Internat. J. Math. & Math. Sci.
Vol. 5 No. 1 (1982) 195-202

RESEARCH NOTES

SOME REMARKS ON THE STABILITY ANALYSIS IN ROBE'S THREE BODY PROBLEM

R. MEIRE
Astronomical Observatory
Ghent State University
Krijgslaan 271 - S9
9000 GENT
BELGIUM

(Received October 12, 1979)

ABSTRACT. An improved technique is presented for the stability analysis of Robe's 3-body problem which gives more accurate results for the transition curves in the parameter plane than does Robe's paper.

A novel property of the system of differential equations describing the motion is used, which reduces the computer time by more than 50%.

KEY WORDS AND PHRASES. *3-body problem, stability, transition curve, Floquet-theory.*
1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 85A05

1. INTRODUCTION.

In a recent paper Robe [1] presented a new kind of restricted three body problem, where one body m_1 is a rigid spherical shell, filled with an homogeneous incompressible fluid of density ρ_1, where a second body m_2 is a mass point outside the shell, and where m_3 is a small solid sphere of density ρ_3, restricted to move inside the shell, its motion determined by the attraction of m_2 and the buoyancy force due to the fluid ρ_1.

There exists a solution with m_3 at the center of the shell while m_2 describes a Keplerian orbit around it. Robe investigated the stability of this configuration under the assumption that the mass of m_3 is infinitesimal. The linearized equations of motion in the neighborhood of this equilibrium are

$$\ddot{x} - 2\dot{y} = \left\{ \frac{1 + 2\mu}{1 + e\cos v} - \frac{K(1 - e^2)^3}{(1 + e\cos v)^4} \right\} x \qquad (1.1)$$

$$\ddot{y} + 2\dot{x} = \left\{ \frac{1 - \mu}{1 + e\cos v} - \frac{K(1 - e^2)^3}{(1 + e\cos v)^4} \right\} y \qquad (1.2)$$

$$\ddot{z} + z = \left\{ \frac{1 - \mu}{1 + e\cos v} - \frac{K(1 - e^2)^3}{(1 + e\cos v)^4} \right\} z \qquad (1.3)$$

where

$$K = \frac{4\pi}{3} \cdot \frac{\rho_1 a^3}{(m_1 + m_2)} \left(1 - \frac{\rho_1}{\rho_3}\right)$$

$$\mu = \frac{m_2}{(m_1^* + m_2)}$$

a = semi-major axis of the Keplerian orbit.

These equations are referred to a coordinate system Oxyz, where 0 is the center of m_1, Ox points to m_2 and Oxy is the plane of the Keplerian orbit.

If $\rho_1 = 0$ (shell empty) or $\rho_1 = \rho_3$, then $K = 0$ and the equations of motion become

$$\ddot{x} - 2\dot{y} = (1 + 2\mu)\, r\, x \qquad (1.4)$$

$$\ddot{y} + 2\dot{x} = (1 - \mu)\, r\, y \qquad (1.5)$$

$$\ddot{z} + z = (1 - \mu)\, r\, z \qquad (1.6)$$

with $r = \dfrac{1}{1 + e\cos v}$.

Equations (1.4) and (1.5) describe the motion in the orbital plane. Robe investigated the stability in the orbital plane by means of the Floquet-theory. However, one can separate the fourth-order system (1.4), (1.5) into two independent second-order systems.

2. THE TRANSFORMATION TO SECOND-ORDER SYSTEMS.

Using $\xi = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\eta = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$,

equations (1.4) and (1.5) can be written

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} 0 & E \\ rC_0 & 2D \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{2.1}$$

where
$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and
$$C_0 = \begin{bmatrix} 1+2\mu & 0 \\ 0 & 1-\mu \end{bmatrix}$$

Now we make the following transformation (Tschauner [2])

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} E & E \\ P_1 & P_2 \end{bmatrix} \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} \tag{2.2}$$

and obtain

$$\begin{bmatrix} \dot{\delta} \\ \dot{\varepsilon} \end{bmatrix} = \frac{1}{P_2 - P_1} \cdot \begin{bmatrix} P_2P_1-rC_0-2DP_1+P_1' & P_2^2-rC_0-2DP_2+P_2' \\ -P_2^2+rC_0+2DP_1-P_1' & -P_1P_2+rC_0+2DP_2-P_2' \end{bmatrix} \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix}$$

Making the nondiagonal elements zero, we obtain

$$\begin{bmatrix} \dot{\delta} \\ \dot{\varepsilon} \end{bmatrix} = \begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix} \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} \tag{2.3}$$

where P_1 and P_2 are two different solutions of the Riccati equation

$$\dot{P} = 2DP - P^2 + rC_0 \, . \tag{2.4}$$

Using
$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},$$

equation (2.4) becomes

$$\left.\begin{aligned}
\dot{p}_{11} &= 2p_{21} - p_{11}^2 - p_{12}p_{21} + r(1 + 2\mu) \\
\dot{p}_{22} &= -2p_{12} - p_{22}^2 - p_{12}p_{21} + r(1 - \mu) \\
\dot{p}_{12} &= 2p_{22} - p_{12}(p_{11} + p_{22}) \\
\dot{p}_{21} &= -2p_{11} - p_{21}(p_{11} + p_{22})
\end{aligned}\right\} \qquad (2.5)$$

Now let

$$\left.\begin{aligned}
p_{11} &= w + z \\
p_{22} &= w - z \\
p_{12} &= u - v + 1 \\
p_{21} &= u + v - 1
\end{aligned}\right\} \qquad (2.6)$$

Then equation (2.5) becomes

$$\left.\begin{aligned}
\dot{w} &= -1 - w^2 - z^2 - u^2 + v^2 + r\left(\frac{2 + u}{2}\right) \\
\dot{z} &= 2u - 2wz + r\,\frac{3}{2}\,\mu \\
\dot{u} &= -2z - 2uw \\
\dot{v} &= -2vw
\end{aligned}\right\} \qquad (2.7)$$

The last two equations yield

$$\left.\begin{aligned}
w &= \frac{v}{2}\left(\dot{\overline{\frac{1}{v}}}\right) \\
z &= -\frac{v}{2}\left(\dot{\overline{\frac{u}{v}}}\right)
\end{aligned}\right\}$$

If we use $p = \frac{1}{v}$ and $q = \frac{u}{v}$, we get

$$\left.\begin{aligned}
w &= \frac{\dot{p}}{2p} \\
z &= -\frac{\dot{q}}{2p}
\end{aligned}\right\} \qquad (2.8)$$

Substituting (2.8) into the first two equations of (2.7), we have

$$p.\ddot{p} - \frac{1}{2}\dot{p}^2 + [2 - r(2 + \mu)]\,p^2 - 2 = -2(q^2 + \frac{1}{4}\dot{q}^2) \tag{2.9}$$

$$\ddot{q} + 4q = -\,3rp\mu \tag{2.10}$$

Now if we let

$$rp = k_0 + k_1 \cos v, \tag{2.11}$$

we find as a solution for (2.10)

$$q = -\frac{3}{4}\mu k_0 - \mu k_1 \cos v + k_2 \cos^2 v \tag{2.12}$$

If we substitute the solutions (2.11) and (2.12) into (2.9), we obtain (by identification) the values for k_0, k_1 and k_2 as functions of μ and e.

For k_0, we obtain

$$k_0 = \pm \frac{4}{c}$$

$$\text{with} \quad c = \sqrt{\frac{4e^4}{(3\mu + 1)^2} + \frac{4e^2\mu}{(3\mu + 1)^2} - (\mu + 3) + \mu(9\mu - 8)} \tag{2.13}$$

This will yield two different solutions, P_1 and P_2, if $c \neq 0$ and $c^2 > 0$.

So $c(\mu,e) = 0$ will give us (analytically) a transtion curve in the μ-e plane, which corresponds to one of the transition curves (IE) that Robe obtained numerically.

This curve can be written as

$$\boxed{\frac{4e^4}{(3\mu + 1)^2} + \frac{4e^2\mu}{(3\mu + 1)^2}\;(\mu + 3) + \mu.(9\mu - 8) = 0} \tag{2.14}$$

The elements p_{ij} of P_1 and P_2 are

$$p_{11} = \frac{-(2\mu + 1)e\,\text{sinv}}{(3\mu + 1 + \text{ecosv})\,(1 + \text{ecosv})} \tag{2.15}$$

$$p_{22} = \frac{-(\mu + 1 + 2e\cos v)\ e\sin v}{(3\mu + 1\ e\cos v)\ (1 + e\cos v)} \tag{2.16}$$

$$p_{12} = \frac{(3\mu + 1)(1 - \frac{3}{4}\mu \pm \frac{c}{4}) + \frac{e^2}{2} + 2e(\mu + 1)\cos v + e^2 \cos 2v}{(3\mu + 1 + e\cos v)\ (1 + e\cos v)} \tag{2.17}$$

$$p_{21} = \frac{-[(3\mu + 1)\ (1 + \frac{3}{4}\mu \pm \frac{c}{4}) + \frac{e^2}{2} + 2(2\mu + 1)e\cos v]}{(3\mu + 1 + e\cos v)\ (1 + e\cos v)} \tag{2.18}$$

3. STABILITY ANALYSIS.

Now we perform a stability analysis using the Floquet theory, on the two independent second order systems (2.3)

$$\dot{\delta} = P_1 \delta \tag{3.1}$$

$$\dot{\varepsilon} = P_2 \varepsilon \tag{3.2}$$

Both equations admit solutions for which

$$u(v + 2\pi) = s_i u(v) \qquad (i = 1,2)$$

where s_i are the roots of the characteristic equation

$$\det\,[X^{-1}(v).X(v + 2\pi) - sE] = 0 \tag{3.3}$$

where X(v) is a fundamental solution matrix of (3.1) (or (3.2)). Equation (3.3) can be written

$$s^2 - 2\alpha s + 1 = 0\ . \tag{3.4}$$

For stable solutions,

$$|\alpha| < 1 \tag{3.5}$$

Thus the transition curves in the (μ-e) plane, separating stable and non-stable regions, will be given by

$$|\alpha| = 1$$

or

$$s = \pm 1 \tag{3.6}$$

Taking X(v = 0) = E, equation (3.3) becomes

$$\det\,[X(2\pi) - sE] = 0$$

where $X(2\pi) = \begin{bmatrix} \alpha & \beta \\ \gamma & \alpha \end{bmatrix}$ is the monodromy-matrix and $\alpha^2 - \beta\gamma = 1$.

It can be shown (R. Meire and A. Vanderbauwhede [3]) that

$$X(2\pi) = SX^{-1}(\pi)SX(\pi) \tag{3.7}$$

where
$$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

This implies that we only have to integrate the equations over π instead of 2π.

One can also prove that for $s = \pm 1$, one of the elements of $X(\pi)$ becomes zero which saves an additional 10% of computer time.

4. RESULTS.

We applied this method to the equations of (3.1) and (3.2). Equation (3.1) yielded two transition curves: FM and FE. Equation (3.2) yielded one transition curve HM. All results are given in Fig. 1.

The curve IE is the analytically obtained curve from equation (2.14). Along

$$\begin{cases} FM & \beta=0 \quad \text{and} \quad \alpha=-1 \\ FE & \gamma=0 \quad \text{and} \quad \alpha=-1 \\ HM & \beta=\gamma=0 \quad \text{and} \quad \alpha=1 \end{cases}$$

The intersection of the curves FE and IE can be obtained very accurately (Robe was not able to give precise coordinates).

The point E is determined as that point on the curve IE for which the characteristic roots of (3.1) are -1. The coordinates of the interesting points in the μ-e plane are

$$\text{point F} \quad \begin{cases} \mu = 0.928053.... = \dfrac{5 + \sqrt{97}}{16} \\ e = 0 \end{cases}$$

$$\text{point I} \quad \begin{cases} \mu = 0.8888.... = \dfrac{8}{9} \\ e = 0 \end{cases}$$

$$\text{point E} \quad \begin{cases} \mu = 0.8596848 \\ e = 0.4531741 \end{cases}$$

The stable region consists of the shaded area in Fig. 1 and is now determined much more accurately than in Robe's paper where the fourth-order system (2.1) was used.

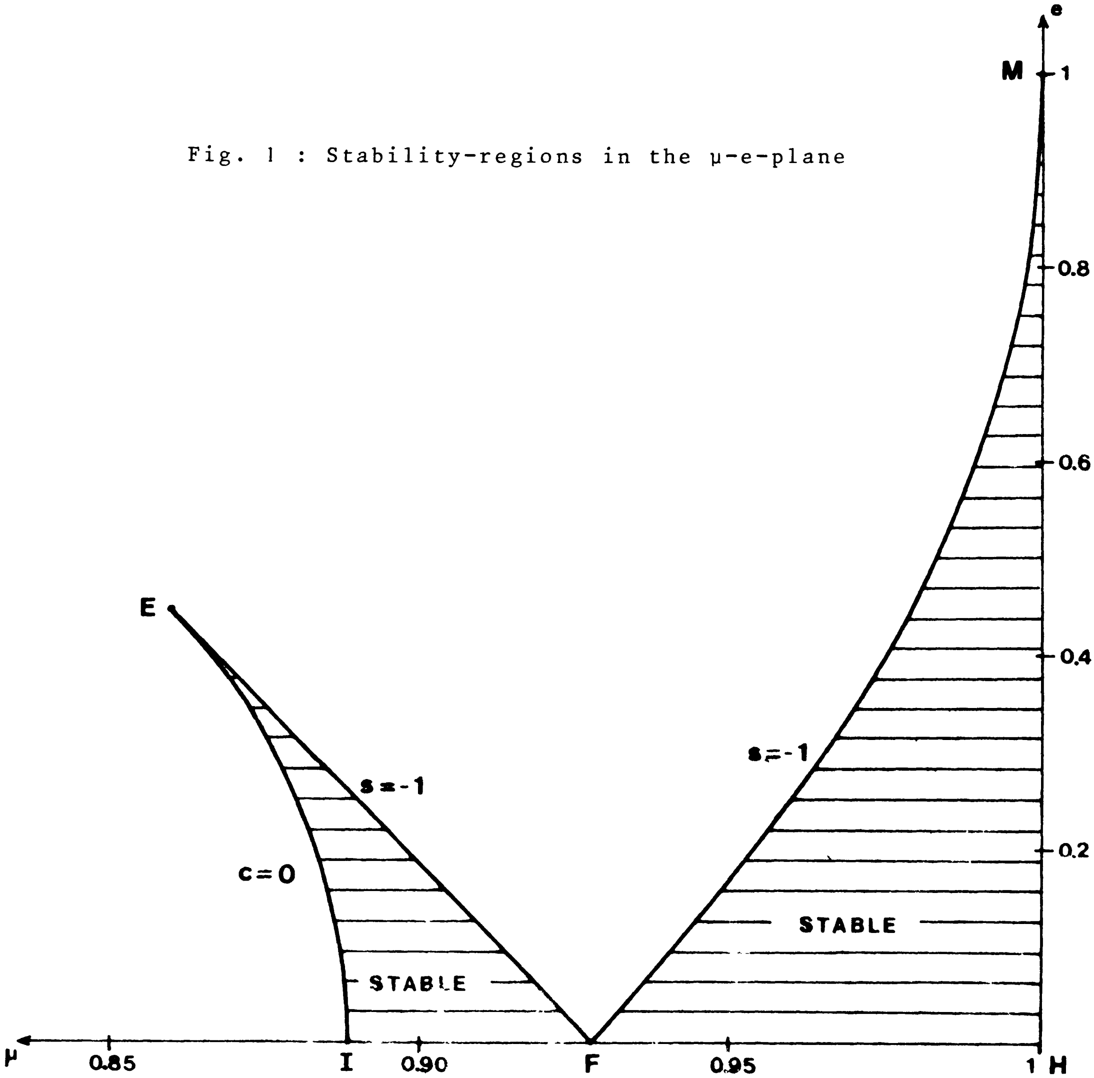

Fig. 1 : Stability-regions in the μ-e-plane

ACKNOWLEDGEMENTS.

The author thanks Prof. Dr. P. Dingens and Prof. Dr. H. Steyaert for reading the manuscript and for their valuable suggestions on this work.

REFERENCES

1. ROBE, H. A new kind of 3-body Problem, Cel. Mech. 16 (1978), pp. 343-351.

2. TSCHAUNER, K. Die Aufspaltung der Variations-gleichungen des Elliptischen Eingeschränkten Dreikörperproblems, Cel. Mech. 3 (1971), pp. 395-402.

3. MEIRE, R. and VANDERBAUWHEDE, A. A useful Result for Certain Linear Periodic Ordinary Differential Equations, J. Comp. and Applied Math. 5, N° 1 (1979), pp. 59-61.