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THE CHROMATIC INDEX OF CYCLIC STEINER 2-DESIGNS

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ABSTRACT. The number of colours needed to colour the blocks of a cyclic Steiner 2-design S(2, k, v) is at most v.

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1. INTRODUCTION.

A Steiner 2-design S(2, k, v) is a pair (V, B); V is a v-set of elements and B is a collection of k-subsets of V called blocks. Each 2-subset of V appears in precisely one block. A colour class in a Steiner system is a set of pairwise disjoint blocks. A k-block colouring of (V, B) is a partitioning of B into k colour classes. The chromatic index of a Steiner system is the least k for which a k-block colouring exists.

Steiner systems with small chromatic index have been studied under the guise of resolvable or nearly resolvable designs (see [1,2] and references therein). Often Steiner and related systems are employed in the scheduling of tournaments or experiments; in these contexts, small chromatic index corresponds to few "rounds". A question of much concern here is: what is the largest possible number of rounds required for a specified Steiner 2-design S(2, k, v)? In other words, what is the upper bound on the chromatic index of a Steiner 2-design? One weak upper bound is immediate.

LEMMA 1: The chromatic index of a Steiner 2-design S(2, k, v) is less than kv/(k-1).

PROOF. Given a Steiner system $S(2, k, v)$, construct its <u>block intersection graph</u> as follows. Each block is represented by a vertex; two vertices are adjacent exactly when the corresponding blocks intersect. The chromatic index of a Steiner 2-design is the chromatic number of its block intersection graph. The block intersection graph has maximum degree less than $kv/(k-1)$. Hence, Brooks' theorem [3] guarantees that the chromatic number is at most $kv/(k-1)$.

We suspect that lemma 1 is quite a weak bound; one reason is that a conjecture of Erdős, Faber, and Lovasz [4] would ensure an upper bound of v on the chromatic index. The purpose of this note is to show that the chromatic index of a <u>cyclic</u> Steiner system $S(2,k,v)$ is at most v.

A Steiner system $S(2,k,v)$ is <u>cyclic</u> if its element set is $\{0,1,\dots,v-1\}$ and the mapping $i \to i+1 \pmod v$ is an automorphism. This automorphism partitions the blocks of the Steiner system into <u>orbits</u>. Each orbit contains v blocks when $v \equiv 1 \pmod{k(k-1)}$. When $v \equiv k \pmod{k(k-1)}$, each orbit except one contains v blocks. The exception, the short orbit, contains v/k blocks. The reader is referred to [5] for a detailed survey of cyclic Steiner 2-designs; the simple introduction here suffices to prove

THEOREM 2. A cyclic Steiner system $S(2,k,v)$ has chromatic index at most v.

PROOF. If there is a short orbit of blocks, we use a single colour for all blocks in this orbit, since they are disjoint [5]. For each "full" orbit of blocks, we consider the subgraph of the block intersection graph induced on this orbit. This subgraph has degree $k(k-1)$. Hence, Brooks' theorem guarantees that it can be coloured in $k(k-1)$ colours, unless it is composed of $(k(k-1)+1)$-cliques. Observe that at most one orbit can induce such a graph, and this can only happen when $k(k-1)+1$ divides v. Thus, for $v \equiv 1 \pmod{k(k-1)}$, we need at most v colours, since we have $(v-1)/(k(k-1))$ full orbits. Similarly, for $v \equiv k \pmod{k(k-1)}$, we have one short orbit, and $(v-k)/(k(k-1))$ full orbits; hence, at most $v-k+2$ colours are needed, completing the proof.

Although cyclic systems comprise a very small fraction of all Steiner systems, we believe that the techniques used in theorem 2 are interesting and have general applicability. Future research might employ different decompositions of the block intersection graph into manageable pieces, such as the orbits used here, and the colouring of the system "piece by piece".

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