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A RESULT ON CO-CHROMATIC GRAPHS

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ABSTRACT. A sufficient condition for two graphs with the same number of nodes to have the same chromatic polynomial is given.

KEY WORDS AND PHRASES. Graph, Co-chromatic, Chromatic polynomial.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES: Primary 05 C15, *Secondary* 26 C10.

1. INTRODUCTION.

We prove a theorem which gives a sufficient condition for two graphs to be co-chromatic i.e. to have the same chromatic polynomial.

The chromatic polynomial $\chi(G;\lambda)$ of a graph G with p nodes has degree p and constant term equal to 0. Hence the chromatic polynomial has p coefficients. If the graph has at least one edge, then the sum of these coefficients is equal to 0. Hence the chromatic polynomial is uniquely determined if p-1 of the coefficients are known. Our result is a generalization of this.

2. MAIN RESULTS.

THEOREM 2. If two graphs with p nodes have chromatic numbers $\geq n$ and have at least $p+1-n$ corresponding coefficients of their chromatic polynomials equal, then they are co-chromatic.

In the proof we shall use a special case of the following Lemma.

LEMMA 2.1

Let $P(x) = c_1x^{n_1} + c_2x^{n_2} + \ldots + c_sx^{n_s}$, where c_i and n_i are real numbers for $i = 1, 2, \ldots, s$. We assume that $c_i \neq 0$ for all i and that $n_i \neq n_j$ for $i \neq j$. Then the equation $P(x) = 0$ has at most s-1 real positive solutions.

PROOF OF THE LEMMA 2.1 By induction over s. For $s = 1$ the statment is obvious. Suppose that it is true for $s - 1$ and let $P(x)$ be the above expression. The expression

$$Q(x) = x^{-n_1} \cdot P(x) = c_1 + c_2x^{n_2-n_1} + \ldots c_sx^{n_s-n_1}$$

has the derivative

$$Q'(x) = c_2(n_2-n_1) \cdot x^{n_2-n_1-1} + \ldots + c_s(n_s-n_1) \cdot x^{n_s-n_1-1}.$$

By induction, $Q'(x) = 0$ for at most s-2 positive x. But between any two positive solutions of $P(x) = x^{n_1} \cdot Q(x) = 0$, there exists a solution of $Q'(x) = 0$. Hence $P(x) = 0$ for at most s-1 positive x.

Q.E.D.

PROOF OF THE THEOREM 2. Let G and H be the two graphs. Let us assume that m of the coefficients of $\chi(G;\lambda)$ and $\chi(H;\lambda)$ are equal. Then $m \geq p + 1 - n$, by our assumption. Let us assume that $m < p$. We can therefore write

$$\chi(G;\lambda) = f(\lambda) + g(\lambda)$$

and

$$\chi(H;\lambda) = f(\lambda) + h(\lambda),$$

where $f(\lambda)$ contains m terms and $g(\lambda)$ and $h(\lambda)$ are the remaining terms of $\chi(G;\lambda)$ and $\chi(H;\lambda)$ respectively.

Since G and H have chromatic numbers $\geq n$, all integers $1, 2, \ldots, n-1$ are roots of $\chi(G;\lambda)$ and $\chi(H;\lambda)$.

Let

$$g(\lambda) = a_1\lambda^{n_1} + a_2\lambda^{n_2} + \dots + a_{p-m}\lambda^{n_{p-m}}$$

and $h(\lambda) = b_1\lambda^{n_1} + b_2\lambda^{n_2} + \dots + b_{p-m}\lambda^{n_{p-m}}$

If r is an integer such that $1 \le r \le n-1$, then $g(r) = h(r)$, i.e.

$$(a_1-b_1)r^{n_1} + (a_2-b_2)r^{n_2} + \dots + (a_{p-m}-b_{p-m})r^{n_{p-m}} = 0.$$

Since $n - 1 \ge p - m$, this is a contradiction by the Lemma.

Q.E.D.

3. ILLUSTRATIONS

We will now illustrate the theorem. We will assume that the chromatic polynomial of a graph G with p nodes is written in descending powers of λ.

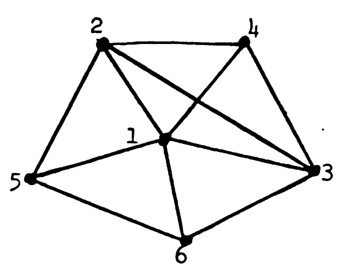
$$\text{i.e.} \quad \chi(G;\lambda) = \sum_{k=0} a_{p-k}\lambda^{p-k} .$$

It is well known that if G contains p nodes and q edges, then a_p, a_{p-1} and a_{p-2} are 1, $-q$ and $\binom{q}{2} - A$ respectively, where A is the number of triangles in G. It was also shown in Farrell [1] (Theorem 1) that

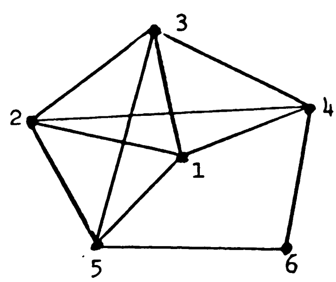
$$a_{p-2} = -\binom{q}{3} + (q-2)A + B - 2C,$$

where B and C are the numbers of subgraphs of G which are quadrilaterals (without diagonals) and complete graphs with four nodes.

Let G_1 and G_2 be the graphs shown below



G_1



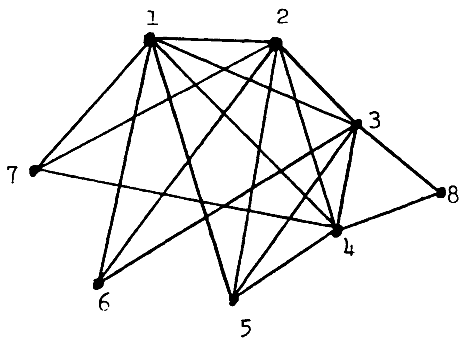
G_2

Let $\chi(G_1;\lambda) = \sum_{k=0} a_{6-k}\,\lambda^{6-k}$ and $\chi(G_2;\lambda) = \sum_{k=0} b_{6-k}\,\lambda^{6-k}$. Then $a_6 = b_6 = 1$ and and $a_5 = b_5 = 11$. Since G_1 and G_2 contain 7 triangles, $a_4 = b_4 = \binom{11}{2} - 7 = 48$. Therefore G_1 and G_2 have 6 nodes, their chromatic number is $\geq 4 = n$ and $(p+1-n) = 3$ of their corresponding coefficients are equal. It follows from the above theorem that G_1 and G_2 are co-chromatic.

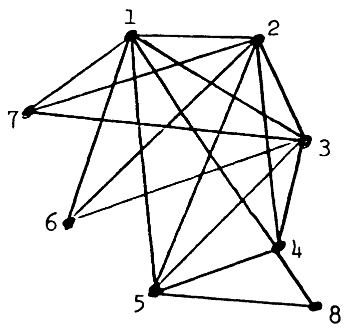
The chromatic polynomial of G_1 and G_2 has been computed. It is

$\chi(G_1;\lambda) = \chi(G_2;\lambda) = \lambda^6 - 11\lambda^5 + 48\lambda^4 - 103\lambda^3 + 107\lambda^2 - 42\lambda.$

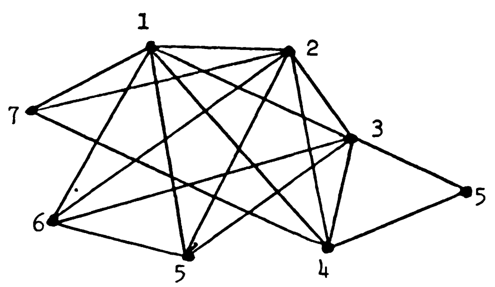
Consider the graphs H_1, H_2 and H_3 shown below.



H_1



H_2



H_3

All three graphs contain 8 nodes and 18 edges. Each contains 17 triangles. Therefore, the third coefficient of their chromatic polynomial is $\binom{18}{2} - 17 = 136$. Finally, each contains 7 subgraphs which are complete graphs with 4 nodes and 0 quadrilaterals without diagonals. Therefore the fourth coefficients are equal.

Hence from the above theorem H_1, H_2 and H_3 are co-chromatic.

The chromatic polynomial of H_1, H_2 and H_3 has been computed. It is

$$\chi(H_1;\lambda) = \chi(H_2;\lambda) = \chi(H_3;\lambda) = \lambda^8 - 18\lambda^7 + 136\lambda^6 - 558\lambda^5 + 1339\lambda^4 - 1872\lambda^3 + 1404\lambda^2 - 432\lambda.$$

REFERENCES

[1] Farrell, E.J., On Chromatic Coefficients, Discrete Mathematics, 29(1980), 257-264.