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FUNCTIONS IN THE SPACE $R^2(E)$ AT BOUNDARY POINTS OF THE INTERIOR

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ABSTRACT. Let E be a compact subset of the complex plane $\mathbb{C}$. We denote by R(E) the algebra consisting of (the restrictions to E of) rational functions with poles off E. Let m denote 2 - dimensional Lebesgue measure. For $p \geq 1$, let $R^p(E)$ be the closure of R(E) in $L^p(E,dm)$.

In this paper we consider the case p = 2. Let $x \in \partial E$ be a bounded point evaluation for $R^2(E)$. Suppose there is a $C > 0$ such that x is a limit point of the set $S = \{y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y - x|\}$. For those $y \in S$ sufficiently near x we prove statements about $|f(y) - f(x)|$ for all $f \in R(E)$.

KEY WORDS AND PHRASES. *Rational functions, compact set L^p - spaces, bounded point evaluation, admissible function.*

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1. INTRODUCTION AND DEFINITIONS

Let E be a compact subset of the complex plane $\mathbb{C}$. We denote by R(E) the algebra consisting of (the restrictions to E of) rational functions with poles off E. Let m denote 2 - dimensional Lebesgue measure. For $p \geq 1$, let $R^p(E)$ be the closure of R(E) in $L^p(E,dm)$. A point $x \in E$ is said to be a bounded point evaluation (BPE) for $R^p(E)$ if there is a constant F such that

$$|f(x)| \leq F \cdot \{\int_E |f(z)|^p dm(z)\}^{\frac{1}{p}} \text{ for all } f \in R(E).$$

In [4] we studied the smoothness properties of functions in $R^p(E)$, $p > 2$, at BPE's. When $p = 2$, the situation is quite different (see Fernström and Polking [2] and Fernström [1]). In [5] we showed that at certain BPE's the functions in $R^2(E)$ have the following smoothness property: Let $x \in \partial E$ be both a BPE for $R^2(E)$ and the vertex of a sector contained in Int E. Let L be a line segment that bisects the sector and has an end point at x. Then for each $\epsilon > 0$ there is a $\delta > 0$ such that if

$y \in L$ and $|y - x| < \delta$, $|f(y) - f(x)| \le \epsilon \|f\|_2$ for all $f \in R(E)$. The goal of this paper is to extend this result to certain cases where there may not be a sector in Int E having vertex at x, but x is still a limit point of Int E.

If $x \in E$ is a BPE for $R^2(E)$, there is a function $g \in L^2(E)$ such that $f(x) = \int_E fg\, dm$ for any $f \in R(E)$. Such a function g is called a representing function for x.

A point $x \in E$ is a bounded point derivation (BPD) of order s for $R^2(E)$ if the map $f \to f^{(s)}(x)$, $f \in R(E)$, extends from $R(E)$ to a bounded linear functional on $R^2(E)$.

Let $A_n(x)$ denote the annulus $\{z \mid 2^{-n-1} \le |z - x| \le 2^{-n}\}$. Let $A_n'(x) = \{z \mid 2^{-n-2} \le |z - x| \le 2^{-n+1}\}$. If $x = 0$, we will denote $A_n(0)$ by A_n and $A_n'(0)$ by A_n'.

For an arbitrary set $X \subset \mathbb{C}$ we let $C_2(X)$ denote the Bessel capacity of X which is defined using the Bessel kernel of order 1 (see [3]).

We say that ϕ is an admissible function if ϕ is a positive, nondecreasing function defined on $(0,\infty)$, and $r \cdot \phi(r)^{-1}$ is nondecreasing and tends to zero when $r \to 0^+$.

Using the techniques of [4] and [2] one can prove:

THEOREM 1.1. Let s be a nonnegative integer and E a compact set. Suppose that x is a BPE for $R^2(E)$ and ϕ is admissible. Then x is represented by a function $g \in L^2(E)$ such that

$$\frac{g}{(z-x)^s \cdot \phi(|z-x|) \cdot} \in L^2(E)$$

if and only if $\sum_{n=0}^{\infty} 2^{2n(s+1)} \phi(2^{-n})^{-2} C_2(A_n(x)-E) < \infty$.

2. THE MAIN RESULTS

Let $x \in \partial E$ be a BPE for $R^2(E)$. We may assume that $x = 0$ and that $E \subset \{|z| < 1\}$. Suppose there is a positive constant C such that 0 is a limit point of the set $S = \{y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y|\}$. We will construct a function $g \in L^2(E)$ which represents 0 for $R^2(E)$ and has support disjoint from S.

LEMMA 2.1. Let $0 \in \partial E$ be a BPE for $R^2(E)$. Suppose there is a positive constant C such that 0 is a limit point of the set $S = \{y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y|\}$. Then there is a function $g \in L^2(E)$ such that:

(i) g represents 0 for $R^2(E)$,

(ii) $m((\text{supp } g) \cap S) = 0$,

(iii) For all $n \geq 2$, $\int_{A_n \cap E} |g|^2 \, dm \leq F \sum_{k=\left[\frac{n-2}{2}\right]}^{k=\left[\frac{n-2}{2}\right]+1} 2^{2k} C_2(A'_{2k+1} - E)$

where F is a constant independent of n.

PROOF. For each i, i = 0,1,2,... consider all the intersections of the set $A_i = \{z | 2^{-i-1} \leq |z| \leq 2^{-i}\}$ with the bounded components of $\mathbb{C} - E$. Let Y_i be the closure of the union of these intersections. Since Y_i is compact, it can be covered by finitely many open discs of radius $< C3^{-1}2^{-i-1}$. Let the union (finite) of these discs be denoted by B_i. The set B_i is bounded by finitely many closed Jordan curves each of which is the union of finitely many circular arcs. Each set B_i is contained in a set C_i bounded by finitely many closed Jordan curves Γ_{ij}, $j = 1,2,..,n_i$ such that if z belongs to any one of these curves, Dist $(z,B_i) = 2C3^{-1}2^{-i-1}$.

Now for each k, $k = 0,1,2,\ldots$ choose a function $\lambda_k \in C_o^1$ such that:

(1) $\operatorname{supp} \lambda_k \subset A'_{2k+1}$

(2) $\lambda_k(z) = 1$ for $z \in \{z | 2^{-2k-2} \le |z| \le 2^{-2k-1}\} \cap B_{2k+1}$

(3) $\lambda_k(z) = 0$ for $z \notin \bigcup_{i=o}^{\infty} C_i$

(4) $\left|\frac{\partial \lambda_k(z)}{\partial x_1}\right| \le F_1 \cdot 2^{2k+1}$, $\left|\frac{\partial \lambda_k(z)}{\partial x_2}\right| \le F_2 \cdot 2^{2k+1}$

where $z = x_1 + ix_2$ and F_1 and F_2 are constants independent of k.

(5) $\lambda_k(z) + \lambda_{k+1}(z) = 1$ for $z \in \{z \mid 2^{-2k-3} \le |z| \le 2^{-2k-2}\} \cap B_{2k+2}$.

Given any $\epsilon > 0$ we use a lemma of Fernström and Polking [2] to obtain functions $\psi_k \in C^\infty$ such that:

(1) $\psi_k(z) \equiv 1$ for z near $A'_k - \{z \mid \operatorname{Dist}(z,E) < \epsilon\}$.

(2) $$\int_{|z| \le 2^{-k+1}} |D^\beta \psi_k(z)|^2 \, dm(z) \le F \cdot 2^{-2k(1-|\beta|)} C_2(A'_k - E)$$

for $\beta = (0,0)$, $(0,1)$, and $(1,0)$. Here the constant F is independent of ϵ and k.

Since $\operatorname{supp} \lambda_k \subset A'_{2k+1}$, we have $\psi_{2k+1} \cdot \lambda_k = \lambda_k$ on the set $\{z \mid \operatorname{Dist}(z,E) \ge \epsilon\}$. Thus $\sum_o^\infty \psi_{2k+1} \cdot \lambda_k \equiv 1$ on $\{|z| \le 4^{-1}\}$ - $\{z \mid \operatorname{Dist}(z,E) < \epsilon\}$. Choose $\chi \in C_o^\infty$ with $\chi(z) \equiv 1$ near E. Set $h(z) = \chi(z) \cdot \frac{1}{\pi z}$. For each double index $\beta = (0,0), (0,1)$, and $(1,0)$ there is a constant F_β such that

$$|D^\beta h(z)| \le F_\beta \cdot |z|^{-1-|\beta|}.$$

$$\text{Set } f_\epsilon = h \cdot \sum_o^\infty \psi_{2k+1} \cdot \lambda_k = \sum_o^\infty \psi_{2k+1} \cdot h_k$$

where $h_k = \lambda_k h$.

Since supp $\lambda_k \subset A'_{2k+1}$, the above inequalities imply that

$$|D^{\beta}h_k(z)| \le F_{\beta}2^{(2k+1)(1+|\beta|)}$$

The subadditivity of C_2 and the convergence of $\sum_0^{\infty} 2^{2k} C_2(A_k - E)$ (see Theorem 1.1) imply that the net $\{f_{\varepsilon}\}$ is bounded in L_1^2. There is a subsequence that converges weakly to a function $f \in L^2_{1,loc}$ which satisfies $f(z) = \frac{\chi(z)}{\pi z}$ for $z \in \mathbb{C} - E$ and $f(z) = 0$ for every $z \in E \cap \{z|\mathrm{Dist}(z,\partial E) \ge C|z|\}$. Set $g = -\frac{\partial}{\partial \bar{z}} f$. Then $g \in L^2(E)$ since $f \in L_1^2(E)$, and g is a representing function for 0. The proof of (iii) proceeds as in [5].

The above lemma can be used to prove the following theorem in almost the same way that in [5] Lemma 5.1 is used to prove Theorem 5.1.

THEOREM 2.1. Let $0 \in \partial E$ be a BPE for $R^2(E)$. Let C be a positive constant such that 0 is a limit point of the set $S = \{y|y \in \mathrm{Int}\, E, \mathrm{Dist}(y,\partial E) \ge C\,|y|\}$. Let g be a representing function for 0 and suppose that $g(z)\cdot\phi(|z|)^{-1} \in L^2(E)$ where ϕ is an admissible function. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $y \in S$ and $|y| < \delta$,

$$|f(y) - f(0)| \le \varepsilon\phi(|y|)\|f\|_2$$

for all $f \in R(E)$.

Using this theorem and the methods in [4] one can prove:

COROLLARY 2.1. Suppose that all the conditions of Theorem 2.1 hold. Suppose, moreover, that s is a positive integer such that $g(z)\cdot z^{-s}\cdot \phi(|z|)^{-1} \in L^2(E)$. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $y \in S$ and $|y| < \delta$,

$$|f(y) - f(0) - \frac{f'(0)}{1!}(y - 0) - \cdots - \frac{f^{(s)}0}{s!}(y - 0)^s| \le \varepsilon|y-0|^s\phi(|y|)\|f\|_2$$

for all $f \in R(E)$.

Finally, there is a corollary with weaker preconditions.

COROLLARY 2.2. Let 0,g, and ϕ be as in Theorem 2.1. Suppose there is a positive constant C such that 0 is a limit point of the set

$$S = \{y \mid y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C \cdot \phi(|y|)|y|\}$$

Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $y \in S$ and $|y| < \delta$,

$$|f(y) - f(0)| \leq \varepsilon \|f\|_2 \text{ for all } f \in R(E).$$

The proof is similar to the proof of Theorem 2.1. One uses the fact that there exists an admissible function $\overline{\phi}$ such that $g.\phi^{-1}. \overline{\phi}^{-1} \in L^2(E)$.

3. EXAMPLES

EXAMPLE 1. We will construct a compact set E such that $0 \in \partial E$, 0 is a BPE for $R^2(E)$, and 0 is a limit point of Int E. Let D = $\{|z| \leq 1\}$. Let D_i, i = 1,2,3,..., be the open disc centered on the positive real axis at $3 \cdot 2^{-i-3}$ and having radius $r_i = \exp(-2^{2i} i^2)$.

Let $E = D - \bigcup_{i=1}^{\infty} D_i$. Then since $C_2(B(x,r)) \leq F(\log \frac{1}{r})^{-1}, r \leq r_o < 1$, (see [3]), we have

$$\sum_{n=1}^{\infty} 2^{2n} C_2(A_n - E) = \sum_{n=1}^{\infty} 2^{2n} C(D_n) \leq F \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Thus 0 is a BPE for $R^2(E)$. If C is a positive constant sufficiently small (any positive number $< \frac{1}{2}$ will do), the set $\{y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C\, |y|\}$ intersects the positive real axis in a sequence of disjoint intervals $[a_n, b_n]$ such that $b_n \to 0$.

EXAMPLE 2. Next we construct a compact set E which is like Example 1 in that 0 is a limit point of Int E and a BPE for $R^2(E)$. In this example, however, there exists no sequence $\{y_n\} \subset$ Int E such that $|f(y_n) - f(0)| \leq \varepsilon \|f\|_2$ for all $f \in R(E)$ if $|y_n| < \delta$. We will use important parts of Fernström's construction in [1]. Let F be a positive constant such that $C_2(B(z,r)) \leq F(\log \frac{1}{r})^{-1}$ for all r, $r \leq r_o < 1$. Choose α, $\alpha \geq 1$ such that

$$\frac{F}{\alpha} \cdot \sum_{n=1}^{\infty} \frac{1}{n\log^2 n} < C_2(B(0,1/2)).$$

Let A_o be the closed unit square with center at 0. Cover A_o with 4^n squares of side 2^{-n}. Call the squares $A_n^{(i)}$, $i = 1,2..,4^n$. In every set $A_n^{(i)}$ put an open disc $B_n^{(i)}$ such that $B_n^{(i)}$ and $A_n^{(i)}$ have the same center, and the radius of $B_n^{(i)}$ is $\exp(-\alpha 4^n n\log^2 n)$. Let D_i, $i = 1,2,3,\ldots$ be an open disc centered on the positive real axis such that $D_i \subset \{z|2^{-i-1} \leq |z| \leq 2^{-i}\}$ and $r_i = \exp(-2^{2i}i^2)$. For each n, $n = 1,2,3,\ldots$, let $G_n = \bigcup_i B_n^{(i)}$ where the summation is over those indices i such that $1 \leq i \leq 4^n$ and $B_n^{(i)} \cap (\bigcup_1^{\infty} D_i) = \emptyset$. Set $E_1 = A_o - \bigcup_{n=2}^{\infty} G_n$. Then $R^2(E_1)$ has no BPE's in ∂E_1 as is shown in [1].

Now replace a suitable number of the discs $B_n^{(i)}, B_n^{(i)} \subset \bigcup_{j=2}^{\infty} G_j$, to obtain a compact set E_2 such that 0 is the only boundary point of E_2 that is a BPE for $R^2(E_2)$, (see [1]). This can be done so that Int $E_2 = \bigcup_1^{\infty} D_i$. If $y \in$ Int E_2, let norm(y) denote the norm of "evaluation at y" as a linear functional on $R^2(E_2)$. Then if $\{y_k\} \subset D_i$, and $y_k \to \partial D_i$, norm$(y_k) \to \infty$; otherwise some point on ∂D_i would be a BPE for $R^2(E_2)$.

For each i choose an open disc $D_i' \subset D_i$ such that D_i' and D_i are concentric and such that if $y \in D_i - D_i'$, then norm(y) for the space $R^2(E_2 - D_i')$ is greater than i.

Now let $E = E_2 - \bigcup_{i=1}^{\infty} D_i'$.

The radii of the D_i^1 are so small that 0 is also a BPE for $R^2(E)$. Let $\{y_n\}$ be any sequence in Int E such that $y_n \to 0$. Let norm(y_n) = norm of "evaluation at y_n" on $R^2(E)$. Then for no $\varepsilon > 0$ is there a $\delta > 0$ such that if $|y_n| < \delta$, $|f(y_n) - f(0)| \le \varepsilon \|f\|_2$ for all $f \in R(E)$.

EXAMPLE 3. Let ϕ be an admissible function. Obtain a compact set E in the same way that the set E_2 was obtained in Example 2 so that:

(1) D_i is centered at $3 \cdot 2^{-i-2}$ and has radius

$r_i = \phi(3 \cdot 2^{-i-2}) \cdot 2^{-i-2}$

(2) $\sum_{n=0}^{\infty} 2^{2n} \cdot \phi(2^{-n})^{-2} C_2(A_n(0) - E) < \infty$, and

(3) $\sum_{n=0}^{\infty} 2^{2n} C_2(A_n(x) - E) = \infty$ for $x \neq 0$, $x \notin \bigcup D_i$.

Let $y_i = 3 \cdot 2^{-i-2}$. Then by the choice of r_i, Dist$(y_i, E) \ge 3^{-1} \cdot \phi(|y_i|)|y_i|$. But there is no $C > 0$ such that Dist$(y_i, \partial E) \ge C|y_i|$ for all i. Hence Corollary 2.2 applies to the sequence $\{y_i\}$ but Theorem 2.2 does not.

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