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THE RADIUS OF CONVEXITY OF CERTAIN ANALYTIC FUNCTIONS II

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ABSTRACT. In [2], MacGregor found the radius of convexity of the functions $f(z) = z+a_2z^2+a_3z^3+\ldots$, analytic and univalent such that $|f'(z) - 1| < 1$. This paper generalized MacGregor's theorem, by considering another univalent function $g(z) = z+b_2z^2+b_3z^3+\ldots$ such that $\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$ for $|z| < 1$. Several theorems are proved with sharp results for the radius of convexity of the subfamilies of functions associated with the cases: $g(z)$ is starlike for $|z| < 1$, $g(z)$ is convex for $|z| < 1$, $\text{Re}\{g'(z)\} > \alpha$ ($\alpha=0$, $1/2$).

KEY WORDS AND PHRASES. Univalent, analytic, starlike, convex, radius of starlikeness and radius of convexity.

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1. INTROUDCTION.

Throughout we suppose that $f(z) = z+a_2z^2+\ldots$ is analytic for $|z| <$ and $g(z) = z + b_2z^2 + \ldots$ is analytic and univalent for $|z| < 1$. In [4] the author solved the following problem: what is the radius of convexity of the family of functions $f(z)$ which satisfy $\mathrm{Re}(\frac{f'(z)}{g'(z)}) > 0$ for $|z| < 1$? The problem was solved also for each of the subfamilies associated with the cases: $g(z)$ is starlike for $|z| < 1$, $\mathrm{Re}\{g'(z)\} > \alpha$ ($\alpha = 0, \frac{1}{2}$) for $|z| < 1$, $g(z)$ is convex of order $\alpha (0 \leq \alpha < 1)$ for $|z| < 1$.

In this paper we consider functions $f(z)$ which satisfy $|\frac{f'(z)}{g'(z)} - 1| < 1$ for $|z| < 1$. The radius of convexity of this family of functions is determined. Also, we find the radius of convexity for the subfamilies associated with each of the cases: $g(z)$ is starlike for $|z| < 1$, $g(z)$ is convex for $|z| < 1$, $\mathrm{Re}\{g'(z)\} > \alpha$ ($\alpha = 0, \frac{1}{2}$). The case $g(z) = z$ has already been proved by MacGregor [2].

2. The following lemmas will be used in the proofs of our theorems.

LEMMA 1. [4] The function $h(z)$ is analytic for $|z| < 1$ and satisfies $h(0) = 1$ and $\mathrm{Re}\{h(z)\} > \alpha$ $(0 \leq \alpha < 1)$ for $|z| < 1$ if and only if $h(z) = \frac{1 + (2\alpha - 1)z\phi(z)}{1 + z\phi(z)}$, where $\phi(z)$ is analytic and satisfies $|\phi(z)| \leq 1$ for $|z| < 1$.

LEMMA 2. If $\phi(z)$ is analytic for $|z| < 1$ and $|\phi(z)| \leq 1$ for $|z| < 1$, then

$$\text{(i)} \quad |\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}$$

$$\text{(ii)} \quad \left|\frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)}\right| \leq \frac{1}{1 - |z|}$$

Part (i) of Lemma 2 is well-known [3], and part (ii) follows easily, by first applying triangular inequalities and then using part (i).

LEMMA 3. If $h(z) = 1 + c_1 z + \dots$ is analytic for $|z| < 1$ and $\mathrm{Re}\{h(z)\} > 0$ for $|z| < 1$, then

$$\mathrm{Re}\{h(z)\} \geq \frac{1 - |z|}{1 + |z|} .$$

This is a well-known result due to C. Carathéodory.

3. THEOREM 1. Suppose $f(z) = z + a_2 z^2 + \dots$ is analytic for $|z| < 1$ and $g(z) = z + b_2 z^2 + \dots$ is analytic and univalent for $|z| < 1$.

if $\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$ for $|z| < 1$, then $f(z)$ maps $|z| < 1/5$ onto a convex domain. The result is sharp.

PROOF. Let $h(z) = \frac{f'(z)}{g'(z)} - 1$. The function $g(z)$ is univalent for $|z| < 1$, therefore $g'(z) \neq 0$ for $|z| < 1$. The function $h(z)$ is analytic for $|z| < 1$, $h(0) = 0$ and $|h(z)| < 1$ for $|z| < 1$. Thus by Schwarz's lemma we have

$$h(z) = z\phi(z) \quad \text{with} \quad |\phi(z)| \leq 1.$$

Therefore

$$f'(z) = g'(z)(1 + z\phi(z)).$$

Taking the logarithmic derivative we obtain

$$\frac{f''(z)}{f'(z)} = \frac{g''(z)}{g'(z)} + \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} .$$

Using lemma 2(ii) we get

$$\text{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} \geq \text{Re}\left\{\frac{zg''(z)}{g'(z)} + 1\right\} - \frac{|z|}{1-|z|} . \tag{3.1}$$

Since $g(z)$ is univalent, we have [1]

$$\text{Re}\left\{\frac{zg''(z)}{g'(z)} + 1\right\} \geq 1 + \frac{|z|(2|z| - 4)}{1 - |z|^2} .$$

Using this estimate in (3.1) we obtain

$$\text{Re}\ \frac{zf''(z)}{f'(z)} + 1\} \geq \frac{1 - 5|z|}{1 + |z|^2} .$$

This last expression is positive if $|z| < 1/5$. Since the condition $\text{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\}$ > 0 for $|z| < r$ is necessary and sufficient for $f(z)$ to map $|z| < r$ onto a convex domain, we conclude that $f(z)$ maps $|z| < 1/5$ onto a convex domain. To show that the estimate obtained in the theorem is sharp, we consider the function $f(z)$ such that $f'(z) = \frac{(1 + z)^2}{(1 - z)^3}$ with $g(z) = \frac{z}{(1 - z)^2}$.

This function $f(z)$ satisfies the hypotheses of the theorem and a short computation shows that $\frac{zf''(z)}{f'(z)} + 1 = \frac{1 + 5z}{1 - z^2}$. This expression vanishes at $z = -1/5$.

THEOREM 2. Let $f(z) = z + a_2z^2 + \ldots$ be analytic for $|z| < 1$ and

$g(z) = z + b_2z^2 + \dots$ be analytic and starlike for $|z| < 1$. If $\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$ for $|z| < 1$, then $f(z)$ maps $|z| < 1/5$ onto a convex domain. The result is sharp.

PROOF: Since $g(z)$ is starlike for $|z| < 1$ implies $g(z)$ is univalent there, the proof of this theorem follows from that of theorem 1.

THEOREM 3. Suppose $f(z) = z + a_2z^2 + \dots$ is analytic for $|z| < 1$ and $g(z) = z + b_2z^2 + \dots$ is analytic and convex for $|z| < 1$. If $\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$ for $|z| < 1$, then $f(z)$ maps $|z| < 1/3$ onto a convex domain. The result is sharp.

PROOF. Since $g(z)$ is convex for $|z| < 1$ it is univalent there. Therefore $g'(z) \neq 0$ for $|z| < 1$ and $\mathrm{Re}\{\frac{zg''(z)}{g'(z)} + 1\} > 0$ for $|z| < 1$.
The function $\frac{zg''(z)}{g'(z)} + 1 = 1 + c_1z + \dots$ is regular for $|z| < 1$ and has positive real part, therefore by lemma 3,

$$\mathrm{Re}\ \{\frac{zg''(z)}{g'(z)} + 1\} \geq \frac{1 - |z|}{1 + |z|} .$$

Using this estimate in (3.1) we get

$$\mathrm{Re}\{\frac{zf''(z)}{f'(z)} + 1\} \geq \frac{1 - |z|}{1 + |z|} - \frac{|z|}{1 - |z|} = \frac{1 - 3|z|}{1 - |z|^2} .$$

This last expression is positive for $|z| < 1/3$. Thus $f(z)$ maps $|z| < 1/3$ onto a convex domain. To see that the estimate obtained is sharp, we consider $f(z)$ such that $f'(z) = \frac{1 + z}{(1 - z)^2}$, with $g(z) = \frac{z}{1 - z}$. Thus $f(z)$ satisfies the hypotheses of the theorem. However $\frac{zf''(z)}{f'(z)} + 1 = \frac{1 + 3z}{1 - z^2}$, which vanishes at $z = -1/3$.

THEOREM 4. Suppose $f(z) = z + a_2z^2 + \dots$ is analytic for $|z| < 1$ and $g(z) = z + b_2z^2 + \dots$ is analytic and $\mathrm{Re}\ g'(z) > 0$ for $|z| < 1$. If $\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$ for $|z| < 1$, then $f(z)$ maps $|z| < (\sqrt{17} - 3)/4$ onto a convex domain. The result is sharp.

PROOF. Since $\mathrm{Re}\ g'(z) > 0$ for $|z| < 1$, it follows from lemma 1, with $\alpha = 0$

that

$$g'(z) = \frac{1 - z\phi(z)}{1 + z\phi(z)} \quad \text{where} \quad |\phi(z)| \leq 1.$$

Taking the logarithmic derivative of this expression we get

$$\frac{g''(z)}{g'(z)} = \frac{-2(z\phi'(z) + \phi(z))}{1 - z^2\phi^2(z)}$$

Using lemma 2 (ii) and simplifying we get

$$\left|\frac{g''(z)}{g'(z)}\right| \leq \frac{2[|z| + |\phi(z)|]}{(1 - |z|^2)(1 + |z|\,|\phi(z)|)} \leq \frac{2}{1 - |z|^2}.$$

Thus

$$\mathrm{Re}\left\{\frac{zg''(z)}{g'(z)} + 1\right\} \geq 1 - \frac{2|z|}{1 - |z|^2}$$

Using this estimate in (3.1) we get

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} \geq 1 - \frac{2|z|}{1 - |z|^2} - \frac{|z|}{1 - |z|} = \frac{1 - 3|z| - 2|z|^2}{1 - |z|^2}$$

This last expression is positive for $|z| < (\sqrt{17} - 3)/4$. To show that the estimate obtained is sharp we consider $f(z)$ such that $f'(z) = \frac{(1 + z)^2}{1 - z}$ with $g(z) = - z - 2\log(1 - z)$. This $f(z)$ satisfies the hypotheses of the theorem. However

$$\frac{zf''(z)}{f'(z)} + 1 = \frac{1 + 3z - 2z^2}{1 - z^2}$$

This last expression vanishes at $z = (3 - \sqrt{17})/4$.

THEOREM 5. Suppose $f(z) = z + a_2z^2 + \ldots$ is analytic for $|z| < 1$ and $g(z) = z + b_2z^2 + \ldots$ is analytic for $|z| < 1$ and $\mathrm{Re}\{g'(z)\} > 1/2$ for $|z| < 1$. If $\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$ for $|z| < 1$, then $f(z)$ maps $|z| < r_0$ onto a convex domain, where r_0 is the smallest positive root of $4 - 4r - 13r^2 - 2r^3 - r^4 = 0$. The result is sharp.

PROOF. Since $\mathrm{Re}\{g'(z)\} > 1/2$ for $|z| < 1$, we have by lemma 1 with $\alpha = 1/2$,

$g'(z) = \frac{1}{1 + z\phi(z)}$. Thus
$$\frac{g''(z)}{g'(z)} = \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)}$$

From (3.1) we get

$$\text{Re}\{\frac{zf''(z)}{f'(z)} + 1\} \geq \text{Re}\{\frac{zg''(z)}{g'(z)} + 1 - \frac{|z|}{1 - |z|}\}$$
$$= \text{Re}\{\frac{zg''(z)}{g'(z)} + \frac{1 - 2|z|}{1 - |z|}\}$$
$$= \text{Re}\{\frac{-z^2\phi'(z) - z\phi(z)}{1 + z\phi(z)} + \frac{1 - 2|z|}{1 - |z|}\} .$$

Therefore, $\text{Re}\{\frac{zf''(z)}{f'(z)} + 1\}$ is positive if

$$\text{Re}\{\frac{1 - 3|z| + |z|(1 - z\phi(z)) - (1 - |z|)z^2\phi'(z)}{(1 - |z|)(1 + z\phi(z))}\} > 0 .$$

This will be true if

$$\text{Re}\{[|z|(1 - z\phi(z)) - \{(3|z| - 1) + (1 - |z|)z^2\phi'(z)\}] [1+z\phi(z)]^*\} > 0,$$

(asterisks denote the conjugate of a complex number)

$$\text{Re}\{|z|(1 - |z|^2|\phi(z)|^2) - [(3|z| - 1) + (1 - |z|)z^2\phi'(z)] [1 + z\phi(z)]^*\} > 0,$$

$$\text{Re}\{[(3|z| - 1) + (1 - |z|)z^2\phi'(z)] [1 + z\phi(z)]^*\} < |z|(1 - |z|^2|\phi(z)|^2).$$

By lemma 2, it is easily seen that this last inequality will be true if

$$3|z| - 1 + (1 - |z|)|z|^2(\frac{1 - |\phi(z)|^2}{1 - |z|^2}) < |z|(1 - |z||\phi(z)|).$$

This inequality is equivalent to showing

$$r + 3r^2 + r^2(1 + r)x - r^2x^2 < 1,$$

where $|z| = r$ $(0 < r < 1)$ and $|\phi(z)| = x$ $(0 \leq x \leq 1)$.

Let $p(x) = r + 3r^2 + r^2(1 + r)x - r^2x^2$.

We see that $p(x)$ attains its maximum value $q(r)$ at $x = \frac{1 + r}{2}$, consequently

$$q(r) = r + 3r^2 + \frac{r^2}{4}(1 + r)^2.$$

Since $r + 3r^2 + \frac{r^2}{4}(1 + r)^2 < 1$ holds for all $r < r_0$, where r_0 is the smallest positive root of the equation $r + 3r^2 + \frac{r^2}{4}(1 + r)^2 = 1$.

This simplifies to

$$4 - 4r - 13r^2 - 2r^3 - r^4 = 0 \tag{3.2}$$

To show that the estimate obtained above is sharp, we let

$g'(z) = \dfrac{1}{1 + z\phi(z)}$, where $\phi(z) = \dfrac{z + b}{1 + bz}$, $b = \dfrac{1}{2 + r_0}$ where r_0 is the smallest positive root of (3.2); and we select f(z) so that $f'(z) = (1 - z)g'(z)$. Since $|\phi(z)| < 1$ for $|z| < 1$, we have Re $g'(z) > 1/2$ for $|z| < 1$. Thus f(z) satisfies the hypotheses of the theorem, and

$$\frac{zf''(z)}{f'(z)} + 1 = \frac{1 + (2b - 2)z + (2b^2 - 5b - 1)z^2 - 4b^2z^3 - bz^4}{(1 - z)\ (1 + bx)(1 + 2bz + z^2)} .$$

Setting $z = r_0$ and $b = \dfrac{1}{2 + r_0}$, we see that the numerator of the above expression is

$$(1 + r_0)\ (4 - 4r_0 - 13r_0^2 - 2r_0^3 - r_0^4)$$

which vanishes.

Theorems 4 and 5 give the radius of convexity for the class of functions f(z) associated with g(z) such that Re $g'(z) > \alpha$ when $\alpha = 0$ and 1/2. For $\alpha \neq 0, 1/2$ our method seems to give only estimates for r_c the radius of convexity and determination of r_c is still open.

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