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FAST METHODS FOR THE SOLUTION OF SINGULAR INTEGRO-DIFFERENTIAL AND DIFFERENTIAL EQUATIONS

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ABSTRACT. Uniform methods based on the use of the Galerkin method and different Chebyshev expansion sets are developed for the numerical solution of linear integro-differential equations of the first order. These methods take a total solution time $0(N^2 \ln N)$ using N expansion functions, and also provide error extimates which are cheap to compute. These methods solve both singular and regular integro-differential equations. The methods are also used in solving differential equations.

KEY WORDS AND PHRASES. Integro-differential equations, numerical solution, Galerkin method, Chebyshev expansion.

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1. INTRODUCTION.

We consider the Galerkin solution for those integro-differential equations of the first order having the form

$$L f(x) = g(x) \qquad x \in [-1,1] = I \tag{1.1}$$

subject to the boundary condition $f(a) = \alpha$, $a \in [-1,1]$.

$$L = \sum_{k=0}^{1} \{P_k(x)\frac{d^k}{dx^k} - \int_{-1}^{1} dy\ K_k(x,y)\frac{d^k}{dy^k}\},$$

where f and g are elements of a Hilbert space $H \subset I^I$, and $L: H \to H$ is linear.

We assume that $K_k(x,y)$ and $P_k(x)$ are either regular or have singularities provided that the singularities are of known and standard form like, for example, weak or logarithmic singularities. For a given Chebyshev expansion set $\{h_i(x)\} \subset H$, the solution f(x) defines approximations

$$f_N(x) = \sum_{i=0}^{N} a_i^{(N)} h_i(x) \tag{1.2}$$

Using Galerkin technique given in Mikhlin [1], we modify the linear system of equations

$$L^{(N)} \underline{a}^{(N)} = \underline{g}^{(N)} \tag{1.3}$$

where $L^{(N)}$ is the $(N+1 \times N+1)$ leading minor of the matrix L with elements

$$L_{ij} = \int_{-1}^{1} dx\ T_i(x)\ \mathcal{L}h_j(x)/\sqrt{1-x^2} \quad i, j = 0,1,\ldots,N \tag{1.4}$$

and $\underline{g}^{(N)}$ is the leading (N + 1)-vector of the vector g with elements

$$g_i = \int_{-1}^{1} dx\ T_i(x)\ g(x)/\sqrt{1-x^2} \quad i = 0,1,\ldots,N \tag{1.5}$$

T_i is the i^{th} Chebyshev polynomial, as usual. Now to determine the vector $a^{(N)}$, we replace the first equation of the system, (1.3), by the boundary condition equation

$$\sum_{j=0}^{N} a_j^{(N)} f_j = \alpha, \quad \text{where} \quad f_j = h_j(a) \tag{1.6}$$

Thus the linear system of equations (1.3) is replaced by

$$L^{*(N)} \underline{a}^{(N)} = \underline{g}^{*(N)} \tag{1.7}$$

where for $i \geq 1$ the i-th equation is the (i - 1)-th equation in (1.3) and the vector $\underline{a}^{(N)}$ can be determined by solving (1.7). According to [2], provided the set $\{h_j\}$ is suitably complete, the exact solution has the expansion

$$f(x) = \sum_{i=0}^{\infty} b_i h_i(x) \tag{1.8}$$

and b satisfies the infinite matrix equation

$$L \underline{b} = \underline{g} \tag{1.9}$$

Further, $f_N \to f$.

In this paper, we consider two different Chebyshev expansion sets:

$$\{h_i(x) = T_i(x)\} \tag{1.10}$$

and

$$\{h_0(x) = 1,\ h_1(x) = x,\ h_j(x) = (1 - x^2)\ T_{j-2}(x),\ j \geq 2\} \tag{1.11}$$

leading to three different methods (I), (II), (III).

Methods based on different techniques have been described before for solving integro-differential equations of the first order; Linz [2], El-Gendi [3], Abd-elal [4] - in all these papers integro-differential equations of the type (1.1) with $K_1(x,y) = 0$ are reduced to integral equations and a quadrature rule is used to establish numerical procedures. All of these methods are limited to integro-differential equations with no f' under the integral sign, also they do not treat boundary conditions in a very uniform way. El-Gendi's method [3] used Chebyshev expansion (1.2, 1.10) in approximating the solution of the equation and produce the solution in time $O(N^3)$. The methods we describe in this paper not only overcome these limitations, but also (the last two methods) produce the solution at a cost of total solution time $O(N^2 \ln N)$ and give reliable error estimates which are cheap to compute. Method (I) is a straightforward method in which a Chebyshev expansion set (1.10) is used to approximate the solution f(x), and then we solve the linear system of equations (1.7) to get the coefficient vector $\underline{a}^{(N)}$; hence, we consider it a standard method. Method (II) uses a modified Chebyshev expansion set (1.11) to approximate f(x) and so we consider it a modified method. Method (III) uses Chebyshev expansion set (1.10) to approximate not only f(x) by expansion (1.2), but also

$$f'(x) = \sum_{i=0}^{\infty} w_i\ T_i(x) \tag{1.12}$$

by

$$f_N'(x) = \sum_{i=0}^{N} d_i^{(N)}\ T_i(x). \tag{1.13}$$

We solve the corresponding linear system

$$L^{*(N)}\ \underline{d}^{(N)} = \underline{g}^{*(N)} \tag{1.14}$$

for the vector $\underline{d}^{(N)}$. An iterative procedure [5] is used to solve the linear systems (1.7) and (1.14).

The three methods effectively handle singularities in any or all of $K_k(x,y)$, $k = 0, 1$, $g(x)$, the solution $f(x)$, and its derivative $f'(x)$, provided that the singularities are of known form and have a known Chebyshev expansion (see [6]). These requirements limit the applicability of the method to those cases where the singularities which appear are of "standard" form - for example, weak singularities or logarithmic singularities. The methods can also treat some other types of singularities modifying the integro-differential equation; for example, a simple pole can be changed to a logarithmic singularity using integration by parts. We give in section 2 the analysis which leads to the structure of the matrix $L^{(N)}$ for the three methods, while a comparison between the convergence rate attained by the methods is given in section 3. Section 4 shows, by example, that in the three cases rapid convergence is obtained.

2. THE MATRIX $L^{(N)}$.

We wish to investigate the construction of the matrix $L^{(N)}$ for the three different methods considered in this paper. Using the expansion (1.2), the matrix $L^{(N)}$ reduces to

$$L^{(N)} = A - B \tag{2.1}$$

with

$$A = A^{(0)} + A^{(1)} \quad \text{and} \quad B = B^{(0)} + B^{(1)}$$

where $A^{(k)}$ and $B^{(k)}$ are $(N + 1 \times N + 1)$ matrices with elements $A_{ij}{}^{(k)}$, $B_{ij}{}^{(k)}$, i, $j = 0,1,\ldots,N$; $k = 0,1$ defined by

$$A_{ij}{}^{(k)} = \int_{-1}^{1} dx\ P_k(x)\ T_i(x)\ \frac{d^k}{dx^k}\{h_j(x)\}\ /\ \sqrt{1-x^2} \tag{2.2}$$

$$B_{ij}{}^{(k)} = \int_{-1}^{1} dx\ T_i(x)/\sqrt{1-x^2} \int_{-1}^{1} dy\ K_k(x,y)\ \frac{d^k}{dy^k}\{h_j(y)\} \tag{2.3}$$

The integrals appearing in Equations (2.2), (2.3) and (1.5) must be approximated numerically. We do this by relating $A_{ij}{}^{(k)}$, $B_{ij}{}^{(k)}$, and g_i, i, $j = 0,1,\ldots,N$ to

Chebyshev coefficients in the expansions of $P_r(x)$, $K_r(x,y)\sqrt{1-y^2}$, and g(x) respectively. These later coefficients are evaluated numerically using the fast algorithm given by Delves, Abd-Elal, and Hendry [6] in which Fourier transform technique is used. This algorithm leads to small quadrature errors whether $K_r(x,y)$, $P_r(x)$, and g(x) are singular or regular functions; also, it takes $O(N^2 \ln N)$ operations for evaluating the coefficients of the expansion for $K_r(x,y)\sqrt{1-y^2}$ and $O(N \ln N)$ for evaluating the coefficients of the expansions for $P_r(x)$ and g(x). Indeed, to evaluate $A_{ij}^{(r)}$, $B_{ij}^{(r)}$ and g_i of Equations (2.2), (2.3), and (1.5) numerically, let us assume that the functions $P_r(x)$ and $K_r(x,y)\sqrt{1-y^2}$ have Chebyshev expansions

$$P_r(x) = \sum_{j=0}^{\infty}{}' \; p_j^{(r)} \; T_j(x) \qquad r = 0,1 \tag{2.4}$$

$$K_r(x,y)\sqrt{1-y^2} = \sum_{i,j=0}^{\infty}{}' \; K_{ij}^{(r)} \; T_i(x) \; T_j(y) \qquad r = 0,1 \tag{2.5}$$

$$g(x) = \sum_{j=0}^{\infty}{}' \; g_j \; T_j(x) \tag{2.6}$$

where the expansion coefficients

$$p_j^{(r)} = \frac{2}{\pi} \int_{-1}^{1} dx \; T_j(x) \; P_r(x)/\sqrt{1-x^2} \qquad r = 0,1 \tag{2.7}$$

satisfy the inequality

$$|p_j^{(r)}| \le C_r \; \hat{j}^{-\xi_r} \qquad r = 0,1 \tag{2.8}$$

and the expansion coefficients

$$K_{ij}^{(r)} = \frac{4}{\pi^2} \int_{-1}^{1} dx \; T_i(x)/\sqrt{1-x^2} \int_{-1}^{1} dy \; K_r(x,y) \; T_j(y) \qquad r = 0,1 \tag{2.9}$$

satisfy the inequality

$$|K_{ij}^{(r)}| \le D_r \; \hat{i}^{-\gamma_r} \; \hat{j}^{-\beta_r} \qquad i,j \ge 0$$

which we can replace by the weaker bounds

$$\left.\begin{aligned} |K_{ij}^{(r)}| &\le D_r \; \hat{i}^{-\gamma_r} \qquad i > j \\ |K_{ij}^{(r)}| &\le D_r \; \hat{j}^{-\beta_r} \qquad j > i . \end{aligned}\right\} \tag{2.10}$$

Also, g_i of Equation (1.5) has the bound

$$|g_i| \le G\, \hat{i}^{-\delta} \qquad i \ge 0$$

$$\hat{j} = \begin{cases} 1 & \text{when } j = 0 \\ j & \text{when } j \ge 1 \end{cases}$$

and C_r, D_r, G are constants. $\sum'$ denotes a sum with first term halved.

Knowing the numerical values for the coefficients $P_j^{(k)}$, $K_{ij}^{(k)}$ using the fast algorithm [6], we can easily calculate the elements $A_{ij}^{(k)}$ and $B_{ij}^{(k)}$, i, j = 0,1,...,N of the matrices $A^{(k)}$ and $B^{(k)}$ for methods I, II, and III as follows:

Method (I). In this method we choose the expansion set (1.10) and by substituting (2.4) in (2.2) we get for i = 0,1,...,N

$$A_{ij}^{(0)} = \frac{\pi}{4}\left(P_{i+j}^{(0)} + P_{|i-j|}^{(0)}\right) \qquad j \ge 0\ ;$$

when $P_0(x) = 1$, then $A_{ij}^{(0)}$ reduces to (2.11)

$$A_{ij}^{(0)} = 0 \text{ for all } i,j \text{ except } A_{00}^{(0)} = \pi,\ A_{ii}^{(0)} = \frac{\pi}{2} \text{ for } i \ge 1$$

Also $A_{i0}^{(1)} = 0$

$$A_{ij}^{(1)} = \frac{\pi}{2}\, j \sum_{r=0}^{\left[\frac{j-1}{2}\right]}{}' \left[P^{(1)}_{i+2r+2\left(\frac{j+1}{2} - \left[\frac{j+1}{2}\right]\right)} + P^{(1)}_{\left|i-2r-2\left(\frac{j+1}{2} - \left[\frac{j+1}{2}\right]\right)\right|}\right],$$

$$j = 1,2,\ldots,N$$

When $P_1(x) = 1$, then $A_{ij}^{(1)}$ reduces to $A_{ij}^{(1)} = 0$ for all i,j except for j > 1, i and j of different parity where by different parity, we mean one even and one odd. $A_{ij}^{(1)} = j$.

[s] is the integer part of s, and $\sum'$ means halving the term with $2r + 2\left(\frac{j+1}{2} - \left[\frac{j+1}{2}\right]\right) = 0$.

Substituting (2.5) in (2.3), we get for i = 0,1,...,N

$$B_{ij}^{(0)} = \frac{\pi^2}{4} K_{ij}^{(0)}, \qquad j = 0,1,\ldots,N \qquad (2.13)$$

$$B_{i0}^{(1)} = 0$$

$$B_{ij}^{(1)} = \frac{\pi^2}{2}\, j \sum_{r=0}^{\left[\frac{j-1}{2}\right]}{}' K_{i,2r+2\left(\frac{j+1}{2} - \left[\frac{j+1}{2}\right]\right)}, \qquad j = 1,2,\ldots,N \qquad (2.14)$$

Notice that we take $O(N^2 \ln N)$ operations to get the matrices $A^{(0)}$, $B^{(0)}$, but from (2.12) and (2.14) it is clear that we take $O(N^3)$ operations to obtain the matrices $A^{(1)}$ and $B^{(1)}$, and in general, this makes method I take $O(N^3)$ operations to set up the matrix $L^{*(N)}$ unless explicit forms for $A_{ij}^{(1)}$ (for example, case $P_1(x) = 1$) and $B_{ij}^{(1)}$ are achieved; then, it takes $O(N^2 \ln N)$ operations to get the matrix $L^{*(N)}$.

Method II. In this method we choose the expansion set (1.11). For $i = 0,1,\dots,N$

$$A_{ij}^{(0)} = \begin{cases} \frac{\pi}{4} \left(p_{i+j}^{(0)} + p_{|i-j|}^{(0)} \right) & j = 0,1 \\ \frac{\pi}{8} \left[\left(p_{i+j-2}^{(0)} + p_{|i-j+2|}^{(0)}\right) - \frac{1}{2} \left(p_{i+j}^{(0)} + p_{|i-j|}^{(0)}\right) - \frac{1}{2} \left(p_{i+|j-4|}^{(0)} + p_{|i-|j-4||}^{(0)}\right)\right] & j \ge 2 \end{cases}$$

When $P_0(x) = 1$, $A_{ij}^{(0)}$ reduces to

$A_{ij}^{(0)} = 0$ for all i,j except, $A_{00}^{(0)} = \pi$, $A_{11}^{(0)} = \frac{\pi}{2}$,

$A_{02}^{(0)} = \frac{\pi}{2}$, $A_{22}^{(0)} = \frac{-\pi}{4}$, $A_{13}^{(0)} = \frac{-3\pi}{8}$, $A_{i,i+2}^{(0)} = \frac{\pi}{4}$ for $i \ge 2$,

$A_{ii}^{(0)} = \frac{-\pi}{8}$ for $i \ge 3$, $A_{i,i+4}^{(0)} = \frac{-\pi}{8}$ for $i \ge 0$ (2.15)

Also

$$A_{i0}^{(1)} = 0$$

$$A_{i1}^{(1)} = \frac{\pi}{2} p_i^{(1)}$$

$$A_{i2}^{(1)} = -\frac{\pi}{2} \left(p_{i+1}^{(1)} + p_{|i-1|}^{(1)}\right)$$

$$A_{ij}^{(1)} = \left[\frac{(j-4)\pi}{8} \left(p_{i+j-3}^{(1)} + p_{|i-j+3|}^{(1)}\right) - \frac{j\pi}{8} \left(p_{i+j-1}^{(1)} + p_{|i-j+1|}^{(1)}\right)\right] \quad j \ge 3 \tag{2.16}$$

When $P_1(x) = 1$, $A_{ij}^{(1)}$ reduces to

$A_{ij}^{(1)} = 0$ for all i,j except $A_{01}^{(1)} = \pi$, $A_{12}^{(1)} = -\pi$, $A_{03}^{(1)} = -\frac{\pi}{2}$,

$A_{i,i+3}^{(1)} = \frac{\pi}{4}(i-1)$ for $i \ge 1$, $A_{i+2,\, i+3}^{(1)} = -\frac{\pi}{4}(i+3)$ for $i \ge 0$

Corollary (II.1).

$$|A_{ij}^{(0)}| \le A_0 (i \hat{-} j)^{-\xi_0} \qquad i > j$$

$$\le A_0 (j \hat{-} i)^{-\xi_0} \qquad j > i$$

Corollary (II.2).

$$|A_{ij}^{(1)}| \le A_1 \ (i \hat{-} j)^{-\xi_1} \qquad i > j$$
$$\le A_1 \ j(j \hat{-} i)^{-\xi_1} \qquad j > i$$

Both Corollaries (II.1,2) follow directly using inequality (2.8) and Equations (2.15,16) respectively. A_0 and A_1 are constants.

$$\left.\begin{aligned} B_{i0}^{(0)} &= \frac{\pi^2}{4} K_{i0}^{(0)} \\ B_{i1}^{(0)} &= \frac{\pi^2}{4} K_{i1}^{(0)} \\ B_{ij}^{(0)} &= \frac{\pi^2}{8} [K_{i,|j-2|} - \frac{1}{2}(K_{ij}^{(0)} + K_{i,|j-4|}^{(0)})] \qquad j \ge 2 \end{aligned}\right\} \tag{2.17}$$

$$\left.\begin{aligned} B_{i0}^{(1)} &= 0 \\ B_{i1}^{(1)} &= \frac{\pi^2}{4} K_{i0}^{(1)} \\ B_{i2}^{(1)} &= -\frac{\pi^2}{4} K_{i1}^{(1)} \\ B_{ij}^{(1)} &= \frac{\pi^2}{4} [(\frac{j}{2} - 2) K_{i,j-3}^{(1)} - \frac{j}{2} K_{i,j-1}^{(1)}] \qquad j \ge 3 \end{aligned}\right\} \tag{2.18}$$

Corollary (II.3).

$$|B_{ij}^{(0)}| \le B_0 \ \hat{i}^{-\gamma_0} \qquad i > j$$
$$\le B_0 \ \hat{j}^{-\beta_0} \qquad j > i$$

Corollary (II.4).

$$|B_{ij}^{(1)}| \le B_1 \ \hat{i}^{-\gamma_1} \qquad i > j$$
$$\le B_1 \ \hat{j}^{-(\beta_1 - 1)} \qquad j > i$$

Corollary (II.3,4) follows directly from inequalities (2.10) and Equations (2.17,18) respectively. B_0 and B_1 are constants.

Method (III). In this method we use expansion set (1.10), and hence approximate f(x) by (1.2) and f'(x) by (1.13). Using the relation connecting a_i and d_i [7]:

$$a_i = (d_{i-1} - d_{i+1})/\ 2i, \qquad i = 1,2,\ldots,N \tag{2.19}$$

hence

$$f_N(x) = a_0 + \sum_{j=0}^{N}{}'' d_j \, [T_{j+1}(x)/\, 2(j+1) - T_{j-1}(x)/\, 2(j-1)] \tag{2.20}$$

where $\sum''$ means that the term $T_{j-1}(x)/\, 2(j-1) = 0$ for $j = 0,1$. Now consider the unknown vector $\underline{c}^{(N)} = [c_0, c_1, \dots, c_{N+1}]$ where $c_0 = a_0$, $c_{j+1} = d_j$, $j = 0,\dots,N$; then the integral equation (1.1) is now reduced to

$$L^{(N)} \; \underline{c}^{(N)} = \underline{g}^{(N)} \tag{2.21}$$

where $L^{(N)}$ is the ($N+1 \;\times N+2$) matrix defined by (2.1); hence we need one equation more, and this comes from the boundary condition $f(a) = \alpha$, which we write in the form

$$\sum_{j=0}^{N+1} c_j^{(N)} \; e_j = \alpha \tag{2.22}$$

where

$$e_0 = 1$$

$$e_{j+1} = \begin{cases} T_{j+1}(a)/\, 2(j+1) & j = 0,1 \\ T_{j+1}(a)/\, 2(j+1) - T_{j-1}(a)/\, 2(j-1) & j = 2,3,\dots,N \end{cases} \tag{2.23}$$

leading to the ($N+2 \times N+2$) linear system of equations

$$L^{*(N)} \; \underline{c}^{(N)} = \underline{g}^{*(N)} \tag{2.24}$$

with elements, $j = 0,1,\dots,N+1$

$$\left.\begin{aligned} L^*_{ij} &= A_{ij}^{(0)} + A_{ij}^{(1)} - B_{ij}^{(0)} - B_{ij}^{(1)} \qquad i = 0,1,\dots,N \\ L^*_{N+1,\,j} &= e_j \end{aligned}\right\} \tag{2.25a}$$

$$g^*_i = g_i, \quad i = 0,1,\dots,N \;, \quad g^*_{N+1} = \alpha \tag{2.25b}$$

where the elements $A_{ij}^{(k)}$, $B_{ij}^{(k)}$, $i = 0,1,\dots,N$; $k = 0,1$ defined by

$$\left.\begin{aligned} A_{i0}^{(0)} &= \int_{-1}^{1} dx\; P_0(x)\; T_i(x)/\sqrt{1-x^2} \\ A_{i,j+1}^{(0)} &= \begin{cases} \dfrac{1}{2(j+1)} \displaystyle\int_{-1}^{1} dx\; P_0(x)\; T_i(x)\; T_{j+1}(x) \;/\; \sqrt{1-x^2} & j = 0,1 \\ \displaystyle\int_{-1}^{1} dx\; P_0(x) T_i(x)\; [T_{j+1}(x)/\, 2(j+1) - T_{j-1}(x)/\, 2(j-1)]/\sqrt{1-x^2} & j = 2,\dots,N \end{cases} \end{aligned}\right\} \tag{2.26}$$

$$\left.\begin{aligned} A_{i0}^{(1)} &= 0 \\ A_{i,j+1}^{(1)} &= \int_{-1}^{1} dx\, P_1(x)\, T_i(x)\, T_j(x)/\sqrt{1-x^2} \qquad j = 0,1,\ldots,N \end{aligned}\right\} \quad (2.27)$$

$$\left.\begin{aligned} B_{i0}^{(0)} &= \int_{-1}^{1} dx\, T_i(x)/\sqrt{1-x^2} \int_{-1}^{1} K_0(x,y)\, dy \\ B_{i,j+1}^{(0)} &= \begin{cases} \dfrac{1}{2(j+1)} \displaystyle\int_{-1}^{1} dx\, T_i(x)/\sqrt{1-x^2} \int_{-1} dy\, K_0(x,y)\, T_{j+1}(y) & j = 0,1 \\ \displaystyle\int_{-1}^{1} dx\, T_i(x)/\sqrt{1-x^2} \int_{-1}^{1} K_0(x,y)\, [T_{j+1}(y)/2(j+1) - T_{j-1}(y)/2(j-1)]dy & j = 2,3,\ldots,N \end{cases} \end{aligned}\right\} \quad (2.28)$$

$$\left.\begin{aligned} B_{i0}^{(1)} &= 0 \\ B_{i,j+1}^{(1)} &= \int_{-1}^{1} dx\, T_i(x)/\sqrt{1-x^2} \int_{-1}^{1} K_1(x,y)\, T_j(y)\, dy, \qquad j = 0,1,\ldots,N \end{aligned}\right\} \quad (2.29)$$

Substituting (2.4) in (2.26, 27) we get for $i = 0,1,\ldots,N$

$$A_{i0}^{(0)} = \frac{\pi}{2} p_i^{(0)}$$

$$A_{i,j+1}^{(0)} = \begin{cases} \dfrac{\pi}{8(j+1)} \left(p_{i+j+1}^{(0)} + p_{|i-j-1|}^{(0)}\right) & j = 0,1 \\ \dfrac{\pi}{8} \left[\dfrac{1}{(j+1)} \left(p_{i+j+1}^{(0)} + p_{|i-j-1|}^{(0)}\right) - \dfrac{1}{(j-1)} \left(p_{i+j-1}^{(0)} - p_{|i-j+1|}^{(0)}\right)\right] & j = 2,3,\ldots,N \end{cases} \quad (2.30)$$

When $P_0(x) = 1$, $A_{ij}^{(0)}$ reduces to

$A_{ij}^{(0)} = 0$ for all i,j except $A_{00}^{(0)} = \pi$, $A_{11}^{(0)} = \pi/4$, $A_{22}^{(0)} = \pi/8$

$A_{jj}^{(0)} = \pi/(4j)$, $j = 3,\ldots,N+1$, $A_{j-1,j+1}^{(0)} = -\pi/(4(j-1))$, $j = 2,\ldots,N$

and

$$A_{i0}^{(1)} = 0$$

$$A_{i,j+1}^{(1)} = \frac{\pi}{4} \left(p_{i+j}^{(1)} + p_{|i-j|}^{(1)}\right) \qquad j = 0,1,2,\ldots,N$$

When $P_1(x) = 1$, $A_{ij}^{(1)}$ reduces to

$A_{ij}^{(1)} = 0$ for all i,j except $A_{01}^{(1)} = \pi$, $A_{j,j+1}^{(1)} = \pi/2$, $j = 1,2,\ldots,N$

Corollary (III.1).

$$|A_{ij}^{(0)}| \le A_0(i \hat{-} j)^{-\xi_0}\, \hat{j}^{-1} \qquad i > j$$
$$\le A_0(j \hat{-} i)^{-\xi_0}\, \hat{j}^{-1} \qquad j > i$$

Corollary (III.2).

$$|A_{ij}^{(1)}| \le A_1(i \hat{-} j)^{-\xi_1} \qquad i > j$$
$$\le A_1(j \hat{-} i)^{-\xi_1} \qquad j > i$$

Both corollaries (III.1,2) follow directly using inequality (2.8) and equations (2.30,31) respectively.

Now substituting (2.5) in equations (2.28,29) we get for $i = 0,1,\ldots,N$

$$\left.\begin{aligned} B_{i0}^{(0)} &= \frac{\pi^2}{4} K_{i0}^{(0)} \\ B_{i,j+1}^{(0)} &= \begin{cases} \dfrac{\pi^2}{8(j+1)} K_{i,j+1}^{(0)} & j = 0,1 \\ \dfrac{\pi^2}{8}\left[\dfrac{1}{(j+1)} K_{i,j+1}^{(0)} - \dfrac{1}{(j-1)} K_{i,j-1}^{(0)}\right] & j = 2,3,\ldots,N \end{cases} \end{aligned}\right\} \tag{2.32}$$

and

$$\left.\begin{aligned} B_{i0}^{(1)} &= 0 \\ B_{i,j+1}^{(1)} &= \frac{\pi^2}{4} K_{ij}^{(1)} \qquad j = 0,1,\ldots,N \end{aligned}\right\} \tag{2.33}$$

Corollary (III.3).

$$|B_{ij}^{(0)}| \le B_0\, \hat{i}^{-\gamma_0}\, j^{-1} \qquad i > j$$
$$\le B_0\, \hat{j}^{-(\beta_0 + 1)} \qquad j > i$$

Corollary (III.4).

$$|B_{ij}^{(1)}| \le B_1\, \hat{i}^{-\gamma_1} \qquad i > j$$
$$\le B_1\, \hat{j}^{-\beta_1} \qquad j > i$$

Both corollaries (III.3,4) follow directly from inequality (2.10) and equations (2.32,33). To evaluate $p_i^{(r)}$, $K_{ij}^{(r)}$, $r = 0,1$ numerically, we refer to the fast algorithm [6].

3. ASYMPTOTICALLY DIAGONAL MATRICES.

Definition: Given an infiniet matrix L, let the matrix F have elements

$$F_{ij} = \frac{|L_{ij}|}{(|L_{ii}||L_{jj}|)^{\frac{1}{2}}} \tag{3.1}$$

The matrix L is said to be "Asymptotically lower diagonal (A.L.D.) of type B(p,r;c)" if constants $p,r \geq 0$, $c > 0$ exist such that

$$F_{ij} \leq c \;\; \hat{i}^{-p} \; (i \hat{-} j)^{-r} \qquad i > j \tag{3.2}$$

and is said to by "Asymptotically upper diagonal" (A.U.D.) of the same type if

$$F_{ij} \leq c \;\; \hat{j}^{-p} \; (j \hat{-} 1)^{-r} \qquad j > i \tag{3.3}$$

In an obvious notation we shall then refer to systems of type B as:

$$\text{Type } B(p_L,\, p_U,\, r_L,\, r_U;\, c_L,\, c_U) \tag{3.4}$$

For systems (A.D.) of type B, Freeman and Delves [8] provide estimates of the convergence rate. In order to compare the convergence rate attained by the three methods of this paper we need to study the matrices given by each method.

We construct the elements of the matrix $L^{*(N)} = A^* - B^*$ where

$$A^* = A^{*(0)} + A^{*(1)} ,$$

$$B^* = B^{*(0)} + B^{*(1)} ;$$

$A^{*(k)}$, $B^{*(k)}$ are (N+1 × N+1) matrices with elements

$$\left.\begin{aligned} &A^{*\,(1)}_{0j} = f_j,\; A^{*\,(0)}_{0j} = B^{*\,(0)}_{0j} = B^{*\,(1)}_{0j} = 0 \\ &A^{*\,(k)}_{ij} = A^{*\,(k)}_{i-1,j} \qquad i = 1,2,\ldots,N \\ &\qquad\qquad\qquad\qquad\qquad\qquad j = 0,1,\ldots,N,\; k = 0,1 \\ &B^{*\,(k)}_{ij} = B^{*\,(k)}_{i-1,j} \qquad i = 1,2,\ldots,N \end{aligned}\right\} \tag{3.5}$$

where $f_0 = 1$, $f_1 = a$, $f_j = (1 - a^2)\, T_{j-2}(a)$, $2 \leq j \leq N(|f_j| \leq 1,\; j = 0,1,\ldots,N)$

$$L^{*\,(N)}_{ij} = \sum_{r=0}^{1} \{A^{*\,(r)}_{ij} - B^{*\,(r)}_{ij}\} ;$$

$$g^{*(N)}_0 = \alpha, \quad g^{*(N)}_i = g^{*\,(N)}_{i-1} \qquad i = 1,\ldots,N$$

Method (I): As is clear from equation (2.12) the matrix $A^{(1)}$ is not (A.D.); indeed, for $j > i$, the elements $A_{ij}^{(1)}$ increase with j. So $L^{*(N)}$ is not (A.U.D.) and hence the analysis of Freeman and Delves [8] is not applicable to method I, so as we will see later we do not suggest a value for the truncation error. Also we can not use the iterative method given by Delves [5] to solve the linear system (1.7) and hence any standard method for solving linear equations (Gauss elimination method) can be used.

Method (II):

Lemma 1. The matrix $L^{*(N)}$ of equation (1.7) is (A.D.) of type

$$B(0,0,\min(\xi_0, \xi_1, \gamma_0, \gamma_1), \min(\xi_0, \xi_1-1, \beta_0, \beta_1-1); L_L, L_U)$$

Proof: From corollaries (II.1, 2, 3, 4) we get

$$|L^*_{ij}| \le A_1\hat{j}(i\hat{-}j)^{-\xi_1} + A_0(i\hat{-}j)^{-\xi_0} + B_0(i\hat{-}j)^{-\gamma_0} + B_1(i\hat{-}j)^{-\gamma_1} \ldots. i > j$$

Hence

$$|L^*_{ij}| \le L_1 \ \hat{j}(i \hat{-} j)^{-\min(\xi_0, \xi_1, \gamma_0, \gamma_1)} \qquad \ldots. i > j \ . \quad (3.6)$$

Also

$$|L^*_{ij}| \le A_1\hat{j}(j \hat{-} 1)^{-\xi_1} + A_0(j\hat{-}1)^{-\xi_0} + B_0(j\hat{-}1)^{-\beta_0} + B_1(j\hat{-}i)^{-\beta_1+1} \ldots. j > i$$

Hence

$$|L^*_{ij}| \le L_2 \ \hat{j}(j \hat{-} i)^{-\min(\xi_1, \xi_0 + 1, \beta_0 + 1, \beta_1)} \qquad \ldots. j > i \quad (3.7)$$

Also

$$|L^*_{ii}| \ge L_3 \ \hat{i} \qquad (3.8)$$

where L_1, L_2, and L_3 are constants.

From (3.6, 3.8), for $i > j$,

$$F_{ij} = \frac{|L^*_{ij}|}{(|L^*_{ii}||L^*_{jj}|)^{\frac{1}{2}}} \le \frac{L_1}{L_3} \hat{i}^{-\frac{1}{2}} \hat{j}^{\frac{1}{2}}(i \hat{-} j)^{-\min(\xi_0, \xi_1, \gamma_0, \gamma_1)}$$

$$\le L_L(i \hat{-} j)^{-\min(\xi_0, \xi_1, \gamma_0, \gamma_1)}, \quad i > j \quad (3.9)$$

Also from (3.7, 3.8), for $i > j$,

$$\le \frac{L_2}{L_3} \hat{i}^{-\frac{1}{2}} \hat{j}^{-\frac{1}{2}} j(j\hat{-}1)^{-\min(\xi_1, \xi_0+1, \beta_0+1, \beta_1)}$$

$$\le L_U(j\hat{-}1)^{-\min(\xi_1-1, \xi_0, \beta_0, \beta_1-1)}, \quad j > i \quad (3.10)$$

since $j(j \mathbin{\hat{-}} 1)^{-1} < 2i$ for $j > i$.

From (3.9) and (3.10), the lemma follows.

Method (III):

Lemma 2. The matrix $L^{*(N)}$ of equations (2.24) is (A.D.) of type $B(0,0,\min(\xi_0,\xi_1,\gamma_0,\gamma_1),\ \min(\xi_0+1,\ \xi_1,\beta_0+1,\beta_1);\ L_L,\ L_U)$

Proof: From corollary (III.1,2,3,4) we get:

$$|L^*_{ij}| \le A_1(i \mathbin{\hat{-}} j)^{-\xi_1} + A_0(i \mathbin{\hat{-}} j)^{-\xi_0}\,\hat{j}^{-1} + B_1(i \mathbin{\hat{-}} j)^{-\gamma_1} + B_0(i \mathbin{\hat{-}} j)^{-\gamma_0}\,\hat{j}^{-1} \qquad i > j$$

$$\le L_4(i \mathbin{\hat{-}} j)^{-\min(\xi_0,\ \xi_1,\ \gamma_0,\ \gamma_1)} \qquad j > i \tag{3.11}$$

$$\le A_1(j \mathbin{\hat{-}} i)^{-\xi_1} + A_0(j \mathbin{\hat{-}} i)^{-\xi_0 - 1} + B_1(j \mathbin{\hat{-}} i)^{-\beta_1} + B_0(j \mathbin{\hat{-}} i)^{-(\beta_0+1)} \qquad j > i$$

$$\le L_5(j \mathbin{\hat{-}} i)^{-\min(\xi_0+1,\ \xi_1,\ \beta_0+1,\ \beta_1)} \qquad j > i \tag{3.12}$$

Also

$$|L^*_{ii}| \ge L_6 \tag{3.13}$$

where L_4, L_5 and L_6 are constants.

From (3.11,3.13)

$$F_{ij} = \frac{|L^*_{ij}|}{(|L^*_{ii}|\,|L^*_{jj}|)^{\frac{1}{2}}} \le L_L(i \mathbin{\hat{-}} j)^{-\min(\xi_0,\ \xi_1,\ \gamma_0,\ \gamma_1)} \qquad i > j \tag{3.14}$$

From (3.12,13)

$$\le L_U(j \mathbin{\hat{-}} i)^{-\min(\xi_0+1,\ \xi_1,\ \beta_0+1,\ \beta_1)} \qquad j > i \tag{3.15}$$

From (3.14,15), the lemma follows.

From Lemma 1,2 the matrix $L^{*(N)}$ is U.A.D. and hence the analysis of Freeman and Delves [8] is applicable to methods II, III and a value of the truncation error is suggested as given later. Also the iterative Method (III) may be used to solve (1.7) and (2.24) in $O(N^2)$ operations.

Now by virtue of Theorems 6 and 7 of Freeman and Delves [8] with normalization we have the following theorem.

THEOREM 1. If L is L.A.D. or U.A.D. of type $B(p_L, p_U, r_L, r_U; c_L, c_U)$ with $r_L, r_U > 1$, $p_L, p_U \geq 0$ and $|L_{ii}| \geq \wedge_i^2$, then if, for some constant c , $|g_i| \leq i^{-\delta}$ with $\delta \geq 1$, there exist some positive constants M_1, M_2 such that

$$|b_i - a_i^{(N)}| \leq M_1 N^{-(p_U + t)} (N + 1 - i)^{-r_U + 1} \wedge_i^{-1} \qquad i \leq N$$

$$|b_i| \leq M_2 i^{-t} \wedge_i^{-1} \qquad i > N$$

$t = \min(\delta, p_L + r_U)$.

Theorem 1 leads us to our two main theorems:

THEOREM 2. In Method (II)

$$|b_i - a_i^{(N)}| \leq M_1 N^{-t}(N + 1 - i)^{-\min(\xi_0 - 1, \xi_1 - 2, \beta_0 - 1, \beta_1 - 2)} \hat{i}^{-\frac{1}{2}} \qquad i \leq N$$

$$|b_i| \leq M_2 i^{-(t + \frac{1}{2})} \qquad i > N$$

where $t = \min(\delta, \min(\xi_0, \xi_1, \gamma_0, \gamma_1))$. This follows directly from Lemma 1 and Theorem 1.

THEOREM 3. In Method (III)

$$|w_i - d_i^{(N)}| \leq M_1 N^{-t} (N + 1 - i)^{-\min(\xi_0, \xi_1 - 1, \beta_0, \beta_1 - 1)} \qquad i \leq N$$

$$|w_i| \leq M_2 i^{-t} \qquad i > N$$

$t = \min(\delta, \min(\xi_0, \xi_1, \gamma_0, \gamma_1))$. This follows directly from Lemma 2 and Theorem 1.

4. ERROR ESTIMATES.

The error in any of the three methods contains three distinct components:

(a) The truncation error due to cutting off the expansion (1.2) at the N^{th} term.

(b) The discretization error stemming from the quadrature errors $\delta A - \delta B$ in the matrix A-B, and δg in the vector g.

(c) There are also in principle errors arising from the numerical solution of the linear equations.

Now according to Delves [9] we measure the error $e_N(x) = f_N(x) - f(x)$, with

$|e_N(x)| \sim |S_1 + S_2 + S_3|$ where the first two summands represent the truncation error (a), and are defined by

$$\left.\begin{aligned} S_1 &= \sum_{i=0}^{N} |\bar{a}_i^{(N)} - b_i| \\ S_2 &= \sum_{i=N+1}^{\infty} |b_i| \end{aligned}\right\} \quad \text{Method (I) and (II)}$$

while

$$\left.\begin{aligned} S_1 &= \sum_{i=0}^{N} |\bar{d}_i^{(N)} - w_i| \\ S_2 &= \sum_{i=N+1}^{\infty} |w_i| \end{aligned}\right\} \quad \text{Method (III)}$$

S_3 is the quadrature error (b), defined by

$$S_3 = \sum_{i=0}^{N} |a_i^{(N)} - \bar{a}_i^{(N)}| \qquad \text{for Method (I) and (II)}$$

$$= \sum_{i=0}^{N} |d_i^{(N)} - \bar{d}_i^{(N)}| \qquad \text{for Method (III).}$$

In the above, $\bar{a}_i^{(N)}$, $\bar{d}_i^{(N)}$ are the computed coefficients.

(a) Truncation error estimates: $(S_1 + S_2)$

From Theorems 2 and 3 and for Methods (II) and (III), S_2 dominates S_1 and so $S_1 + S_2 \sim S_2$.

Hence for Method (II)

$$S_2 \sim M_2 N^{-t+\frac{1}{2}}/(t-1) \sim N\,|a_N|\ , \tag{4.1}$$

while for Method (III)

$$S_2 \sim M_2 N^{-t+1}/(t-1) \sim N\,|d_N|\ . \tag{4.2}$$

For Method (I), since we are unable to apply Theorem 1, we do not suggest a value for $S_1 + S_2$.

(b) Quadrature error estimates: S_3

As given in Delves [9]

$$S_3 \sim ||L^{-1}||\,(||\delta L||\ W + ||\delta g||)/(1 - ||L^{-1}||\ ||\delta L||) \tag{4.3}$$

Here $W = ||a||$ for Method (I) and (II),

$= ||d||$ for Method (III) and a rough estimate for $||L^{-1}||$ is taken to be

$$||L^{-1}|| \sim ||g|| / W$$

We require only an estimate for $||\delta L||$, since we refer to Delves et al [6] to estimate $||\delta g||$.

Evaluation of $||\delta L||$:

Method (I): from Equations (2.11,12), we get

$$||\delta A^{(0)}||_\infty \leq \pi N ||\delta p^{(0)}||_\infty /2, \quad ||\delta A^{(1)}||_\infty \leq \pi N^3 ||\delta p^{(1)}||_\infty /3$$

Hence

$$||\delta A||_\infty \leq \pi(N ||\delta p^{(0)}||_\infty /2 + N^3 ||\delta p^{(1)}||_\infty /3) \tag{4.5}$$

Also, from Equations (2.13,14) we get

$$||\delta B^{(0)}||_\infty \leq \pi^2 ||\delta K^{(0)}||_\infty /4 \;; \quad ||\delta B^{(1)}||_\infty \leq \pi^2 N^2 ||\delta K^{(1)}||_\infty /4$$

Hence

$$||\delta B||_\infty \leq \pi^2 (||\delta K^{(0)}||_\infty + N^2 ||\delta K^{(1)}||_\infty) /4 \tag{4.6}$$

Then $||\delta L||_\infty \sim (4.5) + (4.6)$.

Method (II): from Equations (2.15,16)

$$||\delta A^{(0)}||_\infty \leq \pi N ||\delta p^{(0)}||_\infty /2 \;; \quad ||\delta A^{(1)}||_\infty \leq \pi N^2 ||\delta p^{(1)}||_\infty /4$$

$$||\delta A||_\infty \leq \pi(N ||\delta p^{(0)}||_\infty /2 + N^2 ||\delta p^{(1)}||_\infty /4 \tag{4.7}$$

From Equations (2.17,18),

$$||\delta B^{(0)}||_\infty \leq \pi^2 ||\delta K^{(0)}||_\infty /4 \;; \quad ||\delta B^{(1)}|| \leq N \pi^2 ||\delta K^{(1)}||_\infty /4$$

Hence

$$||\delta B||_\infty \leq \pi^2 (||\delta K^{(0)}||_\infty + N ||\delta K^{(1)}||_\infty) /4 \tag{4.8}$$

$$||\delta L||_\infty \sim (4.7) + (4.8)$$

Method (III): from Equations (2.30,31),

$$||\delta A^{(0)}||_\infty \leq 2\pi(2 + \ln N) ||\delta p^{(0)}||_\infty \;; \quad ||\delta A^{(1)}||_\infty \leq \pi N ||\delta p^{(1)}||_\infty /2$$

Hence

$$||\delta A||_\infty \leq \pi(2(2 + \ln N)) ||\delta p^{(0)}||_\infty + N ||\delta p^{(1)}||_\infty /2 \tag{4.9}$$

from Equations (2.32,33),

$$\| \delta B^{(0)} \|_\infty \le \pi^2 \; \| \delta K^{(0)} \|_\infty /4 \; ; \quad \| \delta B^{(1)} \|_\infty \le \pi^2 \; \| \delta K^{(1)} \|_\infty /4$$

$$\| \delta B \|_\infty \le \pi^2 (\| \delta K^{(0)} \|_\infty + \| \delta K^{(1)} \|_\infty) /4 \qquad (4.10)$$

$$\| \delta L \|_\infty \sim (4.9) + (4.10)$$

We refer to Delves et al [6] for the numerical estimations of $\| \delta p^{(r)} \|_\infty$ and $\| \delta K^{(r)} \|_\infty$.

(c) Error stemming from solving the linear system of equations.

To solve the linear system of equations (1.7) or (2.24), for Method II or III, we use an iterative scheme given in [5] with $O(N^2)$ operations. The error due to this iterative solution is small and so we neglect it, but for Method I and according to Delves [5], we cannot use this iterative method and hence the error could be relatively large due to error cancellation. We will now see that this error cancellation has no serious effect in a numerical example.

5. NUMERICAL EXAMPLE.

We give in this section the numerical results for a singular integro-differential equation of the first order.

$$f'(x) + f(x) = g(x) + \int_{-1}^{1} K_0(x,y) \; f(y) \; dy + \int_{-1}^{1} K_1(x,y) \; f'(y) \; dy$$

$$K_0(x,y) = 2 \ln(x - y) \; (x - 2y + 2xy^2 - 2y^3)$$

$$K_1(x,y) = (x^2 - y^2)$$

$$g(x) = e - 2e(x + 1) \ln(x + 1) + 2xe^{x^2}$$

with boundary condition $f(a) = \alpha$, $a \in [-1,1]$.

It has the exact solution $f(x) = e^{x^2}$.

The computed errors σ_N are defined to be

$$\sigma_N = \left\{ \sum_{j=0}^{N} e_N^2 \; (x_j)/N \right\}^{\frac{1}{2}} \approx \left\{ \int_{-1}^{1} e_N^2 \; (x) \; dx \right\}^{\frac{1}{2}}$$

where $x_j = \cos \; (j\pi/N)$, $j = 0,1,\ldots,N$ and $e_N = f_N - f_{exact}$

Computed results for the numerical example.

N	σ_N Method (I) $f(-1)=e$	σ_N Method (II) $f(-1)=e$	σ_N Method (II) $f(0) = 1$	σ_N Method (III) $f(-1)=e$
3	3.8×10^{-1}	3.7×10^{-1}	3.6×10^{-1}	1.2×10^{-1}
5	4.9×10^{-2}	7.1×10^{-2}	8.5×10^{-2}	1.3×10^{-2}
7	4.6×10^{-3}	6.5×10^{-3}	5.0×10^{-3}	9.1×10^{-4}
9	2.9×10^{-4}	4.1×10^{-4}	2.2×10^{-4}	4.9×10^{-5}
11	1.4×10^{-5}	2.1×10^{-5}	8.6×10^{-6}	2.0×10^{-6}
13	6.0×10^{-7}	8.6×10^{-7}	2.9×10^{-7}	6.8×10^{-8}
15	3.3×10^{-8}	3.2×10^{-8}	8.6×10^{-9}	2.1×10^{-9}
17	2.8×10^{-8}	3.3×10^{-9}	3.8×10^{-10}	2.0×10^{-10}
19	2.8×10^{-8}	2.1×10^{-9}	1.3×10^{-10}	1.1×10^{-10}

6. COMMENTS ON THE METHODS.

(1) As is clear from the analysis, although Method (I) is a standard method, in fact Methods (II) and (III) are preferable because they provide easy error estimates.

(2) All the three methods work very well when applied to the numerical example, and this suggests that Method (I) is probably a stable method.

(3) The three methods can be applied to ordinary differential equations of the first order which have the form (1.1) but with

$$L = \sum_{k=0}^{1} P_k(x) \frac{d^k}{dx^k}$$

Hence the same analysis holds with the matrix $B = 0$.

(4) The three methods represent a uniform way of treating boundary conditions, so we recommend methods (II) and (III) be extended to include integro-differential equations of the second order. We are now working on this.

(5) A standard Galerkin calculation has an operations count of $O(N^3)$ for both setting up and solving the linear equations defining the coefficient vector: however, Methods (II) and (III) of this paper need:

Setting up equations: $0(N^2 \ln N)$ operations

Iterative solution: $0(N^2)$ operations.

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