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REPRESENTATION OF CERTAIN CLASSES OF DISTRIBUTIVE LATTICES BY SECTIONS OF SHEAVES

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ABSTRACT. Epstein and Horn ([6]) proved that a Post algebra is always a P-algebra and in a P-algebra, prime ideals lie in disjoint maximal chains. In this paper it is shown that a P-algebra L is a Post algebra of order $n \geq 2$, if the prime ideals of L lie in disjoint maximal chains each with n-1 elements. The main tool used in this paper is that every bounded distributive lattice is isomorphic with the lattice of all global sections of a sheaf of bounded distributive lattices over a Boolean space. Also some properties of P-algebras are characterized in terms of the stalks.

KEY WORDS AND PHRASES. *Post Algebra, P-algebra, B-algebra, Heyting Algebra, Stone Lattice, Boolean Space, Sheaf of Distributive Lattices Over a Boolean Space, Prime Ideals Lie in Disjoint Maximal Chains, Regular Chain Base.*

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1. INTRODUCTION.

Epstein ([5]) proved that in a Post algebra of order $n \geq 2$ prime ideals lie in disjoint maximal chains each with $n - 1$ elements. He has also proved that if L is a finite distributive lattice and prime ideals of L lie in disjoint maximal chains each with n-1 elements, then L is a Post algebra of order n. Epstein and Horn ([6]) have shown that a Post algebra is always a P-algebra and in a P-algebra prime ideals lie in disjoint maximal chains. It is the main theme of this paper that a P-algebra L is a Post algebra of order $n \geq 2$, if the prime ideals of L lie in disjoint maximal chains each with n-1 elements.

The main tool used in this paper is the fact that every bounded distributive lattice is isomorphic with the lattice of all global sections of a sheaf of bounded distributive lattices over a Boolean space ([15] and [9]). It is well known that a P-algebre is always a (double) Heyting algebra, a (double) L-algebra, a pseudocomplemented lattice, a Stone lattice and a normal lattice. We characterize these properties of P-algebras in detail in terms of the stalks of the corresponding sheaf. We give another characterization of Post algebras by regular chain bases.

Throughout this paper, by L we mean a (nontrivial) bounded distributive lattice $(L, \vee, \wedge, 0, 1)$ and $B = B(L)$ the Boolean algebra of complemented elements of L. For any $a \in B$, we denote the complement of a by a'. For any $x,y \in L$, we denote the largest $z \in L$ such that $x \wedge z \leq y$ (if it exists) by $x \rightarrow y$ and the largest element $a \in B$ such that $x \wedge a \leq y$ (if it exists) by $x \Rightarrow y$. If, for every $x,y \in L$, $x \rightarrow y$ ($x \Rightarrow y$) exists, then we say that L is a Heyting algebra (respectively B algebra). Dually, we define $x \leftarrow y$ and $x \Leftarrow y$ respectively. If in a Heyting algebra (B-algebra), $(x \rightarrow y) \vee (y \rightarrow x) = 1$ ($(x \Rightarrow y) \vee (y \Rightarrow x) = 1$) for every $x,y \in L$, then we say that L is an L-algebra (respectively BL-algebra). For any $x \in L$, if $x \rightarrow 0$ exists, then we say that x has the pseudocomplement and we usually write x^* for $x \rightarrow 0$. If x^* exists for each $x \in L$, then we say that L is pseuodcomplemented. The dual of L is denoted by L^d. If both L and L^d are Heyting algebras (B-algebras, L-algebras, BL-algebras), then we say that L is a double Heyting algebra (respectively double B-algebra, double L-algebra, double BL-algebra). L is said to be a P-algebra if L is a BL-algebra. Epstein and Horn proved that L is a P-algebra if and only if L is a double L-algebra ([6], theorem 3.4). For the elementary properties of these types of lattices, we refer to ([2]) and ([6]).

By a sheaf of bounded distributive lattices we mean a triple $(\mathcal{S},\pi,X)$ satisfying the following:

i) $\mathcal{S}$ and X are topological spaces

ii) $\pi : \mathcal{S} \to X$ is a local homeomorphism

iii) Each $\pi^{-1}(p)$, $p \in X$ is a bounded distributive lattice;

iv) the maps $(x,y) \mapsto x \vee y$ and $(x,y) \mapsto x \wedge y$ from $\mathcal{S} \vee \mathcal{S} = \{(x,y) \in \mathcal{S} \times \mathcal{S} \;/\; \pi(x) = \pi(y)\}$ into $\mathcal{S}$ are continuous and

v) the maps $\hat{0} : p \mapsto 0(p)$ and $\hat{1} : p \mapsto 1(p)$ from $X \rightarrow Q$ are continuous, where $0(p)$ and $1(p)$ are the smallest and largest elements of $\pi^{-1}(p)$ respectively.

We call $\mathcal{S}$ the sheaf space X the base space and π the projection map. We write $\mathcal{S}_p$ for $\pi^{-1}(p)$, $p \in X$ and call $\mathcal{S}_p$ the stalk at p. By a (global) section of the sheaf $(\mathcal{S},\pi,X)$ we mean a continuous map $\sigma : X \rightarrow \mathcal{S}$ such that $\pi \circ \sigma = id_X$. For any sections σ and τ we write $|(\sigma,\tau)|$ for the closed set $\{p \in X | \sigma(p) \neq \tau(p)\}$ and we call $|(\sigma,0)|$ the support of σ and write $|\sigma|$ for $|(\sigma,0)|$. For the preliminary results on sheaf theory, we refer to the pioneering work of Hofmann ([8]).

By Spec L, we mean the (Stone) space Y of all prime ideals of L with the hull-kernel topology; i.e., the topology for which $\{Y_x | x \in L\}$ is a base, where for any $x \in L$, $Y_x = \{P \in \text{Spec } L / x \notin P\}$. Throughout this paper X denotes Spec B which is a Boolean space, ie., a compact, Hausdorff and totally disconnected space. Since $a \mapsto X_a$ is a Boolean isomorphism of B onto the Boolean algebra of all clopen subsets of X, we identify a and X_a and write simply a for X_a. For any $p \in X$, $\mathcal{S}_p$ be the quotient lattice L/θ_p where θ_p is the congruence on L given by

$$(x,y) \in \theta_p \Leftrightarrow x \wedge a = y \wedge a \text{ for some } a \in B-p,$$

and let $\mathcal{S}$ be the disjoint union of all $\mathcal{S}_p$, $p \in X$. For each $x \in L$, define $\hat{x} : X \to \mathcal{S}$ by $\hat{x}(p) = \theta_p(x)$, the congruence class of θ_p containing x. Topologize $\mathcal{S}$ with the largest topology such that each $\hat{x}$, $x \in L$, is continuous. Define $\pi : \mathcal{S} \to X$ by $\pi(s) = p$ if $s \in \mathcal{S}_p$. The following theorem is the main tool used in this paper and is due to Subrahmanyam ([15]) (see also [9]).

THEOREM 1.1 $(\mathcal{S},\pi,X)$ described above is a sheaf of bounded distributive lattices in which each stalk has exactly two complemented elements, viz., 0(p) and 1(p).

1.2 For any $a \in B$, $p \in X$, $\hat{a}(p) = 1(p)$ if $p \in a$ and $\hat{a}(p) = 0(p)$ if $p \notin a$.

1.3 For any $x,y \in L$ and $a \in B$, $\hat{x}/a = \hat{y}/a$ if and only if $x \wedge a = y \wedge a$.

1.4 $x \mapsto \hat{x}$ is an isomorphism of L onto the lattice $\Gamma(x,\mathcal{S})$ of all global sections of the sheaf $(\mathcal{S},\pi,X)$. We identify $\hat{x}$ with x and write simply x for $\hat{x}$.

1.5 For any prime ideal P of L, there exists a unique $p \in X$ such that $\{x(p)/x \in P\}$ is a prime ideal of $\mathcal{S}_p$. On the other hand if Q is a prime ideal of $\mathcal{S}_p$, $p \in X$, then $\{x \in L/x(p) \in Q\}$ is a prime ideal of L. This correspondence exhibits the set of all prime ideals of L as the disjoint union of the sets of prime ideals of the stalks. Moreover, if P and Q are prime ideals of distinct stalks $\mathcal{S}_p$ and $\mathcal{S}_q$, then P and Q are incomparable, when they are regarded as prime ideals of L.

Throughout this paper, by a stalk $\mathcal{S}_p$, $p \in X$, we mean the stalks of the sheaf $(\mathcal{S},\pi,X)$ described above at p.

2. PSEUDOCOMPLEMENTED LATTICES.

It is well known that a bounded distributive lattice is a Heyting algebra if and only if it is relatively pseudocomplemented; i.e., each interval $[x,y]$, $x \le y \in L$ is pseudocomplemented ([1]). Also the class of all distributive pseudocomplemented lattices and the class of all Heyting algebras are equationally definable (see [1] and [11]), when we regard the pseudocomplementation and $(x,y) \mapsto (x \to y)$ as unary and binary operations respectively in L. Thanks to the referee for suggesting a simpler proof of the following.

THEOREM 2.1. L is pseudocomplemtnetd if and only if each stalk $\mathcal{S}_p$, $p \in X$ is pseudocomplemented and the pseudocomplementation $x \mapsto x^*$ is continuous and in this case, the pseudocomplement of $x(p)$ in $\mathcal{S}_p$ is precisely $x^*(p)$ for all $x \in L$.

PROOF. Suppose L is pseudocomplemented. Then it is easily seen that for all x and p, $(x(p))^*_{\mathcal{S}_p}$ exists and is equal to $x^*(p)$. Then it is easy to show that the map $x \mapsto x^*$ is continuous. For the converse, if $x \in L$, the hypothesis implies that the map $f : x \to \mathcal{S}$ defined by $f(p) = (x(p))^*$ is a global section of $\mathcal{S}$. Therefore, $f = \hat{y}$ for some y and it is clear that $y = x^*$.

If L is a Heyting algebra, then each θ_a, $a \in B$, is compatable with the binary operation $(x,y) \mapsto (x \to y)$. For, if $a \in B$ and (x,y) and $(x_1,y_1) \in \theta$ then $(x \to x_1) \wedge y \wedge a = (x \to x_1) \wedge x \wedge a \le x_1 \wedge a = y_1 \wedge a \le y_1$, so that $(x \to x_1)\ a \le (y \to y_1) \wedge a$. Similarly, we have $(y \to y_1) \wedge a \le (x \to x_1) \wedge a$ and hence

$(x \to x_1, y \to y_1) \in \theta_a$. Hence the following theorem and its proof are analogous to the above.

THEOREM 2.2. L is a Heyting algebra if and only if each stalk $\mathcal{J}_p$, $p \in X$ is a Heyting algebra, and the operation $(s,t) \mapsto (s \to t)$ of $\mathcal{J} V \mathcal{J}$ into $\mathcal{J}$ is continuous and in this case $x(p) \to y(p)$ in $\mathcal{J}_p$, $p \in X$, is equal to $(x \to y)(p)$ for all $x,y \in L$.

3. NORMAL LATTICES.

DEFINITION 3.1. (Cornish [4]). L is said to be normal if any two distinct minimal prime ideals of L are comaximal and L is said to be relatively normal if each interval $[x,y]$, $x \leq y \in L$ is normal.

For any $x,y \in L$, let $(x,y)^*_L$ be the ideal $\{z \in L / x \wedge z \leq y\}$ of L. For any $x \in L$, we write $(x)^*_L$ for $(x,0)^*_L$. Cornish ([4]) proved that L is normal if and only if $(x \wedge y)^*_L = (x)^*_L \vee (y)^*_L$ for all $x,y \in L$, and that L is relatively normal if and only if $(x \wedge y,z)^*_L = (x,z)^*_L \vee (y,z)^*_L$ for all $x,y,z \in L$ where $\vee$ stands for the join operation in the lattice of all ideals of L.

THEOREM 3.2. (Speed [13]). A pseudocomplemented distributive lattice is normal if and only if it is a Stone lattice.

THEOREM 3.3. (Balbes and Horn [1]): A Heyting algebra is relatively normal if and only if it is an L-algebra.

THEOREM 3.4. L is normal if and only if each stalk $\mathcal{J}_p$, $p \in X$, is normal.

PROOF. Suppose L is normal and $p \in X$. Let $u,v \in \mathcal{J}_p$ so that $u = x(p)$ and $v = y(p)$ for some $x,y \in L$. Clearly $(u)^*_{\mathcal{J}_p} \vee (v)^*_{\mathcal{J}_p} \subseteq (u \wedge v)^*_{\mathcal{J}_p}$. Let $t(p) \in (u \wedge v)^*_{\mathcal{J}_p}$, $t \in L$. Since, $(x \wedge y \wedge t)(p) = x(p) \wedge y(p) \wedge t(p) = 0(p)$ there exists $a \in B-p$ such that $x \wedge y \wedge t \wedge a = 0$, so that $t \in (x \wedge y \wedge a)^*_L = (x \wedge a)^*_L \vee (y \wedge a)^*_L$ and hence $t = t_1 \vee t_2$ for some $t_1 \in (x \wedge a)^*_L$ and $t_2 \in (y \wedge a)^*_L$. Now $t(p) = t_1(p) \vee t_2(p)$, $t_1(p) \in (u)^*_{\mathcal{J}_p}$ and $t_2(p) \in (v)^*_{\mathcal{J}_p}$. Hence $\mathcal{J}_p$ is normal. Conversely, suppose each stalk $\mathcal{J}_p$, $p \in X$ is normal. Let $x,y \in L$ and

$z \in (x \wedge y)^*_L$. For each $p \in X$, since $z(p) \in (x(p) \wedge y(p))^*_{\mathcal{S}_p} = (x(p))^*_{\mathcal{S}_p} \vee (y(p))^*_{\mathcal{S}_p}$, there exists $a \in B-p$, t and $s \in L$, such that $a \wedge z = a \wedge (t \vee s)$, $t \wedge x \wedge a = s \wedge y \wedge a = 0$. By the compactness of X, it follows that there exists $a_1, a_2, \ldots, a_n \in B$ and $t_1, t_2, \ldots, t_n, s_1, \ldots, s_n \in L$ such that $\bigvee_{i=1}^{n} a_i = 1$, $a_i \wedge z = a_i \wedge (t_i \vee s_i)$, $t_i \wedge x \wedge a_i = 0 = s_i \wedge y \wedge a_i$. Now, Put $t = \bigvee_{i=1}^{n} (t_i \wedge a_i)$ and $s = \bigvee_{i=1}^{n} (s_i \wedge a_i)$ then, $z = \bigvee_{i=1}^{n} (z \wedge a_i) = \bigvee_{i=1}^{n} (a_i \wedge (t_i \vee s_i)) = t \vee s$ and $t \wedge x = \bigvee_{i=1}^{n} (t_i \wedge a_i \wedge x) = 0 = \bigvee_{i=1}^{n} (s_i \wedge a_i \wedge y) = s \wedge y$. Hence $(x \wedge y)^*_L \subseteq (x)^*_L \vee (y)^*_L$ and the otherside inclusion is obvious. Hence L is normal.

The proof of the following theorem is analogus to that of the above.

THEOREM 3.5. L is relatively normal if and only if each stalk $\mathcal{S}_p$, $p \in X$, is relatively normal.

DEFINITION 3.6. (Speed [12]). L is said to be a distributive * lattice and denoted by $L \in \Delta^*$ if, for each $x \in L$, there exists $y \in L$ such that $(x)^{**}_L := \{u \in L \ / \ u \wedge v = 0 \text{ for every } v \in (x)^*_L\} = (y)^*_L$.

Speed ([12] proved that $L \in \Delta^*$ if and only if, for each $x \in L$, there exists $y \in L$, such that $x \wedge y = 0$ and $x \vee y$ is dense; i.e., $(x \wedge y)^*_L = \{0\}$.

THEOREM 3.7. $L \in \Delta^*$ if and only if (i) $\mathcal{S}_p \in \Delta^*$ for each $p \in X$ and (ii) $\{p \in X \ \ x(p) \text{ is dense in } \mathcal{S}_p\}$ is open for each $x \in L$.

PROOF. Suppose $L \in \Delta^*$ and $x \in L$. There exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is dense in L. Let $p \in X$. Clearly, $x(p) \wedge y(p) = 0(p)$. Also, if $z \in L$, such that $((x(p) \vee y(p)) \wedge z(p) = 0(p)$, then $(x \vee y) \wedge z \wedge a = 0$ for some $a \in B-p$ and hence $z \wedge a = 0$, so that $z(p) = 0(p)$. Hence $x(p) \vee y(p)$ is dense in $\mathcal{S}_p$. Therefore $\mathcal{S}_p \in \Delta^*$. Now, suppose $x(p)$ is dense in $\mathcal{S}_p$. It follows that $y(p) = 0(p)$ and hence there exists $a \in B-p$ such that $y \wedge a = 0$.

We claim that $x(q)$ is dense in $\mathcal{S}_q$ for all $q \in a$. For, if $q \in a$ and $z(q) \in \mathcal{S}_q$, $z \in L$, such that $x(q) \wedge z(q) = 0(q)$, then there exists $b \in B - q$ such that $x \wedge z \wedge b = 0$; so that $(x \vee y) \wedge z \wedge a \wedge b = 0$, and hence $z \wedge a \wedge b = 0$ and since $a \wedge b \in B-q$, $z(q) = 0(q)$. Conversely, suppose (i) and (ii) hold and $x \in L$. For each $p \in X$, by (i) and (ii), there exists $y \in L$ and $a \in B-p$ such that $x \wedge y \wedge a = 0$ and $(x \vee y)(q)$ is dense in $\mathcal{S}_q$ for all $q \in a$. By the usual compactness argument, there exists $y_1, y_2, \ldots, y_n \in L$, $a_1, \ldots, a_n \in B$ such that $\bigvee_{i=1}^{n} a_i = 1$, $a_i \wedge a_j = 0$ for $i \neq j$, $x \wedge y_i \wedge a_i = 0$ and $(x \vee y_i)(p)$ is dense in $\mathcal{S}_p$ for all $p \in a_i$. Now put $y = \bigvee_{i=1}^{n} (y_i \wedge a_i)$. Then $x \wedge y = 0$ and $x \vee y$ is dense in L. For, $(x \vee y) \wedge z = 0$ for some $z \in L$, then, for all $p \in a_i$, $0(p) = ((x \vee y) \wedge z)(p) = (x(p) \vee y(p)) \wedge z(p)$ and hence $z(p) = 0(p)$ for all $p \in a_i$ and therefore $z = 0$. Hence $L \in \Delta^*$.

4. STONE LATTICES.

For any $p \in X$, since the stalk $\mathcal{S}_p$ has exactly two complemented elements, $\mathcal{S}_p$ is a Stone lattice if and only if $\mathcal{S}_p$ is dense (i.e., if $x(p) \neq o(p)$, then $(x(p))^*_{\mathcal{S}_p} = \{0(p)\}$). Hence, by theorem 2.1, 3.2, and 3.4, we have the following.

THEOREM 4.1. Suppose L is pseudocomplemented. Then the following are equivalent.

(i) L is a Stone lattice

(ii) L is normal

(iii) Each stalk $\mathcal{S}_p$, $p \in X$, is a normal

(iv) Each stalk $\mathcal{S}_p$, $p \in X$, is a Stone lattice

(v) Each stalk $\mathcal{S}_p$, $p \in X$, is dense.

The following theorem is a consequence of theorem 2.2, 3.3 and 3.5.

THEOREM 4.2. Let L be a Heyting algebra. Then the following are equivalent.

(i) L is an L-algebra

(ii) L is relatively normal

(iii) Each stalk $\mathcal{Y}_p$, $p \in X$, is relatively normal

(iv) Each stalk $\mathcal{Y}_p$, $p \in X$, is an L-algebra.

Since L is an L-algebra if and only if it is relatively Stone lattice (Theorem 4.11 of [1]) (i.e., each interval is a Stone lattice) in view of theorem 4.1, one may suspect that if L is an L-algebra, then each stalk is relatively dense and hence a chain. This is not true (see 4.4 below), though the converse is proved in the following.

THEOREM 4.3. If L is a Heyting algebra and each stalk is a chain, then L is an L-algebra.

PROOF. If each stalk is a chain, then by theorem 1.5, the prime ideals of L lie in disjoint maximal chains and hence L is relatively normal lattice and hence the theorem follows from theorem 2.3.

EXAMPLE 4.4. Let B_4 be the 4-element Boolean algebra and A be the distributive lattice obtained by adjoining an external element to B_4 as the smallest element. Then A is an L-algebra which is not a chain (Thanks to the referee).

Epstein and Horn ([6]) proved that L is a Stone lattice if and only if L^d is pseudosupplemented and $0 \Leftarrow x \wedge y = (0 \Leftarrow x) \wedge (0 \Leftarrow y)$ for all $x, y \in L$. Now, these two necessary and sufficient conditions for L to be a Stone lattice can be viewed in terms of the stalks as follows.

THEOREM 4.5. L^d is pseudosupplemented if and only if $|x|$ is open for each $x \in L$ and in this case $|x| = 0 \Leftarrow x$ for all $x \in L$.

PROOF. Follows from Lemma 5.2.

For any $p \in X$, let (p) be the smallest ideal of L containing p. The proof of the following theorem is simple.

THEOREM 4.6. For any $p \in X$, the stalk $\mathcal{Y}_p$ is dense if and only if (p) is a prime ideal of L.

It can be easily seen that each stalk $\mathcal{Y}_p$, $p \in X$, is dense if and only if $|x \wedge y| = |x| \cap |y|$ for all $x,y \in L$. Hence from theorem 4.5 and 4.6 and lemma 2.9 of ([7]), we have the following.

THEOREM 4.7. L is a Stone lattice if and only if $|x|$ is open for all $x \in L$ and each stalk $\mathcal{Y}_p$, $p \in X$ is dense.

REMARK 4.8. Swamy and Rama Rao ([10] proved that a lattice L is a Stone lattice if and only if L is isomorphic to the lattice of all global sections of a sheaf of dense bounded distributive lattices over a Boolean space in which each section has open support (see also [9]). It can be verified, that when L is a Stone lattice, then our sheaf $(\mathcal{Y},\pi,X)$ coincides with the sheaf constructed in ([10]).

5. P-ALGEBRAS.

The following results interpret B-algebras in sheaf theoretic terminology.

LEMMA 5.1. Let $x,y \in L$. Then $x \Rightarrow y$ exists in B if and only if $\{p \in X \,/\, x(p) \le y(p)\}$ is closed and in this case $x \Rightarrow y = \{p \in X \,/\, x(p) \le y(p)\}$.

PROOF. For any $p \in X$, $x(p) \le y(p)$ if and only if there exists $a \in B-p$ such that $x \wedge a \le y$. If $x \Rightarrow y$ exists in B, then, for any $p \in X$, $x(p) \le y(p)$ if and only if $p \in x \Rightarrow y$. Conversely, if $\{p \in X \,/\, x(p) \le y(p)\}$ is closed, then there exists $a \in B$ such that $p \in a$ if and only if $x(p) \le y(p)$ for all $p \in X$. Hence $a = x \Rightarrow y$.

The proof of the following is easy.

LEMMA 5.2. For any $x,y \in L$, $|(x,y)|$ is open if and only if there exists a largest element a of B such that $x \wedge a = y \wedge a$.

The following theorem is a consequence of the above lemmas.

THEOREM 5.3. The following are equivalent.

1) L is a dual B-algebra

2) For any $x,y \in L$, $\{a \in B \,/\, x \vee a = y \vee a\}$ is a principal filter of B.

3) For any $x,y \in L$, $\{a \in B \,/\, x \wedge a = y \wedge a\}$ is a principal ideal of B.

4) L is a B-algebra

5) $\{p \in X \,/\, x(p) \leq y(p)\}$ is closed for every $x,y \in L$.

6) $|(x,y)|$ is open for every $x,y \in L$.

THEOREM 5.4. Suppose L is a B-algebra. Then the following are equivalent.

1) L is a P-algebra; i.e. L is a BL-algebra

2) Each stalk is a chain

3) For every $x,y \in L$, there exists $a \in B$ such that $x \wedge a \leq y$ and $y \wedge a' \leq x$.

4) For every $x,y \in L$, there exists $a \in B$ such that $x \vee a \geq y$ and $y \vee a' \geq x$.

PROOF. $2 \Leftrightarrow 3$ is proved in ([15]) and $3 \Leftrightarrow 4$ is clear. $1 \Leftrightarrow 2$ follows from lemma 5.1.

6. POST ALGEBRAS.

The following definition is slightly different from that of Chang and Horn ([3]).

DEFINITION 6.1. By a generalized Post algebra, we mean the lattice $\mathcal{C}(Z,C)$ of all continuous maps of a Boolean space Z into a discrete bounded chain C where, the operations are pointwise.

THEOREM 6.2. The following are equivalent

1) L is a generalized Post algebra.

2) There exists a chain C in L such that the natural map $c \mapsto c(p) : C \to \mathcal{J}_p$ is an isomorphism for all $p \in X$.

3) There exists a chain C and, for each $p \in X$, an order isomorphism $\alpha_p : C \to \mathcal{J}_p$ such that for any $c \in C$ and $x \in L$, $\{p \in X / \alpha_p(c) = x(p)\}$ is open in X.

PROOF. $1 \Longrightarrow 2$:

Let $L = \mathcal{C}(Z,D)$ where Z is a Boolean space and D is a discrete bounded chain. It is well known that $a \mapsto \chi_a$ is a Boolean isomorphism of the algebra of all clopen subsets of Z onto B, the centre of L, where χ_a is the characteristic function on a. We identify χ_a with a. Also the Stone space X is homeomorphic with Z.

Let C be the set of all constant maps of Z into D. For any $d \in D$, let $\bar{d}$ denote the constant map which maps every element of Z onto d. Then C is a chain in L. Let $p \in X$. Clearly, the natural map $\emptyset_p: C \to \mathcal{J}_p = L/\theta_p$ is a homomorphism.

If $d_1, d_2 \in D$ such that $\bar{d}_1(p) = \bar{d}_2(p)$ then $\bar{d}_1 \wedge a = \bar{d}_2 \wedge a$ for some $a \in B-p$ and hence $d_1 = d_2$. Now, let $x \in L$. Then if $p \in x^{-1}(d)$ for some $d \in D$, since $x : Z \to D$ is continuous, $x^{-1}(d) \in B-p$ and since $x \wedge x^{-1}(d) = \bar{d} \wedge x^{-1}(d)$, it follows that $(x,\bar{d}) \in \theta_p$. Hence θ_p is an isomorphism.

$2 \Longrightarrow 3$: If C is a chain in L and the natural map $\emptyset_p: C \to \mathcal{J}_p$ is an isomorphism for every $p \in X$, then, for any $c \in C$ and $x \in L$. $\{p \in X \ / \ \alpha_p(c) = x(p)\} = \{p \in X \ / \ c(p) = x(p)\}$ which is open.

$3 \Longrightarrow 1$: We first observe that since $\mathcal{J}_p$ is bounded and α_p is an isomorphism of C onto $\mathcal{J}_p$, C is also bounded. Let X = Spec B. Define $\theta : L \to \mathcal{C}(X,C)$ by $(\theta(x))(p) = \alpha_p^{-1}(x(p))$ for each $x \in L$ and $p \in X$. Let $c \in C$. Then

$$(\theta(x))^{-1}\{c\} = \{p \in X \ / \ \alpha_p^{-1}(x(p)) = c\}$$

$$= \{p \in X \ / \ \alpha_p(c) = x(p)\} \text{ is open by (3) and}$$

hence $\theta(x)$ is continuous. Clearly θ is a homorphism and one-one since α_p^{-1} is so. Now, we will show that θ is onto. Let $f \in \mathcal{C}(X,C)$. Define $\sigma: X \to \mathcal{J}$ by $\sigma(p) = \alpha_p(f(p))$ for every $p \in X$. We will show that σ is a section. Let $x \in L$ and $a \in B$, then

$$\sigma^{-1}(x(a)) = \{p \in a \ / \ \alpha_p \ (f(p)) = x(p)\}$$

$$= a \cap \bigcup_{c \in C} \{p \in X \ / \ f(p) = c\} \cap \{ p \in X \ / \ \alpha_p(c) = x(p)\}.$$

Since f is continusous, it follows that $\sigma^{-1}(x(a))$ is open. Since $\{x(a) \ / \ a \in B$ and $x \in L\}$ is a base for the topology on $\mathcal{Y}$, it follows that σ is continuous and clearly $\pi \circ \sigma = id_X$. Therefore, $\sigma = \hat{x}$ for some $x \in L$ and also $\theta(x) = f$. Hence θ is an isomorphism and therefore L is a generalized Post algebra.

THEOREM 6.3. Let $n \geq 2$ and L a P-algebra. Then the following are equivalent.

1) L is a Post algebra of order n.
2) Spec L is the disjoint union of maximal chains each with n-1 elements.
3) Each stalk is a chain with n elements.

PROOF. $1 \Rightarrow 2$ is proved in ([5]).
Since L is a P-algebra, by theorem 4.4, each stalk $\mathcal{Y}_p$, $p \in X$, a chain. Also, by theorem 0.(5), Spec L is the disjoint union of the chains Spec $\mathcal{Y}_p$, $p \in X$.

If Spec L is the disjoint union of all maximal chains C_α, $\alpha \in \Delta$ each with n-1 elements, then, for any $p \in X$, Spec $\mathcal{Y}_p = C_\alpha$ for some $\alpha \in \Delta$. Hence Spec $\mathcal{Y}_p$ has n-1 elements and therefore $\mathcal{Y}_p$ has n elements and hence $(2) \Rightarrow (3)$.

Now, suppose each stalk is a chain with n elements and C_n is the n-element chain $1 < 2 < \ldots < n$. For any $p \in X$, let $\mathcal{Y}_p = \{0(p) = x_{1p}(p) < x_{2p}(p) < \ldots < x_{np}(p) = 1(p)\}$ where $x_{1p}, x_{2p}, \ldots, x_{np} \in L$. Define for any $p \in X$, $\alpha_p\colon C_n \to \mathcal{Y}_p$ by $\alpha_p(i) = x_{ip}(p)$ for each $i \in C_n$. Clearly, α_p is an order isomorphism. Let $i \in C_n$, $x \in L$ and $p \in X$ such that $\alpha_p(i) = x(p)$. ie., $x_{ip}(p) = x(p)$ so that there exists $a \in B\text{-}p$ such that $x_{ip}(q) = x(q)$ for all $q \in a$. Since L is a B-algebra and $x_{jp}(p) < x_{kp}(p)$ for all $j < k$, by theorem 5.3, there exists $b \in B\text{-}p$ such that $x_{jp}(q) < x_{kp}(q)$ for all $j < k$ and $q \in b$ and hence $x_{ip}(q) = x_{iq}(q)$ for all $i \in C_n$ and $q \in b$. Then $p \in a \wedge b \in B$ and for any $q \in a \wedge b$, $\alpha_q(i) = x_{iq}(q) = x_{ip}(q) = x(q)$. Hence $\{p \in X \ / \ \alpha_p(i) = x(p)\}$ is open for each $i \in C_n$ and $x \in L$.

DEFINITION 6.4. By a chain base C for L we mean a chain C with 0 in L such that L is generated by the centre B and C; i.e., every $x \in L$ can be written in the form $\bigvee_{i=1}^{n} (a_i \wedge c_i)$ for some $a_i \in B$ and $c_i \in C$.

DEFINITION 6.5. A chain base C in L is said to be regular, if, for $c_1 \neq c_2 \in C$ and $a \in B$, $c_1 < c_2$ and $a \wedge c_2 \leq c_1$ imply $a = 0$.

It is proved in ([15]) that a bounded distributive lattice L is a generalized Post algebra if and only if there exists a regular chain base for L. Now, we characterize chain bases and regular chain bases in terms of the stalks. It is also proved in ([15]) that if C is a chain base for L, the natural map $\emptyset_p : C \to \mathcal{S}_p$, defined by $\emptyset_p(c) = c(p)$ is an epimorphism for all $p \in X$. We prove the converse in the following.

THEOREM 6.6. Let C be a chain in L and $0 \in C$. Then $\emptyset_p : C \to \mathcal{S}_p$ is an epimorphism for each $p \in X$, if and only if C is a chain base for L.

PROOF. Suppose $\emptyset_p$ is an epimorphism for each $p \in X$ and let $x \in L$. For each $p \in X$, there exists $c_p \in C$ such that $\emptyset_p(c_p) = x(p)$ ie., $c_p(p) = x(p)$, so that there exists $a \in B\text{-}p$ such that $c_p \wedge a = x \wedge a$. Therefore, there exists a partition $a_1, a_2, \ldots, a_n$ of B and $c_1, c_2, \ldots, c_n \in C$ such that $c_i \wedge a_i = x \wedge a_i$ so that $x = x \wedge 1 = x \wedge \bigvee_{i=1}^{n} a_i = \bigvee_{i=1}^{n} (x \wedge a_i) = \bigvee_{i=1}^{n} (c_i \wedge a_i)$.

Hence C is a chain base for L.

The following theorem is a straight forward verification.

THEOREM 6.7. Let C be a chain in L. Then the following are equivalent.

1) The natural map $\emptyset_p : C \to \mathcal{S}_p$ is one for all $p \in X$.

2) For any $c_1 \neq c_2 \in C$ and $a \in B$, $c_1 < c_2$ and $a \wedge c_2 \leq c_1$ imply $a = 0$.

3) For any $c_1 \neq c_2 \in C$ and $0 \neq a \in B$, $a \wedge c_1 \neq a \wedge c_2$.

By summarizing the above results, we have the following :

THEOREM 6.8. Suppose L is a bounded distributive lattice. Then the following are equivalent.

1) L is a generalized Post algebra

2) There exists a chain C in L such that the natural map $\emptyset_p : C \to \mathcal{S}_p$ is an isomorphism for all $p \in X$.

3) There exists a chain C and for each $p \in X$, an order isomorphism $\alpha_p : C \to \mathcal{S}_p$ such that for any $c \in C$ and $x \in L$, $\{p \in X \ / \ \alpha_p(c) = x(p)\}$ is open in X.

4) L has a regular chain base.

REMARK. The equivalence of (1) and (4) is established in ([15]) by using the Boolean extension techniques.

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