) : w \in f(x_0)\} \leq \inf\{\ell(w,y) : w \in f(x)\}.$$

In particular, if $D = X$, then f is surjective.

PROOF. Since (a) $\Rightarrow$ (b) is trivial, we suppose (b) holds and show that the assumption $y \notin f(D)$ leads to a contradiction. For each $x \in D$, let

$$r(x) = \sup\{r \in (0,1) : B(x;r) \subset D \text{ and } \mathrm{dist}(f(u),f(v)) \geq c(\max\{d(u),d(v)\}d(u,v) \text{ for all } u,v \in B(x;r)\}.$$

By assumption, $r(x) > 0$ for each $x \in D$, and moreover if

$$c = \inf\{c(d(u)) : u \in B(x_0;r(x_0)/2)\},$$

then

$$\epsilon = cr(x_0)/4 > 0.$$

We define a sequence $\{u_n\} \subset D$ as follows. Let $u_1 = x_0$, $t_1 = 0$, and select $w_1 \in f(u_1)$ and a path $\Gamma : [0,1] \to Y$ joining w_1 and y (with $\Gamma(0) = w_1$) such that the length, $\ell(\Gamma)$, of Γ satisfies

$$\ell(\Gamma) \leq \inf\{\ell(w,y) : w \in f(x_0)\} + \epsilon.$$

Let $t_2 = \sup\{t \in [0,1] : \Gamma(t) \in f(B(u_1;r(u_1)/2))\}$, let $\{s_n\} \subset [0,1]$ be such that $s_n \uparrow t_2$, and let $\Gamma(s_n) \in f(v_n)$ where $v_n \in B(u_1;r(u_1)/2)$, $n = 1,2,\cdots$. Since $\Gamma(s_n) \to \Gamma(t_2)$ and

$$d(\Gamma(s_n),\Gamma(s_m)) \geq \mathrm{dist}(f(v_n),f(v_m)) \geq cd(v_n,v_m),$$

it follows that $\{v_n\}$ converges to some point $v \in M$. Since f is a closed mapping, $\Gamma(t_2) \in f(v)$. Also, since $y \notin f(D)$, $y \notin f(B(u_1;r(u_1)/2))$. In view of this, the fact that f is open implies $v \in \partial B(u_1;r(u_1)/2)$. Now set $u_2 = v$ and $w_2 = \Gamma(t_2)$. Similarly, having defined $\{u_i\}$, $\{t_i\}$, and $\{w_i\}$ for $i \in \{1,\cdots,n\}$, let

$$t_{n+1} = \sup\{t \in [0,1] : \Gamma(t) \in f(B(u_n;r(u_n)/2))\}$$

and as above obtain $u_{n+1} \in \partial B(u_n;r(u_n)/2)$ for which $w_{n+1} = \Gamma(t_{n+1}) \in f(u_{n+1})$.

Thus, by induction, sequences $\{u_n\}$, $\{t_n\}$, and $\{w_n\}$ exist satisfying for $n \in \mathbb{N}$,

(i) $t_{n+1} > t_n$;

(ii) $d(u_{n+1},u_n) = r(u_n)/2$;

(iii) $c(\max\{d(u_n),d(u_{n+1})\})d(u_n,u_{n+1}) \le \operatorname{dist}(f(u_n),f(u_{n+1})) \le d(w_n,w_{n+1})$.

Since $\{t_n\}$ is increasing,

(iv) $\sum_{n=1}^{\infty} d(w_n,w_{n+1}) \le \ell(\Gamma) < +\infty$.

If $\{d(u_n)\}$ is unbounded, define $\bar{c}(s) = c(s-1)$ for $s > 1$ and select $\{u_{i_k}\}_{k=1}^{\infty}$ so that $i_1 = 1$ and i_{k+1} is the smallest integer j such that $d(u_j) \le d(u_{n+1})$ if $d(u_{i_k+1}) \le d(u_{i_k})$; otherwise, take $i_{k+1} = i_k+1$. Then

$$
\begin{aligned}
\sum_{n=1}^{\infty} c(\max\{d(u_n),d(u_{n+1})\})d(u_n,u_{n+1}) \\
&\ge \sum_{n=1}^{\infty} c(d(u_n))|d(u_{n+1})-d(u_n)| \\
&\ge \sum_{k=1}^{\infty} c(d(u_{i_k}))(d(u_{i_k+1})-d(u_{i_k})) \\
&\ge \int^{+\infty} \bar{c}(s)ds \\
&= +\infty.
\end{aligned}
$$

This contradicts (iii) and (iv). Thus $s = \sup\{d(u_i) : i = 1,2,\cdots\} < +\infty$ and (iii) implies

$$c(s)d(u_n,u_{n+1}) \le d(w_n,w_{n+1}), \quad n = 1,2,\cdots.$$

In conjunction with (iv), the above in turn implies that $\{u_n\}$ is a Cauchy sequence. Since X is complete, $u_n \to x \in \bar{D}$. Moreover, since $r(u_n) = 2d(u_n,u_{n+1}) \to 0$, it follows that x is not in D. Also, since $t_n \uparrow t \in [0,1]$, $w_n \to w^* = \Gamma(t)$, and the assumption that f is closed implies $w^* \in f(x)$. To complete the proof, observe that

$$
\begin{aligned}
cr(x_0)/2 &\le c(\max\{d(u_1),d(u_2)\})d(u_1,u_2) \\
&\le \operatorname{dist}(f(u_1),f(u_2)) \\
&\le d(w_1,w_2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\inf\{\ell(w,y) : w \in f(x)\} &\le \ell(w^*,y) \\
&\le \ell(\Gamma) - d(w_1,w_2) \\
&\le \inf\{\ell(w,y) : w \in f(w_0)\} + \epsilon - d(w_1,w_2)
\end{aligned}
$$

$$\le \inf\{\ell(w,y) : w \in f(x_0)\} - cr(x_0)/4$$
$$< \inf\{\ell(w,y) : w \in f(w_0)\},$$

and, since $x \in X\backslash D$, this contradicts (b).

The final assertion of the theorem follows from the fact that, if $D = X$, then (b) is satisfied vacuously.

3. APPLICATIONS TO ACCRETIVE MAPPINGS

Let X be a real Banach space and $D \subset X$. We recall that a mapping $A : X \longrightarrow 2^X$ is said to be accretive if for each $x,y \in D$, $u \in A(x)$, $v \in A(y)$:

$$\langle u-v,x-y\rangle_+ \equiv \sup\{\langle u-v,j\rangle : j \in J(x,y)\} \ge 0.$$

Since the unit ball of X^* is weak* compact, the above supremum is attained and thus, by Lemma 1.1 of Kato [5], $\langle u-v,x-y\rangle_+ \ge 0$ iff for each $\lambda \ge 0$,

$$\|x-y\| \le |(x-y) + \lambda(A(u)-A(v))|.$$

Therefore $A : D \longrightarrow 2^X$ is accretive if for each $\lambda > 0$, $J_\lambda \equiv (I+\lambda A)^{-1}$ is a nonexpansive mapping of $(I+\lambda A)(D)$ onto D. If $(I+\lambda A)(D) = X$ for some (hence all) $\lambda > 0$, then A is said to be m-accretive.

Finally, $A : D \longrightarrow 2^X$ is said to be strongly accretive if $A-cI$ is accretive for some $c > 0$.

For our first application we require the following version of Deimling's domain invariance theorem of [6]. Schöneberg's modification (see [7]) of the Crandall-Pazy proof ([8]) of this result carries over from point-valued mappings to set-valued mappings without essential change.

THEOREM 3.1 (cf. [7]). Let X be a Banach space, $\mathcal{B}(X)$ the nonempty bounded closed subsets of X, and H the Hausdorff metric on $\mathcal{B}(X)$. Suppose $U \subset X$ is open, and let $T : U \longrightarrow \mathcal{B}(X)$ be continuous (relative to H) and satisfy for some $c > 0$,

(i) $|T(x)-T(y)| \ge c\|x-y\|$;

(ii) the mapping $R : U \longrightarrow \mathcal{B}(X)$ defined for fixed $y_0 \in X$ by

$$R(x) = c^{-1}(T(x)-y_0) - x \quad (x \in U) \text{ satisfies}$$
$$\|u-v\| \le |(u-v) + t(R(u)-R(v))| \quad (u,v \in U,\ t \ge 0).$$

Then $T(U)$ is an open subset of X.

This theorem can be proved as follows. Let $x_1 \in U$ and $y_1 \in T(x_1)$. Choose

$r > 0$ and $\rho > 0$ so that $B(x_1;r+\rho) \subset U$. Fix $y \in B(y_1;cr)$ and define $R : U \longrightarrow \mathcal{B}(X)$ as in (ii). It must be shown that there exists $x \in U$ such that $0 \in x + R(x)$; thus, $0 \in T(x)-y$ and $y \in T(x)$, from which $B(y_1;cr) \subset T(U)$.

Let $\psi : [0,1] \longrightarrow [0,1)$ satisfy $\int_1^\infty \psi(s)ds \le \rho$. For $u \in U$, $v \in R(u)$ and $\Sigma > 0$, let

$$\Lambda(u,v,\Sigma) = \{c \in [0,1] : (1-c)u-cv \in U \text{ and } H(R((1-c)u-cv),R(u)) < \psi(\Sigma+1)\},$$

and let $\lambda(u,v,\Sigma) = \sup \Lambda(u,v,\Sigma)$. (Since U is open and R continuous, $\Lambda(u,v,\Sigma) \neq \emptyset$.)

Now let $c_1 = 1$ and $v_1 = c^{-1}(y_1-y)-x_1$, and select $c_2 \in \Lambda(x_1,v_1,1)$ so that $2c_2 \ge \lambda(x_1,v_1,1)$. Next, select $v_2 \in R((1-c_2)x_1-c_2v_1)$ so that $\|v_2-v_1\| < \psi(c_1+1)$, and define $\{x_n\}$, $\{c_n\}$, and $\{v_n\}$ recursively by taking

$$x_{n+1} = (1-c_{n+1})x_n-c_{n+1}v_n,$$

where $c_{n+1} \in \Lambda(x_n,v_n,\sum_{j=1}^{n} c_j)$ is chosen so that $2c_{n+1} \ge \lambda(x_n,v_n,\sum_{j=1}^{n} c_j)$, and then select $v_{n+1} \in R(x_{n+1})$ so that

$$\|v_{n+1}-v_n\| \le \psi(1+\sum_{j=1}^{n} c_j).$$

From this point on it is possible, except for obvious modifications (generally, replacing $R(x_i)$ with v_i), to follow Schöneberg's proof and obtain a point $x \in U$ for which $|x+R(x)| = 0$. Since $R(x)$ is closed, $0 \in x+R(x)$. We refer to [7] for the details.

We now prove the analog of Theorem 3 of [1].

THEOREM 3.2. Let X be a Banach space with D an open subset of X, let $c : [0,\infty) \longrightarrow [0,\infty)$ be a continuous nonincreasing function for which $\int^{+\infty} c(s)ds = +\infty$, and suppose $T : \overline{D} \longrightarrow \mathcal{B}(X)$ is continuous on $\overline{D}$ and locally strongly accretive on D in the following sense: Each point $z \in D$ has a neighborhood N such that for each $x,y \in N$, if $u \in T(x)$ and $v \in T(y)$, then for some $j \in J(x,y)$,

$$\langle u-v,j\rangle \ge c(\max\{\|x\|,\|y\|\})\|x-y\|^2. \qquad (*)$$

Then the following are equivalent:

(a′) $0 \in T(D)$.

(b′) There exists $x_0 \in D$ such that $|T(x_0)| \le |T(x)|$ for each $x \in \partial D$.

PROOF. Let $z \in D$ and let N be a bounded neighborhood of z for which $(*)$

holds for all $x,y \in N$. Then the assumptions on c imply $\underline{c} = \inf\{c(\|u\|) : u \in N\} > 0$. If $u \in T(x)$ and $v \in T(y)$ for $x,y \in N$, for suitable $j \in J(x,y)$,

$$\langle u-v-\underline{c}(x-y), j\rangle \geq 0.$$

Thus, by Lemma 1.1 of Kato [5], for each $\lambda \geq 0$,

$$\|x-y+\lambda((u-v)-\underline{c}(x-y))\| \geq \|x-y\|,$$

and since this is true for all $u \in T(x)$ and $v \in T(y)$,

$$|x-y+\lambda(T(x)-T(y))-\underline{c}(x-y))| \geq \|x-y\| \quad (x,y \in N,\ \lambda \geq 0).$$

Taking $\lambda = \underline{c}^{-1}$ in the above,

$$|T(x)-T(y)| \geq \underline{c}\|x-y\| \quad (x,y \in N).$$

Also, if $R : N \longrightarrow \mathcal{B}(X)$ is defined by $R(x) = \underline{c}^{-1}(T(x)-y_0)-x$ (for fixed $y_0 \in X$), then $T(x)-T(y) = \underline{c}(R(x)-R(y)) + \underline{c}(x-y)$, and it follows that for each $t \geq 0$,

$$|x-y+t(R(x)-R(y))| \geq \|x-y\| \quad (x,y \in N).$$

Therefore, by Theorem 3.1, T maps open subsets of N (hence open subsets of D) onto open sets in X. Since (*) implies

$$|T(x)-T(y)| \geq c(\max\{\|x\|,\|y\|\})\|x-y\| \quad (x,y \in N),$$

and since (b′) implies that (b) of 2.1 holds for $y = 0$, we conclude: (b′) $\Rightarrow$ (a′). The reverse implication is obvious.

Our second application involves the so-called ϕ-accretive mappings ([4]). Let X and Y be Banach spaces and ϕ a mapping of X onto a dense subset of Y^* which satisfies

$$\|\phi(x)\| \leq \|x\| \quad \text{and} \quad \phi(\xi x) = \xi\phi(x) \quad (x \in X,\ \xi \geq 0).$$

THEOREM 3.3. Let X and Y be Banach spaces and suppose Y has an equivalent Fréchet differentiable norm with respect to which Y^* is strictly convex. Let $\phi : X \longrightarrow Y^*$ be as above, let $c : [0,\infty) \longrightarrow [0,\infty)$ be a continuous nonincreasing function for which $\int^{+\infty} c(x)dx = +\infty$, and suppose $T : X \longrightarrow \mathcal{C}(Y)$ is locally lipschitzian and satisfies: For each $z \in X$ there is a neighborhood $N = N(z)$ such that for each $x,y \in N$ and each $u \in T(x)$, $v \in T(y)$,

$$\langle u-v,\phi(x-y)\rangle \geq c(\max\{\|x\|,\|y\|\})\|x-y\|^2. \tag{**}$$

Then for each open set $D \subset X$ the following are equivalent:

(a″) $0 \in T(D)$.

(b″) There exists $x_0 \in D$ such that $|T(x_0)| \leq |T(x)|$ for each $x \in \partial D$.

PROOF. Since $\langle u-v, \phi(x-y)\rangle \le \|u-v\|\|\phi(x-y)\| \le \|u-v\|\|x-y\|$, condition (**) implies that

$$|T(x)-T(y)| \ge c(\max\{\|x\|,\|y\|\})\|x-y\| \quad (x,y \in N).$$

Also the local ϕ-accretive assumption on T of Theorem 3.3 implies that T is locally strongly ϕ-accretive in the sense of Definition 2.1 of Downing and Ray. Thus by Theorem 2.1 of [9], T maps open subsets of D onto open sets in Y. The result now follows from Theorem 2.1 as in the proof of Theorem 3.2.

Our final application of the above development is a global result patterned after the approach of [10].

THEOREM 3.4. Let X be a Banach space with $D \subset X$ bounded and open, let $A : D \to \mathcal{B}(X)$ be continuous and accretive, and suppose there exists $z \in D$ such that

$$|A(z)| < \inf\{|A(x)| : x \in \partial D\}.$$

Then there exists a (single-valued) nonexpansive mapping $f : \overline{D} \to D$ whose fixed points are zeros of A.

PROOF. Since D is bounded, it is possible to choose $\alpha \in (0,1)$ so near 1 that

$$\alpha|A(z)| + (1-\alpha)\|z-y\| < \inf\{\alpha|A(x)| - (1-\alpha)\|x-y\| : x \in \partial D\}$$

for each $y \in \overline{D}$. Fix $w \in \overline{D}$ and define $T_w : \overline{D} \to 2^X$ by

$$T_x(x) = (1-\alpha)(x-w) + \alpha A(x), \quad x \in \overline{D}.$$

Then, if $\overline{x} \in \partial D$,

$$\begin{aligned} |T_w(z)| &= |(1-\alpha)(z-w) + \alpha A(z)| \\ &\le (1-\alpha)\|z-w\| + \alpha|A(z)| \\ &< \inf\{\alpha|A(x)| - (1-\alpha)\|x-w\| : x \in \partial D\} \\ &\le \inf\{|\alpha A(x) + (1-\alpha)(x-w)| : x \in \partial D\} \\ &\le |T_w(\overline{x})|. \end{aligned}$$

Also, if $u \in T_w(x)$ and $v \in T_w(y)$, then, for some $j \in J(x-y)$,

$$\langle u-v, j\rangle \ge (1-\alpha)\|x-y\|^2;$$

so by Theorem 3.2, there exists $z_w \in D$ such that $0 \in T_w(z_w) = (1-\alpha)(z_w-w) + \alpha A(z_w)$; i.e.,

$$z_w \in w - \lambda A(z_w) \quad (\lambda = \alpha/(1-\alpha)).$$

By accretivity of A,

$$\|z_u - z_v\| \le |(z_u - z_v) + \lambda(A(z_u) - A(z_v))| \quad (u,v \in \overline{D}).$$

But $z_u \in u-\lambda A(z_u)$ and $z_v \in v-\lambda A(z_v)$. Thus

$$|(z_u-z_v)+\lambda(A(z_u)-A(z_v))| \le \|u-v\|,$$

proving that the mapping $u \longmapsto z_u$ is nonexpansive. Finally, if $u = z_u$ for $u \in D$, then $u \in u-\lambda A(u)$, proving $0 \in A(u)$.

COROLLARY 3.1. Let X be a Banach space for which the closed balls have the fixed point property for nonexpansive self-mappings. Suppose $A : X \longrightarrow \mathcal{B}(X)$ is continuous and accretive, and satisfies

$$\lim_{\|x\|\to\infty} |A(x)| = +\infty$$

Then $A(X) = X$.

PROOF. Fix $y \in X$ and define $\tilde{A} : X \longrightarrow \mathcal{B}(X)$ by $\tilde{A}(x) = A(x)-y$. Choose $\delta > 0$ so that

$$C = \{x \in X : |\tilde{A}(x)| \le \delta\} \ne \emptyset.$$

Since $|A(x)| - \|y\| \le |\tilde{A}(x)| \implies |\tilde{A}(x)| \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$, C is bounded, and moreover for $r > 0$ sufficiently large and $x_0 \in C$,

$$|\tilde{A}(x_0)| < \inf\{|\tilde{A}(x)| : \|x\| = r\}.$$

Thus, $0 \in \tilde{A}(x)$ for some $x \in B_r(0)$; hence, $y \in A(x)$.

The analog of Corollary 3.1 for m-accretive operators is proved in [11].

4. <u>REMARKS</u>.

(1) As Torrejón observes in [3], the assumption that c is nonincreasing in Theorem 2.1 (hence in Theorems 3.2, 3.3) is not really essential. To see this, define $\{u_{i_k}\}$ as in the proof of Theorem 2.1, fix $k \ge 1$, and use the fact that the image of $\Gamma([0,1])$ under the inverse of the restriction of f to $B(u_{i_k};r(u_{i_k})/2)$ is a path. Consequently, it is possible to obtain points $\{s_1^{(k)}, s_2^{(k)}, \cdots, s_{n_k}^{(k)}\}$ in $B(u_{i_k};r(u_{i_k})/2)$ such that the numbers $d(u_{i_k}) = d(s_1^{(k)}) < d(s_2^{(k)}) < \cdots < d(s_{n_k}^{(k)}) = d(u_{i_k+1})$ induce a partition of $[d(u_{i_k}), d(u_{i_k+1})]$, while at the same time $\Gamma(t_i^{(k)}) \in f(s_i^{(k)})$ where $t_{i_k} = t_1^{(k)} < t_2^{(k)} < \cdots < t_{n_k}^{(k)} = t_{i_k+1}$. Moreover, if $\epsilon_k > 0$, then the above partition may be further refined so that

$$\sum_{i=1}^{n_k-1} c(d(s_{i+1}^{(k)}))(d(s_{i+1}^{(k)}) - d(s_i^{(k)})) \ge \int_{d(u_{i_k})}^{d(u_{i_k+1})} c(s)ds - \epsilon_k.$$

Since the left side of the above is bounded by the length of Γ from t_{i_k} to t_{i_k+1}, by choosing $\{\epsilon_k\}$ so that $\Sigma\epsilon_k < \infty$, it is possible to proceed as in the proof of Theorem 2.1 to obtain (if $\{d(u_n)\}$ is unbounded) the contradiction:

$$+\infty > \ell(\Gamma) \geq \sum_{k=1}^{\infty} \sum_{i=1}^{n_k-1} \operatorname{dist}(f(s_i^{(k)}), f(s_{i+1}^{(k)}))$$

$$\geq \sum_{k=1}^{\infty} \sum_{i=1}^{n_k-1} c(d(s_{i+1}^{(k)}))(d(s_{i+1}^{(k)}) - d(s_i^{(k)}))$$

$$\geq \int^{+\infty} c(s)ds - \Sigma\epsilon_k = +\infty.$$

(Note that this argument is merely a reworking of that of Browder [4, Theorem 4.9].)

(2) We note also that Theorem 2.1 has the following corollary (cf. [4, Theorem 4.10]).

COROLLARY. Let X and Y be Banach spaces, $c : [0,\infty) \longrightarrow (0,\infty)$ a continuous (nonincreasing) mapping for which $\int^{+\infty} c(s)ds = +\infty$, and $T : X \longrightarrow 2^Y$ a closed mapping which maps open subsets of X onto open sets in Y. Suppose each point $z \in X$ has a neighborhood N such that for each $x,y \in N$,

$$|T(x)-T(y)| \geq c(\max\{\|x\|,\|y\|\})\|x-y\|.$$

Then $T(X) = Y$.

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