Internat. J. Math. & Math Sci.
Vol. 5 No. 4 (1982) 729-736

DISTRIBUTIONAL AND ENTIRE SOLUTIONS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

JOSEPH WIENER

Department of Mathematics
Pan American University
Edinburg, Texas 78539

(Received February 26, 1982)

ABSTRACT. A unified approach to the study of generalized-function and entire solutions to linear functional differential equations with polynomial coefficients is suggested.

KEY WORDS AND PHRASES. Functional Differential Equations, Distributional Solutions, Entire Solutions.

1980 Mathematics Subject Classification Codes. 34K25, 34K99, 46F10, 30D05.

1. INTRODUCTION AND PRELIMINARIES.

An interesting survey of recent results on entire solutions of ordinary differential equations with polynomial coefficients is given in [1]. In the present paper we continue the study of distributional solutions to linear functional differential equations (FDE) in accordance with the concepts outlined in [2] and [3]. There are profound and close links between spaces of generalized and entire functions [4]. Therefore, the basic ideas in the method of proof may also be applied to the study of entire solutions to linear FDE, especially with linear transformations of the argument. We investigate such linear homogeneous FDE with infinitely smooth coefficients that have solutions in the class of singular functionals which is impossible for analogous ordinary differential equations. Solutions of linear FDE with countable sets of variable argument deviations are considered in the generalized-function space $(S_0^\beta)'$ conjugate to the space S_0^β of testing functions $\phi(t)$ that satisfy the restriction [4]

$$|\phi^{(n)}(t)| \leq ac^n n^{n\beta}, \ \beta > 1.$$

In the sequel, $\delta^{(n)}$ denotes the nth derivative of the Dirac δ measure and $\langle f, \phi \rangle$ is the value of the functional f applied to the testing function ϕ of the real variable t. The norm of a matrix is defined to be

$$\| A \| = \max_i \sum_j |a_{ij}| \quad ,$$

and E is the identity matrix. In [2], it has been proved that under certain conditions the system

$$\sum_{i=0}^{\infty} \sum_{j=0}^{m} (A_{ij} + tB_{ij}) x^{(j)}(\lambda_j t) = tx(\lambda t)$$

has a solution

$$x(t) = \sum_{n=0}^{\infty} x_n \delta^{(n)}(t) \tag{1.1}$$

in $(S_0^\beta)'$ with arbitrary $\beta > 1$. To ensure the convergence of series (1.1), it is sufficient to require that for $n \to \infty$ the vectors x_n satisfy the inequalities

$$\| x_n \| \le bd^n \, n^{-n\rho}, \ \rho > 1 \tag{1.2}$$

since

$$\| \sum_{n=0}^{\infty} \langle x_n \, \delta^{(n)}(t), \, \phi(t) \rangle \| = \| \sum_{n=0}^{\infty} (-1)^n \, \phi^{(n)}(0) \, x_n \| \le$$

$$\le \sum_{n=0}^{\infty} |\phi^{(n)}(0)| \, \| x_n \| \le ab \sum_{n=0}^{\infty} (cdn^{\beta-\rho})^n < \infty \ ,$$

for $\beta < \rho$. If series (1.1) converges, its sum represents the general form of a linear functional in $(S_0^\beta)'$ with the support $t = 0$ [5]. Some recent developments in astrophysics posed new problems about the existence of distributional solutions also to certain integral equations [2].

2. EXISTENCE OF DISTRIBUTIONAL SOLUTIONS.

We look for solutions of the form (1.1) to the system

$$\sum_{i=0}^{\infty} \sum_{j=0}^{m} A_{ij}(t) \, x^{(j)}(\lambda_{ij}(t)) = 0. \tag{2.1}$$

LEMMA. If t_0 is a fixed point of the function $\lambda(t) \in C^1(-\infty, \infty)$ and $\lambda'(t_0) \neq 0$, then in some neighborhood of t_0,

$$\delta^{(n)}(\lambda(t) - t_0) = \delta^{(n)}(t - t_0) \, / \, (\lambda'(t_0))^n \, |\lambda'(t_0)| \tag{2.2}$$

PROOF. Since $\lambda(t_0) - t_0 = 0$,

$$\lambda(t) - t_0 = (t - t_0)\,\psi(t),\ \psi(t_0) = \lambda'(t_0).$$

There exists a neighborhood D_0 of t_0 at all points of which $\psi(t) \neq 0$, $t \neq t_0$, for assuming the opposite we find a sequence $t_{0\nu} \to t_0$ such that $\lambda(t_{0\nu}) - \lambda(t_0) = 0$ and, hence, $\lambda'(t_0) = 0$.

In D_0 for $t \neq t_0$,

$$\delta^{(n)}(\lambda(t) - t_0) = \delta^{(n)}(\psi(t)(t - t_0)) = 0.$$

But for $t = t_0$,

$$\delta^{(n)}(\lambda(t) - t_0) = \delta^{(n)}(\lambda'(t_0)(t - t_0)). \tag{2.3}$$

Thus, (2.3) holds for all $t \in D_0$ and it remains to observe that $\delta^{(n)}(ct) = \delta^{(n)}(t) \,/\, c^n\,|c|$.

THEOREM 2.1. Let (2.1), in which $x(t)$ is an r-vector and $A_{ij}(t)$ are $r \times r$ - matrices, satisfy the following hypotheses.

(i) The real-valued functions $\lambda_{ij}(t) \in C^1$ have a common fixed point t_0 and $0 < |\lambda'_{00}\,(t_0)| < 1$, $|\lambda'_{ij}\,(t_0)| \geq 1$, $i + j \geq 1$.

(ii) The coefficients $A_{ij}(t)$ are polynomials of degree not exceeding p:

$$A_{ij}(t) = \sum_{k=0}^{p} A_{ijk}(t - t_0)^k,\ A_{00}(t) = A(t - t_0)^p,\ p \geq 1.$$

(iii) The series $\sum_{i=1}^{\infty} \lambda_i^{-1} A_i$ converges where

$$A_i = \max_{j,k} \| A_{ijk} \|,\ \lambda_i = \inf_{j} |\lambda'_{ij}(t_0)\ ,\quad i + j \geq 1.$$

(iv) The matrix A is nonsingular and

$$c = |\lambda'_{00}(t_0)|^{-p-1}\ \| A \| - \sum_{i \geq 1} |\lambda'_{io}(t_0)|^{-p-1}\ \| A_{iop} \| > 0.$$

Then, in some neighborhood of t_0, there exists a solution $x(t) \in (S_0^{\beta})'$ with arbitrary $\beta > 1$:

$$x(t) = \sum_{n=0}^{\infty} x_n\, \delta^{(n)}(t - t_0).$$

PROOF. By virtue of (2.2) and the formula $t^k\,\delta^{(n)}(t) = (-1)^k n!\,\delta^{(n-k)}(t) \,/\, (n-k)!$, for $n \geq k$, and 0 for $n < k$, we obtain the equations

$$\sum_{i,j,k} (-1)^k A_{ijk} \sum_{n+j\geq k} (n+j)!\, x_n \delta^{(n+j-k)}(t-t_0) \,/\, (n+j-k)!\, |\alpha_{ij}|\, \alpha_{ij}^{n+j} = 0,$$

$$\alpha_{ij} = \lambda'_{ij}(t_0)$$

for the unknowns x_n of the solution $x(t)$. Hence,

$$\Sigma\, (-1)^k (n+k)!\, |\alpha_{ij}|^{-1} \alpha_{ij}^{-n-k} A_{ijk}\, x_{n+k-j} = 0,\ n \geq 0$$

which can be written as

$$\sum_{k-j<p} (-1)^{p-k}(n+k)!\, |\alpha_{ij}|^{-1} \alpha_{ij}^{-n-k} A_{ijk}\, x_{n+k-j} \,/\, (n+p)! \,+$$

$$+ \Big(\sum_{i\geq 0} |\alpha_{i0}|^{-1} \alpha_{i0}^{-n-p} A_{i0p} \Big) x_{n+p} = 0 .$$

Since $A_{00k} = 0 (k < p)$, the first sum does not include terms with α_{00}. According to (iv), the coefficients of x_{n+p} are nonsingular matrices and

$$\Big\| \Big(\sum_{i=0}^{\infty} |\alpha_{i0}|^{-1} \alpha_{i0}^{-n-p} A_{i0p} \Big)^{-1} \Big\| \leq c\, \| A \|\, |\alpha_{00}|^n .$$

Consequently,

$$\| x_{n+p} \| \leq \mu\, q^{n+p} \sum_{k=0}^{m+p-1} \| x_{n+k-m} \|,\ 0 < q < 1 \tag{2.4}$$

where μ is some positive constant. Using the notation

$$M_n = \max_{0\leq i\leq n} \| x_i \|, \tag{2.5}$$

we conclude from (2.4) that

$$\| x_{n+p} \| \leq \mu(m+p) q^{n+p} M_{n+p-1} .$$

For large n, there is $\mu(m+p)q^{n+p} \leq 1$. Hence, $\| x_{n+p} \| \leq M_{n+p-1}$ and $M_{n+p} = M_{n+p-1}$. Thus, starting with some N,

$$M_n = M_N ,\ n \geq N. \tag{2.6}$$

The application of (2.6) to (2.4) successively yields:

$$\| x_{N+p+k} \| \leq \mu(m+p) q^{N+p} M_N ,$$

$$\| x_{N+p+(m+p)+k} \| \leq \mu^2 (m+p)^2 q^{N+p} q^{N+p+(m+p)} M_N \ ,$$

$$\| x_{N+p+2(m+p)+k} \| \leq \mu^3 (m+p)^3 q^{N+q} q^{N+p+(m+p)} q^{N+p+2(m+p)} M_N$$

$$(0 \leq k \leq m+p-1).$$

The conjecture

$$\| x_{N+p+n(m+p)+k} \| \leq \mu^{n+1} (m+p)^{n+1} q^{n(N+p)+n(n+1)(m+p)/2} M_N \tag{2.7}$$

may readily be ascertained by induction for all n and the mentioned values of k, and proves the theorem since the condition $0 < q < 1$ makes it more restrictive than (1.2).

3. EXISTENCE OF ENTIRE SOLUTIONS.

We apply the method of the previous section to prove the existence of entire solutions of linear FDE with polynomial coefficients and to evaluate their order of growth.

THEOREM 3.1. Suppose the system

$$X^{(P)}(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{p} Q_{ij}(t) X^{(j)}(\lambda_{ij} t), \ X^{(j)}(0) = X_j \ , \ j = 0, \ldots, p-1 \tag{3.1}$$

in which Q_{ij} and X are $(r \times r)$ - matrices, satisfies the following conditions:

(i) $Q_{ij}(t)$ are polynomials of degree not exceeding m;

(ii) λ_{ij} are complex numbers such that

$$0 < q_1 \leq |\lambda_{ij}| \leq 1, \ (j=0, \ldots, p-1), \ 0 < q_2 \leq |\lambda_{ip}| \leq q_3 < 1;$$

(iii) the series $\Sigma\, Q^{(i)}$ converges, where $Q^{(i)} = \max\limits_{j,k} \| Q_{ijk} \|$ and Q_{ijk} are the coefficients of $Q_{ij}(t)$, and $\sum\limits_{i=0}^{\infty} \| Q_{ip}(0) \| < 1$.

Then the problem has a unique holomorphich solution, which is an entire function or order not exceeding m + p.

PROOF. The expansions

$$Q_{ij}(t) = \sum_{k=0}^{m} Q_{ijk} t^k, \ X(t) = \sum_{n=0}^{\infty} X_n t^n$$

imply that

$$X^{(p)}(t) = \sum_{n=0}^{\infty} X_{n+p} t^n (n+p)! \,/\, n! \ ,$$

$$x^{(j)}(\lambda_{ij}t) = \sum_{n=0}^{\infty} \lambda_{ij}^{n} X_{n+j} t^{n}(n + j)! / n! ,$$

$$Q_{ij}(t)\, x^{(j)}(\lambda_{ij}t) = \sum_{n=0}^{\infty} t^{n} \sum_{k=0}^{m} \lambda_{ij}^{n-k} Q_{ijk} X_{n+j-k}(n + j - k)! / (n - k)!$$

and yield the following recursion relations for the matrices X_n:

$$(E - \sum_{i=0}^{\infty} \lambda_{ip}^{n} Q_{ip0})\, X_{n+p} = \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} \sum_{k=0}^{m} \frac{n!(n + j - k)!}{(n+p)!(n-k)!} \lambda_{ij}^{n-k} Q_{ijk} X_{n+j-k}$$

$$+ \sum_{i=0}^{\infty} \sum_{k=1}^{m} \frac{n!(n + p - k)!}{(n+p)!(n-k)!} \lambda_{ip}^{n-k} Q_{ipk} X_{n+p-k} , \; n \geq 0 \tag{3.2}$$

Hypotheses (ii) and (iii) ensure the existence of the inverse matrices $(E - \sum_{i=0}^{\infty} \lambda_{ip}^{n} Q_{ip0})^{-1}$ for all n:

$$(E - \sum_{i=0}^{\infty} \lambda_{ip}^{n} Q_{ip0})^{-1} = \sum_{k=0}^{\infty} (\sum_{i=0}^{\infty} \lambda_{ip}^{n} Q_{ip0})^{k} ,$$

$$\| (E - \sum_{i=0}^{\infty} \lambda_{ip}^{n} Q_{ip0})^{-1} \| \leq (1 - \sum_{i=0}^{\infty} \| Q_{ip0} \|)^{-1} .$$

Therefore, formulas (3.2) determine the coefficients X_n uniquely and, since

$$n!(n + j - k)! / (n + p)!(n - k)! \leq (n + p)^{-1}, \; 0 \leq j \leq p - 1$$
$$n!(n + p - k)! / (n + p)!(n - k)! \leq 1,$$

we obtain, by virtue of (iii),

$$\| X_{n+p} \| \leq \frac{a}{n + p} \cdot \sum_{j=0}^{p-1} \sum_{k=0}^{m} \| X_{n+j-k} \| + bq_3^{n} \sum_{k=1}^{m} \| X_{n+p-k} \|$$

For large n, there is $q_3^n \leq (n + p)^{-1}$ and

$$\| X_{n+p} \| \leq c(n + p)^{-1} \sum_{k=0}^{m+p-1} \| X_{n+k-m} \|. \tag{3.3}$$

Here a, b, c are some positive constants. With the notation (2.5), it follows from (3.3) that

$$\| X_{n+p} \| \leq c(m + p)M_{n+p-1} / (n + p).$$

Starting with some N,

$$c(m + p) / (n + p) \leq 1, \; \| X_{n+p} \| \leq M_{n+p-1} , \; M_{n+p} = M_{n+p-1}$$

and it remains to apply (2.6) successively to (3.3):

$$\| X_{N+p+k} \| \leq c(m + p)M_N / (N + p),$$

$$\|X_{N+p+(m+p)+k}\| \leq c^2(m+p)^2 M_N \,/\, (N+p)(N+(m+p)+p) ,$$

$$\|X_{N+p+2(m+p)+k}\| \leq c^3(m+p)^3 M_N \,/\, (N+p)(N+(m+p)+p)(N+2(m+p)+p)$$

$$(0 \leq k \leq m+p-1).$$

Now it can be proved easily that, for all n,

$$\|X_{N+p+n(m+p)+k}\| \leq c^{n+1}(m+p)^{n+1} M_N \,/\, \prod_{i=0}^{n} (N+i(m+p)+p).$$

Thus,

$$\|X_{N+p+n(m+p)+k}\| \leq c^{n+1}(m+p) M_N \,/\, n! ,$$

and the solution $X(t)$ is an entire function whose order of growth does not exceed $m + p$.

THEOREM 3.2. If, in addition to the hypotheses of Theorem 3.1, the parameters $\lambda_{ij}(0 \leq j \leq p-1)$ are separated from unity: $0 < q_1 \leq |\lambda_{ij}| \leq q_4 < 1$, the solution of (3.1) is an entire function of zero order.

PROOF. Under the conditions of Theorem 2.1, the system

$$\sum_{i=0}^{\infty} \sum_{j=0}^{m} A_{ij}(t) X^{(j)}(\alpha_{ij} t) = 0 \tag{3.4}$$

with real constants α_{ij} ,

$$0 < |\alpha_{00}| < 1, \ |\alpha_{ij}| \geq 1, \ i + j \geq 1$$

has a distributional solution (1.1), the coefficients X_n of which satisfy inequalities (1.2) and are determined with the exactness to arbitrary $X_0, \ldots, X_{p-1}$. We apply to (3.4) the Laplace transformation assuming α_{ij} positive and retaining the same notation for $X(t)$ and its transform:

$$X^{(p)}(s/\alpha_{00}) + \alpha_{00}^{p+1} A^{-1} \sum_{i+j>0} \sum_{k=0}^{p} (-1)^{p-k} \alpha_{ij}^{-j-1} A_{ijk}(s^j X(s/\alpha_{ij}))^{(k)} = 0$$

The substitutions $s/\alpha_{00} = t$, and $\alpha_{00}/\alpha_{ij} = \lambda_{ij}$ reduce this equation to the form (3.1). This proves the theorem, since the transform of $\delta^{(n)}(t)$ is s^n and the coefficients X_n satisfy (2.7). These estimates use only the moduli of α_{ij}; hence, the parameters λ_{ij} may be complex. Theorems 3.1 and 3.2 generalize the results of [6].

THEOREMS 3.3. The problem

$$F'(z) = \sum_{i=0}^{\infty} A_i(z)\, F(z - a_i) + \sum_{i=0}^{\infty} B_i(z)\, F'\,(z - b_i), \tag{3.5}$$

$$\lim_{\operatorname{Re} z \to -\infty} F(z) = F_0$$

with $(r \times r)$ - matrices A, B, F has a unique holomorphic solution which is an entire function if:

(i) $A_i(z) = \sum_{k=1}^{m} A_{ik} e^{kz}$, $B_i(z) = \sum_{k=0}^{m} B_{ik} e^{kz}$;

(ii) a_i, b_i are complex numbers such that

$$0 \leq \operatorname{Re} a_i \leq M_1 < \infty,\ 0 < M_2 \leq \operatorname{Re} b_i \leq M_3 < \infty \ ;$$

(iii) the series $\Sigma\, A^{(i)}$ and $\Sigma\, B^{(i)}\, e^{-\operatorname{Re} b_i}$

converge where $A^{(i)} = \max_k \| A_{ik} \|$, $B^{(i)} = \max_k \| B_{ik} \|$,

and $\sum_{i=0}^{\infty} \| B_i(0) \|\, e^{-\operatorname{Re} b_i} < 1.$

PROOF. The substitutions $t = e^z$, $\alpha_i = e^{-a_i}$, $\beta_i = e^{-b_i}$, and $F(z) = X(t)$ reduce (3.5) to (3.1) of the first order with the initial condition $X(0) = F_0$

REFERENCES

1. Fricke, G.H.; Roy, Ranjan and Shah, S. M. Bounded index entire solutions of ordinary differential equations and summability methods, Internat. J. Math. & Math. Sci. 4(3), (1981), 417-434.

2. Wiener, Joseph. Generalized-function solutions of linear systems, J. Differ. Equat. 38(2), (1980), 301-315.

3. Wiener, Joseph. Generalized-function solutions of differential and functional differential equations, J. Math. Anal. and Appl. 88(1), (1982), 170-182.

4. Gel'fand, I.M. and Shilov, G.E. Generalized functions, Vol. 2, Academic Press, New York, 1968.

5. Mityagin, B.S. On the infinitely differentiable function with the given values of the derivatives at a point, Dokl. Akad. Nauk SSSR 138(2), (1961), 289-292.

6. Wiener, Joseph. On the entire solutions of a linear functional differential system, Differ. Uravn. 13(3), (1977), 551-555.