Internat. J. Math. & Math. Sci.
Vol. 6 No. 2 (1983) 371-386

THE COMBINATIONAL STRUCTURE OF NON-HOMOGENEOUS MARKOV CHAINS WITH COUNTABLE STATES

A. MUKHERJEA and A. NAKASSIS
University of So. Florida, Tampa, FL. 33620
and American University, Washington, D. C. 20016

(Received June 16, 1982)

ABSTRACT. Let P(s,t) denote a non-homogeneous continuous parameter Markov chain with countable state space E and parameter space [a,b], $-\infty < a < b < \infty$. Let $R(s,t) = \{(i,j) : P_{ij}(s,t) > 0\}$. It is shown in this paper that R(s,t) is reflexive, transitive, and independent of (s,t), $s < t$, if a certain weak homogeneity condition holds. It is also shown that the relation R(s,t), unlike in the finite state space case, cannot be expressed even as an infinite (countable) product of reflexive transitive relations for certain non-homogeneous chains in the case when E is infinite.□

KEYWORDS AND PHRASES. Non-homogeneous Markov chains, reflexive and transitive relations, homogeneity condition.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 60J27, 60C05.

1. INTRODUCTION AND STATEMENTS OF RESULTS

Throughout this paper, P(s,t) will denote a non-homogeneous continuous parameter Markov chain with countable state space E and parameter space [a,b], $-\infty < a < b < \infty$, such that P is a function from the domain space

$$D = \{(s,t): a \leq s \leq t \leq b\}$$

into S, the set of countable stochastic matrices with state space E such that the following conditions hold:

(i) $P_{ij}(s,t)$ is separately continuous in s and in t;

(ii) $P(s,t) = P(s,u)P(u,t)$ if $a \leq s \leq u \leq t \leq b$;

(iii) $P_{ij}(t,t) = \delta_{ij}$.

One important result for homogeneous Markov chains (i.e. when P(s,t) above is a function of t-s alone) is the classical Austin-Ornstein result, namely that if $\overline{P}(u) = P(s,s+u)$, then for $i,j \in E$,

$$\overline{P}_{ij}(u) > 0 \text{ for some } u => \overline{P}_{ij}(u) > 0 \ \forall\ u.$$

This means that the relation

$$R = \{(i,j): \overline{P}_{ij}(u) > 0\} \tag{1.1}$$

is independent of u; it also follows that R is reflexive and transitive. Conversely, given any reflexive and transitive relation R on E, there exists a standard homogeneous Markov chain $\overline{P}(t)$ on E satisfying (1.1). Thus, it is natural to ask what an analogous result for non-homogeneous Markov chains should be. Kingman and Williams (see Theorem 3, [2]) have shown when E is finite that the relation R(s,t) defined by

$$R(s,t) = \{(i,j): P_{ij}(s,t) > 0\} \tag{1.2}$$

can be expressed as a finite product of reflexive and transitive relations on E. It was also mentioned in [2] that "Our main result is Theorem 3, ... The methods depend heavily on the finiteness of E, and a generalization to infinite state spaces would require new techniques." Our aim in this paper is to tackle the case when E is infinite.

Before we state our main results, let us point out that with no loss of generality, the non-homogeneous chain P defined on D can be considered as defined on the domain

$$D' = \{(s,t): -\infty < s \leq t < \infty\}$$

in the following way. Define

$$\begin{aligned} P(s,t) &= P(s,b) \text{ if } a \leq s \leq b \leq t; \\ &= I \text{ if } b \leq s \leq t; \\ &= P(a,t) \text{ if } s \leq a \leq t; \\ &= I \text{ if } s \leq t \leq a; \\ &= P(a,b) \text{ if } s \leq a \leq b \leq t. \end{aligned}$$

Notice that with this definition P is a non-homogeneous chain on D' satisfying again conditions (i), (ii) and (iii).

Let us also point out that Theorem 1 in [2] (with the same proof) remains true even for E infinite so that for $s \leq t$ and $(s,t) \subseteq (u,v)$,

$$R(s,t) \text{ is reflexive and } R(s,t) \subseteq R(u,v). \tag{1.3}$$

In the rest of this section, we state our results. The state space E is always infinite unless otherwise mentioned. Our main results are Theorems 1 and 2. Theorem 2 shows that the Kingman-Williams result for finite non-homogeneous Markov chains is false in the infinite case. (This problem, though mentioned in [2], was left unsolved in [2].) Theorem 1 presents a necessary and sufficient condition for an Austin-Ornstein type theorem for non-homogeneous Markov chains with countable states in terms of a weak homogeneity condition. It is doubtful to us if this condition can be any further weakened while maintaining the same conclusion. Among other results, there is a proposition in section 2 that holds even in the infinite case and gives a simple proof of the main result in [2]. Finally, in section 3, we present several results for infinite products of reflexive transitive relations on positive integers. Here are our results.

THEOREM 1. (a) Let s be a fixed time parameter. Suppose that for each positive β, there is a h, $0 < h < \beta$, such that for each positive integer m, the following condition holds:

$$R(s+mh, s+(m+1)h) \subseteq R(s+(m-1)h, s+mh). \tag{1.4}$$

Then the relation $R(s,t)$ is reflexive, transitive, and independent of t (for $t > s$).

(b) Consider the following weak homogeneity condition: for every real s and and for each positive β, there is a h (depending on s) such that $0 < h < \beta$ and for each positive integer m,

$$R(s+(m-1)h, s+mh) = R(s+mh, s+(m+1)h). \tag{1.5}$$

Then the relation $R(s,t)$ is reflexive, transitive, and independent of (s,t), $s < t$, iff condition (1.5) holds.□

THEOREM 2. There are non-homogeneous Markov chains P where the relation $R(s,t)$ cannot be expressed as a finite product of reflexive and transitive relations.□

Our next theorem gives a sufficient condition for $R(s,t)$ to be a product of reflexive, transitive relations. The conditions (i) and (ii) considered in this theorem are natural in the sense that they hold in the finite dimensional situation (see Theorem 5, [2]).

THEOREM 3. Suppose that for each t, there exists $h_t > 0$ such that

(i) $t-h_t \leq t' < t \Rightarrow R(t',t) = R(t-h_t,t)$ and

(ii) $t < t'' \leq t+h_t \Rightarrow R(t,t'')=R(t,t+h_t)$

hold. Then for $(s,t) \in D$, there exist reflexive, transitive relations $T_1,T_2,\ldots,T_m$ such that $R(s,t)=T_1T_2\ldots T_m$.□

THEOREM 4. Let $t_n < t_{n+1} \to t$ as $n \to \infty$. Then for $i \neq j$,

$$\sum_{n=1}^{\infty} P_{ij}(t_n,t_{n+1}) < \infty.$$

Also if $s_n < s_{n-1} \to s$ as $n \to \infty$, then for $i \neq j$,

$$\sum_{n=1}^{\infty} P_{ij}(s_{n+1},s_n) < \infty.$$

If $\varliminf_{s \to t} P_{ij}(s,t)=0$ uniformly in all i different from j (for each j), then for $t_n < t_{n+1} \to t$, we have: for each i,

$$\sum_{n=1}^{\infty} \sum_{k \neq i} P_{ik}(t_n,t_{n+1}) < \infty. \square$$

We remark that though Theorem 4 is not combinatorial in nature and therefore does not blend well in this respect with out other results, we include it here since it uncovers a structural property of a non-homogeneous chain which is by no means obvious and seems to be missing in the literature even in the homogeneous case.

THEOREM 5. There exists a non-homogeneous Markov chain P such that the relation R(s,t) cannot be expressed as an infinite forward product $T_1T_2\ldots.T_n\ldots\ldots$, of reflexive transitive relations on E.□

THEOREM 6. There exists a non-homogeneous Markov chain P such that the relation R(s,t) cannot be expressed as an infinite backward product $\ldots.T_{-n}T_{-n+1}\ldots T_{-1}$ of reflexive transitive relations on E.□

THEOREM 7. There exists a non-homogeneous Markov chain P such that the relation R(s,t) cannot be expressed as an infinite 2-sided product $\ldots.T_{-n}T_{-n+1}\ldots T_{-1}T_0T_1\ldots.T_{n-1}T_n\ldots.$ of reflexive transitive relations on E.□

THEOREM 8. Let $(T_n)_{n=1}^{\infty}$ be a sequence of reflexive transitive relations on E. Then there is a non-homogeneous Markov chain P on [a,b], uniformly continuous separately in each variable, such that $R(a,b) = T_1T_2\ldots.T_n\ldots\ldots$□

One can also obtain theorems analogous to Theorem 8 for infinite backward as well as two-sided products of reflexive transitive relations. The proofs of Theorems 5, 6 and 7 are contained in the example considered in section 3. The proof of Theorem 8 is also contained in section 3. The proofs of Theorems 1, 2, 3 and 4 are given in section 2.

2. DISCUSSION AND PROOFS (of the first four theorems)

Notice that there are simple examples of non-homogeneous Markov chains where $R(s,t)$ is not independent of t for a given s. For example, consider the two standard homogeneous chains $Q(t)$ and $S(t)$ with state space $\{1,2\}$ such that

$$Q(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S(t) = \begin{pmatrix} e^{-t} & 1-e^{-t} \\ 0 & 1 \end{pmatrix}$$

Let the non-homogeneous chain P be defined by

$$\begin{aligned} P(s,t) &= Q(t-s) \text{ if } s \le t \le 1; \\ &= S(t-s) \text{ if } 1 \le s \le t; \\ &= Q(1-s)S(t-1) \text{ if } s \le 1 \le t. \end{aligned}$$

Let $s < 1$. Then $P_{12}(s,1) = 0$, whereas $P_{12}(s,2) > 0$.
As we can see in this example (see also [2]), in the finite dimensional case the non-homogeneous chains are in principle, formed by taking together several homogeneous chains in a manner shown in the example. In the infinite dimensional case, however, it will appear from our results in this paper that non-homogeneous Markov chains are, in general, results of infinite products (forward, backward or two-sided) of homogeneous Markov chains.

Before we go into the proofs of our main results, let us present a simple proposition. The main result in [2] follows immediately from this proposition.

PROPOSITION 1. Suppose that $R(s,t)$ is not transitive. Then there exists v such that $s < v < t$ and $R(s,t)=R(s,v)R(v,t)$ (see [2] for definition of composition of relations), where $R(s,v)$ as well as $R(v,t)$ is properly included in $R(s,t)$.□

PROOF. Write R for $R(s,t)$. There exist i,j,k in E such that $(i,j) \in R$, $(j,k) \in R$, but $(i,k) \notin R$. By separate continuity, there exists u, $s < u < t$, such that $P_{jk}(u,t) > 0$. Since $P(s,t)=P(s,u)P(u,t)$ and $(i,k) \notin R$, we must have $P_{ij}(s,u)=0$.

Let

$$v = \sup\{u: s < u < t \text{ and } P_{ij}(s,u)=0\}.$$

Then, $P_{ij}(s,v)=0$ and $v < t$. Also, $P_{jk}(v,t)=0$, since otherwise there exists w, $v < w < t$, such that $P_{jk}(w,t) > 0$ and also (because of the supremum property of v) $P_{ij}(s,w) > 0$, which mean that $P_{ik}(s,t) > 0$, contradicting the assumption that $(i,k) \notin R$. This means that $R(s,v)$ and $R(v,t)$ are both properly included in R.□

PROPOSITION 2. Let E be finite. Then given $(s,t) \in D$, there exist

$$s < u_1 < u_2 < \ldots\ldots < u_m < t$$

such that $R(s,t)=R(s,u_1)R(u_1,u_2)\ldots.R(u_m,t)$ where each relation on the right side is reflexive and transitive.□

PROOF. This proposition follows by applying Proposition 1 repeatedly.

Now we present proofs of our main results.

PROOF OF THEOREM 1. With no loss of generality, we assume that $P(s,t)$ is defined on $D = \{(s,t): -\infty < s \leq t < \infty\}$. We follow closely the proof given on pages 126-128 of [1] (see Theorem 5, p. 126, [1]). We briefly sketch the proof. By (3), if $x < y < z$, then $R(x,y) \subset R(x,z)$. Therefore, let us suppose, if possible, that there exist $i,j \in E$ such that

$$P_{ij}(s,t)=0 \text{ whenever } s < t \leq t_o,$$

where

$$t_o = \sup\{t: P_{ij}(s,t)=0\}.$$

We assume that t_o is finite. The theorem will be proven by reaching a contradiction. Choose t' such that $t_o < t' < 2t_o-s$. Then $P_{ij}(s,t') > 0$.

Let $c > 0$ such that

$$P_{ij}(s,t')=2c. \tag{2.1}$$

Choose $\beta > 0$ such that $P_{ij}(s,t) \geq c$ if $s < t'-\beta < t < t' +\beta$ and such that $t' + \beta < 2t_o-s$. Note that for sufficiently large m,

$$(0,\frac{t'-s}{4m}) \subset \bigcup_{n=m}^{\infty} (\frac{t'-s}{4n}, \frac{t'-s+\beta}{4n}).$$

This means that using compactness of the interval $[s,t']$ and separate continuity of the mapping $t \to P(s,t)$, it follows that there exists a sufficiently large positive

N such that $\forall\, t \in [s,t']$,

$$\sum_{k>N} P_{ik}(s,t) < \frac{c}{4} \tag{2.2}$$

and the interval $(\frac{t'-s}{4N}, \frac{t'-s+\beta}{4N})$ contains a positive h that satisfies (1.4) and

$$P_{ij}(s,s+4Nh) \geq c. \tag{2.3}$$

Now we define the set A_m, $\forall\, m \geq 1$, by

$$A_m=\{k: P_{ik}(s,s+mh) > 0\}. \tag{2.4}$$

Then $A_m \subset A_{m+1}$ and it follows by assumption (1.4) that for $m \geq 2$, $k \in A_{m-1}$ and $k' \notin A_m$, we have:

$$P_{kk'}(s+mh,s+(m+1)h) = 0 \tag{2.5}$$

since

$$0=P_{ik'}(s,s+mh)= \sum_{k\in A_{m-1}} P_{ik}(s,s+(m-1)h)P_{kk'}(s+(m-1)h,s+mh).$$

Using (2.5) and noting that $j \in A_{4N}$ while $j \notin A_{2N}$, it follows as in the proof in [1] that the sets

$$B_1 \equiv A_1 \text{ and } B_m \equiv A_m - A_{m-1} (2 \leq m \leq 2N)$$

are all nonempty and pairwise disjoint. One can then reach a contradiction by following the same procedure as given in the proof in [1]. This completes the proof of part (a).

To prove part (b) of the theorem, let us first show that under condition (1.5) we have: for $s < u < t$, $R(s,t)=R(u,t)$. To this end, let $0 < \beta < \min\{u-s,t-u\}$, where $s < u < t$. By assumption then there exists a h, $0 < h < \beta$, satisfying (1.5). Let m be the smallest positive integer such that

$$s < s+mh < u \leq s+(m+1)h < t.$$

Then we have:

$$P_{ij}(s,t) > 0$$

$$\Rightarrow \quad P_{ij}(s,s+h) > 0 \text{ (by part (a) of this theorem)}$$

$\Rightarrow$ $P_{ij}(s+(m+1)h, s+(m+2)h) > 0$ (by condition (1.5))

$\Rightarrow$ $P_{ij}(s+(m+1)h, t) > 0$ (the reason being that part (a) applies if $t < s+(m+2)h$, and (1.3) applies otherwise)

$\Rightarrow$ $P_{ij}(u,t) > 0$ (by (1.3)).

This proves that $R(s,t)=R(u,t)$ whenever $s < u < t$ and condition (1.5) holds. This result along with part (a) and the reflexivity property in (1.3) implies immediately the conclusion in part (b).□

Now we prove the following lemma which will be needed in the proof of Theorem 2.

LEMMA. Let n be a given positive integer. Then there exists a non-homogeneous Markov chain P with state space $\{1,2,\ldots,n\}$ such that for some $s < t$, $R(s,t)$ cannot be written as a product $T_1T_2\ldots T_m$, where each T_i is reflexive and transitive and $m < n-1$.□

PROOF. Let us define the relations $R_1, R_2, \ldots, R_{n-1}$ as follows:

$$R_k=\{(i,i):\ 1 \le i \le n\} \cup \{(n-k, n-k+1)\},\ 1 \le k \le n-1. \tag{2.6}$$

Then each R_k is reflexive and transitive, and the product

$$R=R_1R_2\ldots R_{n-1}=\{(i,i):\ 1 \le i \le n\} \cup \{(i,i+1):\ 1 \le i \le n-1\ . \tag{2.7}$$

By line 21, p. 82 in [2], R is embeddable; that is, there exists a non-homogeneous Markov chain P such that $R(a,b)=R$.

Now suppose that there are m, $m < n-1$, reflexive transitive relations $T_1, T_2, \ldots, T_m$ such that

$$R=T_1T_2\ldots T_m. \tag{2.8}$$

We claim that

$$(i,i+1) \notin T_1 \text{ if } i+1 < n; \text{ also } T_1 \subseteq R. \tag{2.9}$$

Notice that $T_1T_1=T_1$ so that $T_1R \subseteq R$. If $(i,i+1) \in T_1$ for some $i < n-1$, then since $(i+1,i+2) \in R$ for $i \le n-2$, $(i,i+2) \in R$ for some $i < n-1$. This contradicts (2.6). Thus (2.9) is verified. Now it follows from (2.7), (2.8) and (2.9) that the relation $R^{(1)}$ defined by

$$R^{(1)}=\{(i,i+1):\ 1 \le i \le n-2\} \subseteq T_2T_3\ldots T_m \subseteq R. \tag{2.10}$$

It again follows as in (2.9) that

$$(i,i+1) \notin T_2 \text{ if } i+1 < n-1$$

so that the relation $R^{(2)}$ defined by

$$R^{(2)}=\{(i,i+1): 1 \le i \le n-3\} \subseteq T_3 \dots T_m \subseteq R.$$

Continuing, we see that if $m < n-1$, then

$$\{(1,2),(2,3)\} \subseteq T_m \subseteq R.$$

Since T_m is transitive, $(1,3) \in T_m \subseteq R$. This contradicts (2.7) and the lemma follows.□

PROOF OF THEOREM 2. Define the sets A_i, $i \ge 1$, as follows:

$$A_i=\{2^{i-1},2^{i-1}+1,\dots,2^i-1\}$$

Consider the state space $E = \bigcup_{i=1}^{\infty} A_i$. For each positive integer k, let $P^{(k)}$ be a non-homogeneous Markov chain with state space A_k defined as in the lemma. Define the non-homogeneous Markov chain P with E as state space as follows:

$$P_{ij}(s,t)=P_{ij}^{(k)}(s,t) \text{ if } (i,j) \in A_k \times A_k;$$

$$=0 \text{ if } (i,j) \in A_{k_1} \times A_{k_2} \quad (k_1 \ne k_2).$$

Note that by construction, the relation $R^{(k)}(a,b)$, a and b remaining the same for all k, cannot be expressed as a product of fewer than $2^{k-1}-1$ reflexive transitive relations on A_k. Now if we write

$R(a,b)=T_1T_2\dots T_m$ (the T_i's are reflexive transitive), then since each $T_i \subseteq R(a,b)$, $(i,j) \notin T_i$ if $(i,j) \in A_{k_1} \times A_{k_2}$ $(k_1 \ne k_2)$.

This means that if $T_i^{(k)}$ is the restriction of T_i on $A_k \times A_k$, then

$$R^{(k)}(a,b)=T_1^{(k)}\ T_2^{(k)}\ \dots\ T_m^{(k)}.$$

This is a contradiction since then m cannot be finite.□

PROOF OF THEOREM 3. Observe that if h_t is as in conditions (i) and (ii), then the relations $R(t',t)$ as well as $R(t,t'')$ is reflexive and transitive, where t' and t'' are as described in the theorem. Reflexivity follows from (1.3). For the transitive property, suppose that $P_{ij}(t',t) > 0$ and $P_{jk}(t',t) > 0$. By separate continuity, there exists, u, $t' < u < t$, such that $P_{ij}(t',u)$ 0. Since by condition (i),

$R(t',t)=R(u,t)$, $P_{jk}(u,t) > 0$. Therefore, $P_{ik}(t',t) \geq P_{ij}(t',u)P_{jk}(u,t) > 0$. Thus, $R(t',t)$ is transitive. Similarly, $R(t,t'')$ is also transitive. The theorem now follows easily by using compactness of the interval $[s,t]$.□

PROOF OF THEOREM 4. Write $P_n=P(t_n,t_{n+1})$. Then $Q_k=\lim_{n\to\infty} P_{k+1}P_{k+2}\cdots P_n$ exists and equals $P(t_{k+1},t)$ so that $\lim_{k\to\infty} Q_k = I$. We can also make similar observations using backward products. The theorem now follows immediately from related results proven in detail in [3].□

3. INFINITE PRODUCTS OF REFLEXIVE AND TRANSITIVE RELATIONS

In this section, we will consider a second example to show that for a non-homogeneous Markov chain P with countable states, the relation $R(s,t)$ need not be even of the form

$$T_1T_2\cdots T_n\cdots$$

where each T_i is reflexive and transitive. Here an infinite product simply means a relation that is the union of all the finite partial products of the T_i's in the same order. It will be clear that the same result remains true even if we consider products of the form

$$T_{-n}T_{-n+1}\cdots T_{-1} \text{ or } \cdots T_{-n}T_{-n+1}\cdots T_{-1}T_0T_1\cdots T_nT_{n+1}\cdots$$

where each T_i is reflexive and transitive. We will also show that given any such infinite product, there is always a non-homogeneous Markov chain P on $[a,b]$ such that $R(a,b)$ is the given infinite product.

First, the example. Let $E=\{1,2,3,\ldots\}$ be the state space and let

$s < s_{n+1} < s_n < \cdots < s_2 < s_1=t$ and $\lim_{n\to\infty} s_n = s$. We know that there exists on each interval $[s_{n+1},s_n]$ a homogeneous Markov chain $P^{(n)}$ with state space E such that

$$R^{(n)}(s_n-s_{n+1}) = \{(n+1,n+2)\}\cup U_0,$$

where $U_0=\{(i,i): 1 \leq i < \infty\}$

(Notice that above one could actually define $P^{(n)}(u)$ as follows:

$$P^{(n)}_{ij}(u) = 1 \text{ if } i=j\neq n+1,$$

$$= \exp(-u) \text{ if } i=j=n+1,$$

$= 1- \exp(-u)$ if $i=n+1$ and $j=n+2)$.

Let us now define P on subintervals of $[s,t]$ as follows:

Case 1. Let $s < u \leq v \leq t$. In this case, there exist positive integers n_1, n_2 such that

$$s_{n_1+1} < u \leq s_{n_1} \leq \ldots\ldots \leq s_{n_2} \leq v < s_{n_2-1}.$$

We define $P(u,v)= P^{(n_1)}(s_{n_1}-u)P^{(n_1-1)}(s_{n_1-1}-s_{n_1})\ldots.P^{(n_2-1)}(v-s_{n_2})$.

Case 2. $s=u < v$. Suppose that $s_{m+1} < v \leq s_m$. Note that because of our definition in Case 1, we have for $n > m$:

$$\begin{aligned} P_{ij}(s_n,v) &= 1 \text{ if either } i=j < m+1 \text{ or } i=j > n, \\ &= \exp(-(v-s_{m+1})) \text{ if } i=j=m+1, \\ &= \exp(-(s_k-s_{k+1})) \text{ if } m+1 < i=j=k+1 \leq n. \end{aligned}$$

This clearly shows that $\lim_{n\to\infty} P(s_n,v)$ exists and we define

$$P(s,v)= \lim_{n\to\infty} P(s_n,v).$$

It is now easy to see that P as defined above is a separately continuous non-homogeneous Markov chain and $R(s,t)$ is given by

$$R(s,t) = U_0 \cup \{(i,i+1): i \in E\}.$$

We claim that $R(s,t)$ cannot be written in the form T_1T_2, where T_1 is reflexive and transitive, $T_1 \neq U_0$, and T_2 any reflexive relation. To see this, notice that if $R=T_1T_2$ as above, then $T_1R=R$ since $T_1=T_1^2$. Since $T_1 \neq U_0$ and $T_1 \subset R$, there is a $i \in E$ such that $(i,i+1) \in T_1$. Since $(i+1,i+2) \in R$, this means that $(i,i+2) \in R$ and this is a contradiction. Thus, $R(s,t)$ cannot be written as an infinite product of reflexive transitive relations. We also claim that this $R(s,t)$ cannot be written even in the form

$$\ldots.T_{-n}T_{-n+1}\ldots.T_{-1}T_0T_1\ldots.T_nT_{n+1}\ldots..$$

where the T_i's are reflexive and transitive and different from U_0. To prove this claim, let us suppose that R has this form. Since $T_{-1} \neq U_0$ and $T_{-1} \subset R(s,t)$, there is an $i_0 \in E$ such that $(i_0,i_0+1) \in T_{-1}$. The element (i_0+1, i_0+2), being an element of R, must be in T_m for some m, but this m must be less than -1 since otherwise the element (i_0,i_0+2) would be in R. Repeating this argument, it follows that

$$\{(j,j+1): j \geq i_0\} \subset \bigcup_{i=-\infty}^{-1} T_i.$$

Since for each $i \in E$, $(i,i+1) \in T_m$ for some m, this means that there is a positive integer N such that

$$R(s,t) \subset \bigcup_{i=-\infty}^{N} T_i.$$

Since we can assume with no loss of generality that any two consecutive T_i's are distinct, there is a $m > N$ such that $(i,i+1) \in T_m$ for some $i > 1$. But since $(i-1,i) \in T_n$ for some $n \leq N$, this means that $(i-1,i+2) \in R$, and this is a contradiction. Let us point out that in the above example the relation R(s,t) can be written, however, in the form $\ldots T_{-n}T_{-n+1}\ldots T_{-1}$, where $T_{-n} = U_0 \cup \{(n,n+1)\}$ is reflexive and transitive for each n. However, if we modify the above example so as to have

$$R^{(n)}(s_{n+1},s_n) = \{(n+1,n)\} \cup U_0,$$

then again we have a non-homogeneous Markov chain P such that R(s,t) cannot be expressed in the form $\ldots T_{-n}T_{-n+1}\ldots T_{-1}$, where each T_i is reflexive and transitive.

In what follows, we show that given an infinite product (forward, backward or two-sided) of reflexive and transitive relations on E, we can always construct a non-homogeneous Markov chain P on a given interval [a,b] such that R(a,b) is the given infinite product. First, a useful proposition.

PROPOSITION 3. Let $0 < c < 1$. Suppose that T is any given reflexive and transitive relation on E. Then there is a homogeneous Markov chain P(t) on $[0,\infty)$ such that $\lim_{t\to 0+} P(t) = P(0)$, $R(t)=T$, and

$$0 < t < \frac{1}{2c} \Rightarrow ||P(t)-I|| \leq 2ct. \square$$

(Here, $|| A || = \sup_{i,j} |A_{ij}|$.)

Proof. Let $0 < c_n < 1$ such that $\sum_{n=1}^{\infty} c_n = \frac{c}{2} < \frac{1}{2}$. Define the matrix D with state space E such that

$$D_{ij} = c_j \text{ if } (i,j) \in T \text{ and } i \neq j;$$

$= 0$ if $(i,j) \notin T$;

$= -\sum_{k \neq i} c_k$ if $i=j$.

Notice that it follows easily by induction that

(1) $\sum_{j=1}^{\infty} (D^n)_{ij} = 0$ for each positive integer n;

(2) $||D^n|| \leq c^n$ for each positive integer n.

Define $P(t) = \exp(tD)$. Then $P(t)$ is well-defined and a stochastic matrix for each t It is also verified easily that $T=R(t)$. Notice that if $0 < 2ct < 1$, then

$$||P(t)-I|| \leq ct.[1+ct+c^2t^2+\ldots]$$

$$= ct/[1-ct] < 2ct. \square$$

PROOF OF THEOREM 7. Choose a sequence (s_n) such that

$a=s_1 < s_2 < \ldots.< s_{n+1} \to b$. For each T_n, define $P^{(n)}$, a homogeneous Markov chain as constructed in Prop. 3, such that $R^{(n)}(s_{n+1}-s_n) = T_n$. We now define a non-homogeneous Markov chain P on subintervals of $[a,b)$ as follows:

If u, v are such that

$$s_m \leq u < s_{m+1} < \ldots. < s_{m+p} \leq v < s_{m+p+1},$$

then define

$$P(u,v)=P^{(m)}(s_{m+1}-u)P^{(m+1)}(s_{m+2}-s_{m+1})\ldots.P^{(m+p)}(v-s_{m+p}).$$

We claim that $\lim_{v \to b-} P(u,v)$ exists and is a stochastic matrix. Once we prove this, we will define $P(a,b)$ as the limit of $P(a,v)$ as $v \to b-$.

It will be sufficient to show that for each $u \in [a,b)$,

(i) $\lim_{v \to b-} P_{ij}(u,v)$ exists (for i,j in E) and (ii) given $\varepsilon > 0$ and $i \in E$, there is a positive integer N and a $\delta > 0$ such that

$$\sum_{j=1}^{N} P_{ij}(u,v) > 1 - \varepsilon$$

whenever $u \leq v$ and $b - \delta < v < b$.

To establish the above results, we choose m_o such that

$$m \geq m_C \Rightarrow b - s_m < \frac{1}{2c}.$$

Let $n \geq m_0$ and $s_n \leq v \leq v' \leq s_{n+1}$. Then for $u \leq v$,

$$|P_{ij}(u,v) - P_{ij}(u,v')$$

$$= |P_{ij}(u,v) - \sum_k P_{ik}(u,v)P_{kj}(v,v')|$$

$$\leqq P_{ij}(u,v).|1 - P_{ij}(v,v')| + \sum_{k\neq j} P_{ik}(u,v)P_{kj}(v,v')$$

$$\leqq 2c.(v'-v)$$

so that

$$||P(u,v) - P(u,v')|| \leq 2c(v'-v). \tag{3.1}$$

Now if $s_n \leq v \leq s_{n+1}$ and $s_{n+p} \leq v' \leq s_{n+p+1}$, then

$$||P(u,v) - P(u,v')||$$

$$\leq ||P(u,v) - P(u,s_{n+1})||$$

$$+ ||P(u,s_{n+1}) - P(u,s_{n+2})||$$

$$+\ldots\ldots+||P(u,s_{n+p}) - P(u,v')||$$

$$\leq 2c(v'-v), \text{ by (3.1)}.$$

Taking $u = v$, we have

$$||I - P(v,v')|| < \varepsilon \tag{3.2}$$

whenever $v'-v < \varepsilon/_{2c}$.

Choose $\delta > 0$ such that $b - \delta \leq v \leq v' < b \Rightarrow v' - v < \varepsilon/_{2c}$.

Then for $v = b - \delta$, let N be such that

$$\sum_{j=1}^{N} P_{ij}(u,v) > 1 - \varepsilon. \tag{3.3}$$

Since

$$\sum_{j=1}^{N} P_{ij}(u,v') \geq \sum_{j=1}^{N} P_{ij}(u,v)P_{jj}(v,v')$$

$$> 1 - 2\varepsilon, \text{ by (3.2) and (3.3)}$$

the theorem follows.□

REFERENCES

1. Chung, K. L.: Markov chains with stationary transition probabilities. Berlin-Heidelberg-New York. Second Edition, Springer 1967.

2. Kingman, J.F.C. and David Williams.: The combinatorial structure of non-homogeneous Markov chain. Z. Wahrscheinlichkeitstheorie verw. Geb. 26, 77-86, 1973.

3. Mukherjea, A. and A. Nakassis.: Convergence of non-homogeneous stochastic chains with at most countable states, to appear in the J. of Multivariate Analysis (Academic Press).