# ON INTEGRAL TRANSFORMS

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ABSTRACT.  A number of interesting theorems have recently been added to the
existing ones in the field of integral transforms.  The new arrivals can be
broadly classified under two categories.  The first expresses relationship be-
tween images and originals of related functions in the transforms concerned while
the second reveals interconnections existing between images of related functions
in the respective transforms.  An attempt has been made in this paper to unify and
extend all such results by establishing two most general theorems, embracing each
of the two types mentioned above, and of elucidate their basic underlying principle.
Certain special cases of these theorems which are of interest in themselves and
do not seem to be recorded in the literature have also been given.  Exact refer-
ence to twenty four theorems which follow as simple special cases of our theorems
has also been given.

1. UNDERLINE INTRODUCTION.

The linear integral transform $T[f(x); p]$ of the function $f(x)$ is defined by

the integral equation:

$$T[f(x); p] = \int_a^b k(x,p) \, f(x) \, dx = \emptyset(p) \quad \text{(say)} \tag{1.1}$$

wherein the class of functions and the domain of p are so prescribed that the in-

tegral exists. In (1.1), $k(x,p)$ is known as the kernel of the transform, $\emptyset(p)$

the image of $f(x)$ in the said transform, and $f(x)$ the original. The following

important theorem which is an analogue of the well-known Parseval-Goldstein

theorem will be required in the sequel.

If

$$T[f_1(x); p] = \emptyset_1(p)$$

and

$$T[f_2(x); p] = \emptyset_2(p)$$

then

$$\int_0^\infty f_1(x) \, \emptyset_2(x) \, dx = \int_0^\infty f_2(x) \, \emptyset_1(x) \, dx \tag{1.2}$$

provided that the various integrals involved in the theorem converge absolutely.

2. THEOREM 1.

If

$$h_1(p) = T_1[h_2(x) \, g(x); p] = \int_0^\infty k_1(px) \, h_2(x) \, g(x) \, dx \tag{2.1}$$

and

$$h_2(p^\sigma) = T_2[f(x); p] = \int_0^\infty k_2(px) \, f(x) \, dx \tag{2.2}$$

then

$$h_1(p) = \sigma \int_0^\infty f(x) \, \emptyset(x,p) \, dx \tag{2.3}$$

where

$$\emptyset(p,\alpha) = T_2[x^{\sigma-1} \, g(x^\sigma) \, k_1(\alpha x^\sigma); p]$$

$$= \int_0^\infty k_2(px) \, x^{\sigma-1} \, g(x^\sigma) \, k_1(\alpha x^\sigma) \, dx \tag{2.4}$$

provided that the integrals involved in the equations (2.1) to (2.4) are absolute-

ly convergent, $\alpha$ is independent of p, and $\sigma$ is a non-zero real number.

PROOF. Applying generalized Parseval-Goldstein formula given by (1.2) to

operational pairs given by (2.2) and (2.4), we get

$$\int_0^\infty x^{\sigma-1} \, g(x^\sigma) \, k_1(\alpha x^\sigma) \, h_2(x^\sigma) \, dx \;=\; \int_0^\infty \emptyset(x,\alpha) \, f(x) \, dx \qquad (2.5)$$

Now, replacing $\alpha$ by p in (2.5), changing the variable of integration slightly on its left-hand side and interpreting the result thus obtained in terms of (2.1), we easily arrive, after a little simplification, at the required result (2.3).

The above theorem is general in nature and generalizes a large number of interesting theorems given from time to time by several authors lying scattered in the literature. Thus, on making suitable choices of transforms $T_1$, $T_2$, function g(x) and the quantity $\sigma$ in it, we easily get the theorems obtained earlier by Agarwal [1, p. 538], Bose [3, p. 59], Kaushik [10, p. 7], Jaiswal [9, p.85], Kesarwani [11, p. 298], Sharma [22, p.114], K.M. Saksena [16, p.102], Arya [2, p.40], R.K. Saxena [18, p.404]. Gupta [6, p.3], Varma [26, p. 204], Srivastava and Vyas [23, p. 140], Mittal [14, p.33], Koul [12, p.14], S.K. Srivastava [24, p. 61], and many others as special cases of Theorem 1.

THEOREM 1(a). If in Theorem 1, we put $\sigma = 1$ and take both the transforms $T_1$ and $T_2$ as Laplace transforms, $\emptyset(p,\alpha)$ occurring in (2.4) can be given in a slightly altered form with the help of a known property of Laplace transform. In such a case we have (for $\sigma = 1$)

$$\emptyset(p,\alpha) \;=\; L \, [\, g(x) \, e^{-\alpha x} \; ; \; p\,] \;=\; \Theta(p + \alpha)$$

where $\Theta(p) = L \, [\, g(x); \, p\,]$ and the theorem assumes the following form:

If
$$h_1(p) = L \, [\, g(x) \, h_2(x); \, p\,] \;=\; \int_0^\infty e^{-px} \, g(x) \, h_2(x) \, dx \qquad (2.6)$$

and
$$h_2(p) = L \, [\, f(x); \, p\,] \;=\; \int_0^\infty e^{-px} \, f(x) \, dx \qquad (2.7)$$

then
$$h_1(p) = \int_0^\infty f(x) \, \Theta(x + p) \, dx \qquad (2.8)$$

where
$$\Theta(p) \;=\; L \, [\, g(x); \, p\,] \;=\; \int_0^\infty e^{-px} \, g(x) \, dx, \qquad (2.9)$$

Re(p) > 0, and the various integrals involved in equations (2.6) to (2.9) are absolutely convergent.

Theorem 1(a) is of interest in itself. It is also quite general in nature

and generalizes the theorems established earlier by Saxena [19, p.230] , Malco

[13, p.844] , Saxena and Gupta [21, p.160] and several other authors.

3.  __THEOREM 2__

If
$$h_1(p) = T_1 [x^{(c/\sigma)-1} f(x); p] = \int_0^\infty k_1(px) \, x^{(c/\sigma)-1} f(x) \, dx \qquad (3.1)$$

and
$$h_2(p) = T_2 [f(x^{-\sigma}); p] = \int_0^\infty k_2(px) \, f(x^{-\sigma}) \, dx \qquad (3.2)$$

then
$$h_1(p) = \sigma \int_0^\infty h_2(x) \, \emptyset \, (x,p) \, dx \qquad (3.3)$$

where
$$p^{-c}{}^{-1} k_1(\alpha p^{-\sigma}) = T_2 [\emptyset(x,\alpha); p] \qquad (3.4)$$

$\sigma$ is a non-zero real number, $\alpha$ is independent of $p$ and the integrals involved in

the equations (3.1) to (3.4) are absolutely convergent.

PROOF. The above theorem easily follows on applying (1.2) to the pairs given

by (3.2) and (3.4) and proceeding on lines similar to those indicated in the proof

of Theorem 1.

Theorem 2 given above is also sufficiently general in nature to generalize

a number of theorems scattered in the literature on making suitable choice of

transforms $T_1$, $T_2$, and $\sigma$ in it. Thus theorems obtained earlier by Bose [4, p. 18],

Tiwari [25, p. 33], Rathie [15, p. 385], Saxena and Gupta [20, p. 712], Saxena

[17, p. 166] and several others follow as special cases of this theorem.

Now we give below a special case of theorem 2 which is believed to be new

and of interest in itself.

COROLLARY 1.

If
$$h_1(p) = H [x^{(c/\sigma)-1} f(x); p] = \int_0^\infty x^{(c/\sigma)-1} f(x) \, H_{t,q}^{m,n} \left[px \, \middle| \, \begin{matrix} (a_j, \alpha_j)_{1,t} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] dx \quad (3.5)$$

and
$$h_2(p) = K [f(x^{-\sigma}); \nu; p] = \int_0^\infty (px)^{1/2} \, K_\nu \, (px) \, f(x^{-\sigma}) \, dx \qquad (3.6)$$

then
$$h_1(p) = \sigma \, 2^{(1/2)-c} \int_0^\infty x^c \, h_2(x) \, H_{t,q+2}^{m,n} \left[p\left(\tfrac{x}{2}\right)^\sigma \, \middle| \, \begin{matrix} (a_j, \alpha_j)_{1,t} \\ (b_j, \beta_j)_{1,q}; (\tfrac{1}{4} - \tfrac{c}{2} \pm \tfrac{\nu}{2}, \tfrac{\sigma}{2}) \end{matrix}\right] dx \quad (3.7)$$

where the integrals involved in (3.5) to (3.7) are absolutely convergent, $Re(p) >$

0, $\sigma > 0$ and

$$- \sigma \min_{1 \leq j \leq m} [\text{Re}(b_j/\beta_j)] < \min_j \text{Re}(c \pm \nu + 3/2).$$

PROOF. The above corollary follows from Theorem 2, if we take in it $T_1$ as the H-function transform defined by (3.5) (introduced by Gupta and Mittal [8, p.142]), $T_2$ as the well-known Bessel function transform defined by (3.6) and make use of the following formula easily obtainable from the known results [7, Eq.(5.1), (4.4) and (2.1)]

$$K\left[ 2^{(1/2)-c} \, x^c \, H_{t,q+2}^{m,n} \left[ \alpha(\tfrac{x}{2})^\sigma \middle| \begin{array}{c} (a_j,\alpha_j)_{1,t} \\ (b_j,\beta_j)_{1,q}; \, (\tfrac{1}{4} - \tfrac{c}{2} \pm \tfrac{\nu}{2}, \tfrac{\sigma}{2}) \end{array} \right]; \nu \, ; \, p \right]$$

$$= p^{-c-1} \, H_{t,q}^{m,n} \left[ \alpha p^{-\sigma} \middle| \begin{array}{c} (a_j,\alpha_j)_{1,t} \\ (b_j,\beta_j)_{1,q} \end{array} \right] \tag{3.8}$$

The H-function occurring in the corollary stands for Fox's H-function. It is quite general in nature and includes most of the commonly used functions as its special cases. It is defined and represented as follows:

$$H_{t,q}^{m,n} \left[ x \middle| \begin{array}{c} (a_j,\alpha_j)_{1,t} \\ (b_j,\beta_j)_{1,q} \end{array} \right] = H_{t,q}^{m,n} \left[ x \middle| \begin{array}{c} (a_1,\alpha_1),\ldots,(a_t,\alpha_t) \\ (b_1,\beta_1),\ldots,(b_q,\beta_q) \end{array} \right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+\beta_j s) \prod_{j=n+1}^{t} \Gamma(a_j-\alpha_j s)} x^s \, ds \tag{3.9}$$

The nature of contour L, various conditions satisfied by its parameters for the convergence of the above integral and some of the special cases of the H-function can be referred to in the paper by Gupta and Jain [7].

If in Corollary 1, we reduce each of the transforms involved to Laplace transform, we get, after a little simplification a theorem obtained earlier by H.C. Gupta [5, p.140]. Several other theorems connecting different transforms can also be obtained from our corollary, by reducing the H-function transform involved in it to other simpler transforms mentioned by Gupta and Mittal [8]. We however omit the details.

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