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RANDOM FIXED POINT THEOREMS FOR ASYMPTOTIC 1-SET CONTRACTION OPERATORS

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Random fixed point theorems for condensing 1-set contraction selfmaps are known. But no random fixed point theorem for more general asymptotic 1-set contraction selfmaps is yet available. The purpose of this paper is to prove random fixed point theorems for such maps.

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1. Introduction. Random fixed point theorems are stochastic versions of (classical or deterministic) fixed point theorems and are required for the theory of random equations. The study of random fixed point theorems for contraction maps in separable complete metric spaces was initiated by Spacek [8] and Hans [3]. In separable Banach spaces, Itoh [4] proved a random fixed point theorem for condensing selfmaps. This Itoh's result was extended by Xu [11] to condensing non-selfmap T, by assuming an additional condition that T satisfies either weakly inward (see [11]) or the Leray-Schauder condition (see [11]). Lin [5] proved a random fixed point theorem for 1-set contraction selfmap T, by assuming that $I-T$ is demiclosed at zero, where I denotes the identity map.

Rao [6] obtained a probabilitic version of Krasnoselskii's theorem which states that a sum $T+S$ of a contraction random operator $T:\Omega\times K\to X$, and a compact random operator $S:\Omega\times K\to X$ with $T(\omega,x)+S(\omega,y)\in K$ for all $x,y\in K$, $\omega\in\Omega$, has a random fixed point, where K is a closed convex subset of a separable Banach space X and (Ω,Σ) is a measurable space. Some results on random fixed points of Krasnoselskii type were obtained by Bharucha-Reid [1, 2].

Itoh [4] extended Rao's result to a sum $T+S$ of a nonexpansive random operator $T:\Omega\times K\to X$ and a completely continuous random operator $S:\Omega\times K\to X$ with $T(\omega,x)+S(\omega,x)\in K$ for each $x\in K$, $\omega\in\Omega$, where K is a weakly compact convex subset of a separable uniformly convex Banach space X. This Itoh's result was extended by Lin [5] to a sum $T+S$ of a locally almost nonexpansive (LANE) (cf. [5]) random operator $T:\Omega\times K\to K$ and a completely continuous random operator $S:\Omega\times K\to K$, where K is a nonempty closed convex bounded subset of a separable uniformly convex Banach space X.

Shahzad [7] extended Itoh's and Lin's results to a sum of two random nonself operators, by assuming an additional condition that the sum of these operators satisfies either weakly inward or the Leray-Schauder condition. The class of 1-set contraction random operators includes the classes of condensing, nonexpansive, and LANE random operators. In this paper, we prove random fixed point theorems of Krasnoselskii type for sum of an asymptotic 1-set contraction and a compact (completely continuous) map from which we will be able to deduce random fixed point theorems for asymptotic 1-set contraction in the setting of separable Banach spaces, and we also prove random fixed point theorems of Krasnoselskii type for sum of a 1-set contraction and a compact (completely continuous) map.

2. Preliminaries. Throughout this paper, (Ω,Σ) denotes a measurable space with Σ, a sigma algebra of subsets of Ω. Let 2^X be the family of all subsets of a Banach space X. A mapping $F:\Omega\to 2^X$ is called measurable if for each open subset A of X, $F^{-1}(A)\in\Sigma$. A mapping $\phi:\Omega\to X$ is called a measurable selector of a measurable mapping $F:\Omega\to 2^X$, if ϕ is measurable and for any $\omega\in\Omega$, $\phi(\omega)\in F(\omega)$. Let K be a nonempty subset of X. A map $T:\Omega\times K\to X$ is called a random operator if for each fixed $x\in K$, the map $T(\cdot,x):\Omega\to X$ is measurable. A measurable map $\phi:\Omega\to K$ is a random fixed point of a random operator T if $T(\omega,\phi(\omega))=\phi(\omega)$ for each $\omega\in\Omega$. A random operator $T:\Omega\times K\to K$ is called continuous (k-set contraction, condensing, 1-set contraction, asymptotic 1-set contraction, asymptotically regular, uniformly asymptotically regular (UAR), nonexpansive, locally almost nonexpansive, asymptotically nonexpansive, compact, completely continuous, weakly inward, demiclosed, etc.) if the map $T_\omega:K\to K$, defined by $T_\omega(x)=T(\omega,x)$ for all $x\in K$, is so, for each $\omega\in\Omega$.

Let A be a nonempty bounded subset of X. The Kuratowski's measure of noncompactness of A is defined as the number $\alpha(A)=\inf\{\varepsilon>0: A$ can be covered by a finite number of sets of diameter $\le\varepsilon\}$. Let K be a nonempty subset of X and let T be a continuous mapping of K into K. If there exists k, $0\le k<1$, such that for each nonempty bounded subset A of K, we have $\alpha(T(A))\le k\alpha(A)$, then T is called a k-set contraction. If $k=1$, then T is called a 1-set contraction. If for every nonempty bounded subset A of K with $\alpha(A)>0$, we have $\alpha(T(A))<\alpha(A)$, then T is called a condensing map. It is clear that a k-set contraction map is a condensing map and a condensing map is a 1-set contraction map. If there is a sequence $\{k_n\}$ of real numbers with $k_n\ge 1$, $k_n\ge k_{n+1}$, and $k_n\to 1$ as $n\to\infty$ such that $\alpha(T^n(A))\le k_n\alpha(A)$ for $n\ge 1$ and for every nonempty bounded subset A of K, then T is called an asymptotic 1-set contraction (see [9]). As shown in [9, Theorem 2.2 and Remark 2.1] the sum of an asymptotic 1-set contraction and a compact map is an asymptotic 1-set contraction.

If T maps weakly convergent sequences into strongly convergent sequences, then T is called a completely continuous or strongly continuous map. If T is continuous and $T(K)$ is precompact, then T is called a compact map. If for any sequence $\{x_n\}$ in K, the conditions $\{x_n\}$ converge weakly to x in K and $T(x_n)$ converges to y in X, which imply that $T(x)=y$, then T is said to be demiclosed at y.

A map $T:K\to X$ is said to be nonexpansive if $\|T(x)-T(y)\|\le\|x-y\|$ for all $x,y\in K$. A selfmap T of K is said to be asymptotically nonexpansive if there is a sequence $\{k_n\}$ of real numbers with $k_n\ge 1$, $k_n\ge k_{n+1}$, and $k_n\to 1$ as $n\to\infty$ such that $\|T^n(x)-T^n(y)\|\le k_n\|x-y\|$ for all $x,y\in K$ and for each integer $n\ge 1$. The class of asymptotically nonexpansive maps includes properly the class of nonexpansive maps. A selfmap T of K is called an asymptotically regular map if for any $x \in K$, $\|T^n(x) - T^{n+1}(x)\| \to 0$ as $n \to \infty$. If T and S map K into X, then T is called a uniformly asymptotically regular (UAR) map with respect to S [10], if for each $\varepsilon > 0$ there exists an integer N such that $\|T^n(x) - T^{n-1}(x) + S(x)\| < \varepsilon$ for all $n \geq N$ and for all $x \in K$. If T and S are random operators of $\Omega \times K$ into X, then we say that T is UAR with respect to S if T_ω is UAR with respect to S_ω for each $\omega \in \Omega$ in the above classical definition. The map T is UAR if T is UAR with respect to the zero operator. It is clear that uniform asymptotic regularity is stronger than asymptotic regularity.

3. Main results. We will need the following theorem to prove random fixed point theorems of Krasnoselskii type.

Theorem 3.1 (see [5]). *Let K be a nonempty weakly compact convex subset of a separable Banach space X. If $T : \Omega \times K \to K$ is a continuous 1-set contraction random operator such that $I - T_\omega$ is demiclosed for each $\omega \in \Omega$, then T has a random fixed point.*

Theorem 3.2. *Let K be a nonempty weakly compact convex subset of a separable Banach space X, $T : \Omega \times K \to K$ an asymptotic 1-set contraction random operator. Let $S : \Omega \times K \to K$ be a compact random operator. Assume further that T is UAR with respect to S such that $I - (T + S)$ is demiclosed at zero, $T_\omega^n(K) + S_\omega(K) \subseteq K$ for $n \geq 1$ and for each $\omega \in \Omega$. Then $T + S$ has a random fixed point.*

Proof. Let v be a fixed element of K. For each n, define a map $T_n : \Omega \times K \to K$ by

$$T_n(\omega, x) = a_n v + (1 - a_n)\left[T_\omega^n(x) + S_\omega(x)\right] \quad \forall x \in K,\ \omega \in \Omega, \tag{3.1}$$

where $\{a_n\}$ is a sequence of real numbers in $(0,1)$ with $1 - 1/k_n \leq a_n \leq 1 - 1/k_{n-1}$ and $\{k_n\}$ is as in the definition of asymptotic 1-set contraction.

Let M be a bounded subset of K. Then

$$\begin{aligned} \alpha(T_n(\omega, M)) &\leq a_n \alpha(\{v\}) + (1 - a_n)\left[\alpha(T_\omega^n(M) + S_\omega(M))\right] \\ &\leq (1 - a_n) k_n \alpha(M) \quad \text{since } T \text{ is asymptotic 1-set contraction, } S \text{ is compact} \\ &\leq \alpha(M). \end{aligned} \tag{3.2}$$

Therefore each T_n is a 1-set contraction random operator. By Theorem 3.1, T_n has a random fixed point ϕ_n, that is, there is a measurable map $\phi_n : \Omega \to K$ such that $T_n(\omega, \phi_n(\omega)) = \phi_n(\omega)$ for each $\omega \in \Omega$.

Let $\mathfrak{K}$ be the family of all nonempty weakly compact subsets of K.

Let $\mathrm{w}-\mathrm{cl}(A)$ denote the weak closure of A. For each n, define a map $F_n : \Omega \to \mathfrak{K}$ by $F_n(\omega) = w - \mathrm{cl}\{\phi_i(\omega) : i \geq n\}$.

Define a map $F : \Omega \to \mathfrak{K}$ by $F(\omega) = \bigcap_{n=1}^{\infty} F_n(\omega)$.

Since K is weakly compact and X is separable, it follows that the weak topology on K is a metric topology. Since K and $F_n(\omega)$ are weakly compact, F is measurable with respect to the weak topology on K. Then, as in Itoh [4, Proof of Theorem 2.5], there is a ω-measurable selector ϕ of F.

For each $x^* \in X^*$, dual of X, $x^*(\phi(\cdot))$ is measurable as a numerically-valued function in Ω. Since X is separable, ϕ is measurable [1, pages 14–16].

Now we prove that ϕ is a random fixed point of $T+S$.

Fixing any $\omega \in \Omega$, since $\phi(\omega) \in F(\omega)$, there is a subsequence $\{\phi_m(x)\}$ of $\{\phi_n(x)\}$ that converges weakly to $\phi(\omega)$.

Since $\phi_m(\omega) = T_m(\omega, \phi_m(\omega)) = a_m v + (1-a_m)[T_\omega^m(\phi_m(\omega)) + S_\omega(\phi_m(\omega))]$, it follows that

$$\begin{aligned} &\phi_m(\omega) - T_\omega^m(\phi_m(\omega)) - S_\omega(\phi_m(\omega)) \\ &\qquad = a_m[v - T_\omega^m(\phi_m(\omega)) - S_\omega(\phi_m(\omega))] \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \end{aligned} \tag{3.3}$$

Since T_ω is UAR with respect to S_ω for each $\omega \in \Omega$,

$$T_\omega^m(\phi_m(\omega)) - T_\omega^{m-1}(\phi_m(\omega)) + S_\omega(\phi_m(\omega)) \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{3.4}$$

Therefore $\phi_m(\omega) - T_\omega^{m-1}(\phi_m(\omega)) \to 0$ as $m \to \infty$.

Now,

$$\begin{aligned} &||\phi_m(\omega) - T_\omega(\phi_m(\omega)) - S_\omega(\phi_m(\omega))|| \\ &\quad = ||\phi_m(\omega) - T_\omega^m(\phi_m(\omega)) - S_\omega(\phi_m(\omega)) + T_\omega^m(\phi_m(\omega)) \\ &\qquad + S_\omega(\phi_m(\omega)) - T_\omega(\phi_m(\omega)) - S_\omega(\phi_m(\omega))|| \\ &\quad \le ||\phi_m(\omega) - T_\omega^m(\phi_m(\omega)) - S_\omega(\phi_m(\omega))|| + ||T_\omega^m(\phi_m(\omega)) - T_\omega(\phi_m(\omega))|| \\ &\quad \le ||\phi_m(\omega) - T_\omega^m(\phi_m(w)) - S_\omega(\phi_m(\omega))|| + k_1||T_\omega^{m-1}(\phi_m(\omega)) - \phi_m(w)|| \\ &\quad \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \end{aligned} \tag{3.5}$$

Since $I - (T_\omega + S_\omega)$ is demiclosed at zero for each $\omega \in \Omega$, $[I - (T_\omega + S_\omega)](\phi(\omega)) = 0$ and therefore $\phi(\omega) = T(\omega, \phi(\omega)) + S(\omega, \phi(\omega))$. □

Corollary 3.3. *Let K be a nonempty closed convex bounded subset of a separable reflexive Banach space X, $T : \Omega \times K \to K$ an asymptotic 1-set contraction random operator. Let $S : \Omega \times K \to K$ be a completely continuous random operator. Assume further that T is UAR with respect to S such that $I - (T+S)$ is demiclosed at zero and that $T_\omega^n(K) + S_\omega(K) \subseteq K$ for $n \ge 1$ and for each $\omega \in \Omega$. Then $T+S$ has a random fixed point.*

Proof. Since K is a closed convex bounded subset of a reflexive Banach space X, K is weakly compact and hence every sequence $\{x_n\}$ in K has a weakly convergent subsequence $\{x_m\}$. Since S is completely continuous, $S(\omega, x_m) \to S(\omega, x)$ as $m \to \infty$. Hence S is compact. All the conditions of Theorem 3.2 are satisfied. The result follows from Theorem 3.2. □

Remark 3.4. If $S = 0$ in Theorem 3.2 and Corollary 3.3, then random fixed point theorems for asymptotic 1-set contraction are obtained. It is known in [9] that every asymptotically nonexpansive selfmap of a nonempty subset K of a Banach space X is an asymptotic 1-set contraction on K. Therefore, we obtain the following result.

Theorem 3.5. *Let K and X be as in Corollary 3.3, $T:\Omega\times K\to K$ an asymptotically nonexpansive, UAR random operator such that $I-T$ is demiclosed at zero. Then T has a random fixed point.*

Corollary 3.6. *Let K be a nonempty closed convex bounded subset of a separable uniformly convex Banach space X, $T:\Omega\times K\to K$ an asymptotically nonexpansive, UAR random operator. Then T has a random fixed point.*

Proof. Since X is uniformly convex and $T_\omega : K\to K$ is asymptotically nonexpansive, it follows as in Xu [12] that $I-T_\omega$ is demiclosed at zero for each $\omega\in\Omega$. The remaining part of proof follows from Theorem 3.5. □

In Theorem 3.2 and Corollary 3.3 if T is a 1-set contraction random operator, then uniform asymptotic regularity of T with respect to S is not required.

Theorem 3.7. *Let K be a weakly compact convex subset of a separable Banach space X. Let $T:\Omega\times K\to X$ be a 1-set contraction random operator and $S:\Omega\times K\to X$ a compact random operator. Suppose that for any $\omega\in\Omega$, $T_\omega(x)+S_\omega(y)\in K$ for all $x,y\in K$ and $I-(T+S)$ is demiclosed at zero. Then $T+S$ has a random fixed point.*

Proof. Motivated by Itoh [4, Theorem 3.5], choose an element v in K and a sequence $\{a_n\}$ of real numbers such that $0<a_n<1$ and $a_n\to 0$ as $n\to\infty$. For each n, define a map $T_n:\Omega\times K\to K$ by

$$T_n(\omega,x)=a_nv+(1-a_n)\big[T_\omega(x)+S_\omega(x)\big]\quad \forall x\in K,\ \omega\in\Omega, \tag{3.6}$$

then T_n is a $(1-a_n)$-set contraction random operator. The remaining part of proof of this theorem is almost identical to [4, Proof of Theorem 3.5]. □

Corollary 3.8. *Let K be a nonempty closed convex bounded subset of a separable uniformly convex Banach space X. Let T be as in Theorem 3.7 such that $I-T$ is demiclosed at zero. Let $S:\Omega\times K\to X$ be a completely continuous random operator. Suppose that for any $\omega\in\Omega, T_\omega(x)+S_\omega(y)\in K$ for all $x,y\in K$. Then $T+S$ has a random fixed point.*

The proof of this corollary follows from the proof of Theorem 3.7.

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