

## HARDY-LITTLEWOOD TYPE INEQUALITIES FOR LAGUERRE SERIES

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Let  $\{c_j\}$  be a null sequence of bounded variation. We give appropriate smoothness and growth conditions on  $\{c_j\}$  to obtain the pointwise convergence as well as  $L^r$ -convergence of Laguerre series  $\sum c_j \mathcal{L}_j^a$ . Then, we prove a Hardy-Littlewood type inequality  $\int_0^\infty |f(t)|^r dt \leq C \sum_{j=0}^\infty |c_j|^r \bar{j}^{1-r/2}$  for certain  $r \leq 1$ , where  $f$  is the limit function of  $\sum c_j \mathcal{L}_j^a$ . Moreover, we show that if  $f(x) \sim \sum c_j \mathcal{L}_j^a$  is in  $L^r$ ,  $r \geq 1$ , we have the converse Hardy-Littlewood type inequality  $\sum_{j=0}^\infty |c_j|^r \bar{j}^\beta \leq C \int_0^\infty |f(t)|^r dt$  for  $r \geq 1$  and  $\beta < -r/2$ .

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**1. Introduction.** Given complex numbers  $\{c_j\}_{j \in \mathbb{Z}}$  satisfying  $\sum |c_j|^r (|j| + 1)^{r-2} < \infty$  for some  $r \geq 2$ , Hardy and Littlewood [4] (see also [14, Theorem 3.19, page 109]) proved in 1926 that  $c_j$ 's are the Fourier coefficients of an  $f$  in  $L^r$ , and

$$\int_0^{2\pi} |f(t)|^r dt \leq A_r \sum_{j=-\infty}^\infty |c_j|^r (|j| + 1)^{r-2}. \quad (1.1)$$

Also, they proved that if  $f(x) \sim \sum c_j e^{ijx}$  is in  $L^r$ ,  $1 < r \leq 2$ , then

$$\sum_{j=-\infty}^\infty |c_j|^r (|j| + 1)^{r-2} \leq A_r \int_0^{2\pi} |f(t)|^r dt. \quad (1.2)$$

Later on, Paley [10] (see also [14, Theorem 5.1, page 121]) extended Hardy and Littlewood's results to general systems of orthonormal and uniformly bounded functions over an interval. In this paper, we concentrate on the Laguerre system, and prove the similar inequalities

$$\begin{aligned} \int_0^\infty |f(t)|^r dt &\leq C \sum_{j=0}^\infty |c_j|^r \bar{j}^{1-r/2} \quad \text{for certain } r \leq 1, \\ \sum_{j=0}^\infty |c_j|^r \bar{j}^\beta &\leq C \int_0^\infty |f(t)|^r dt \quad \text{for } r \geq 1, \beta < -\frac{r}{2}, \end{aligned} \quad (1.3)$$

where  $\bar{\xi}$  means  $\max\{\xi, 1\}$ .

For  $a > -1$ , the Laguerre polynomials of type  $a$  are defined by the formula

$$L_n^a(t) = \frac{1}{n!} t^{-a} e^t \frac{d^n}{dt^n} (t^{n+a} e^{-t}), \quad n = 0, 1, 2, \dots \quad (1.4)$$

Each  $L_n^a$  is a polynomial of degree  $n$ , whose explicit expression is

$$L_n^a(t) = \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n+a}{n-k} t^k. \tag{1.5}$$

The Laguerre polynomials form a complete orthogonal system in  $L^2(\mathbb{R}^+, t^a e^{-t} dt)$ . Hence, if we define  $\mathcal{L}_n^a(t)$  by

$$\mathcal{L}_n^a(t) = \sqrt{\frac{n!}{\Gamma(n+a+1)}} e^{-t/2} t^{a/2} L_n^a(t), \tag{1.6}$$

then they form an orthonormal basis in  $L^2(\mathbb{R}^+, dt)$  with the inner product  $\langle f, g \rangle = \int_0^\infty f(t)g(t)dt$ .

A number of authors have studied the problems of pointwise convergence and mean convergence for different types of Laguerre series. Of particular interest are the results of Askey and Wainger [1], Chen and Lin [2], Długosz [3], Muckenhoupt [7, 8, 9], and Stempak [11, 12, 13]. However, all of them started at a given function  $f$  to get the Laguerre coefficients  $\{c_j\}$ , and proved the pointwise convergence or mean convergence of the series  $\sum c_j \mathcal{L}_j^a$ . In this paper, we start from  $\{c_j\}$  satisfying

$$\sum_{j=0}^\infty |\Delta^p c_j| \bar{j}^{p/2-1/4} < \infty, \tag{1.7}$$

$$|c_j| \bar{j}^{p/2-1/4} (\overline{\log j})^{1+\epsilon} = O(1) \quad \text{as } j \rightarrow \infty, \tag{1.8}$$

for some  $p \in \mathbb{N}$  and  $\epsilon > 0$ , and prove the pointwise convergence of Laguerre series  $\sum c_j \mathcal{L}_j^a$ . Here,  $\Delta^p c_j$  denotes the finite-order difference

$$\Delta^0 c_j = c_j, \quad \Delta^p c_j = \Delta^{p-1} c_j - \Delta^{p-1} c_{j+1} \quad \text{for } p \in \mathbb{N}. \tag{1.9}$$

Then, we strengthen the assumptions on  $\{c_j\}$  such that the Laguerre series  $\sum c_j \mathcal{L}_j^a$  converges not only pointwise but also in  $L^r$ -metric. In addition, we obtain the Hardy-Littlewood type inequalities.

**THEOREM 1.1.** *Let  $a \geq 0$ . Assume that  $\{c_j : j \geq 0\}$  satisfies*

$$\sum_{j=0}^\infty |\Delta^p c_j|^r \bar{j}^{1-r/2} < \infty, \tag{1.10}$$

$$|c_j| \bar{j}^{2/r-3/2+\epsilon} = O(1) \quad \text{as } j \rightarrow \infty, \tag{1.11}$$

for some  $p \in \mathbb{N}$ ,  $\epsilon > 0$ , and  $r \leq \min\{1, 4/(1+2p)\}$ . Then, the Laguerre series  $\sum c_j \mathcal{L}_j^a(t)$  converges to  $f \in L^r(\mathbb{R}^+)$  pointwise and in  $L^r$ -metric, where

$$f(t) = e^{-t/2} t^{a/2} \sum_{j=0}^\infty (\Delta^p b_j) L_j^{a+p}(t) \tag{1.12}$$

and  $b_j = c_j \sqrt{j!/\Gamma(j+a+1)}$ .

**COROLLARY 1.2.** *Under the same assumptions of Theorem 1.1, there is a constant  $C$  independent of  $f$  such that*

$$\int_0^\infty |f(t)|^r dt \leq C \sum_{j=0}^\infty |c_j|^r \bar{j}^{1-r/2}. \tag{1.13}$$

We also prove the above converse inequality in the following theorem.

**THEOREM 1.3.** *Let  $a \geq 0$ . If  $f \in L^r(\mathbb{R}^+)$ ,  $r \geq 1$ , then there is a constant  $C$  independent of  $f$ , such that*

$$\sum_{j=0}^\infty |c_j|^r \bar{j}^\beta \leq C \int_0^\infty |f(t)|^r dt \quad \forall \beta < -\frac{r}{2}, \tag{1.14}$$

where  $c_j \equiv \int_0^\infty f(t) \mathcal{L}_j^a(t) dt$ .

**REMARK 1.4.** For  $1 \leq r < 4/3$ , we can find a  $\beta < -r/2$  such that  $\beta > r - 2$ . Thus, Theorem 1.3 improves Paley’s result for Laguerre system  $\{\mathcal{L}_j^a\}$ . Moreover, Kanjin [5] showed that, for  $f(t) \sim \sum_{j=0}^\infty c_j \mathcal{L}_j^a(t)$  in  $H^1(\mathbb{R}^+)$ ,  $\sum_{j=0}^\infty c_j (j+1)^{-1} \leq C \|f\|_{H^1(\mathbb{R}^+)}$ , which is the special case of Theorem 1.3 for  $r = 1$  and  $\beta = -1$ .

In the next section, we first give some estimates of Laguerre functions and talk about the pointwise convergence and  $L^r$ -convergence of Laguerre series. Then we prove Corollary 1.2 and Theorem 1.3 in Section 3. Finally, we mention that  $C$ , possibly with subscripts, denotes a constant which may stand for a different number from one appearance to another.

**2. Pointwise convergence and mean convergence.** It is known that the Laguerre functions satisfy the estimates

$$|\mathcal{L}_j^a(t)| \leq \begin{cases} Ct^{a/2} \nu^{a/2}, & \text{if } 0 \leq t \leq \frac{1}{\nu}; \\ Ct^{-1/4} \nu^{-1/4}, & \text{if } \frac{1}{\nu} < t \leq \frac{\nu}{2}; \\ C\nu^{-1/4} (\nu^{1/3} + |t - \nu|)^{-1/4}, & \text{if } \frac{\nu}{2} < t \leq \frac{3\nu}{2}; \\ Ce^{-\nu t}, & \text{if } \frac{3\nu}{2} < t < \infty, \end{cases} \tag{2.1}$$

where  $\nu = 4j + 2a + 2$ , and both  $C$  and  $\gamma$  are positive constants independent of  $j$  and  $t$  (cf. [1, 9]). Hence, by a straightforward calculation, we have

$$|L_j^\alpha(t)| \leq Ce^{t/2} t^{-\alpha/2-1/4} \bar{j}^{\alpha/2-1/4} (1+t)^{1/6} \tag{2.2}$$

for all  $j \geq 0$ , all  $t \geq 0$ , and  $\alpha = a, a + 1, \dots, a + p$ . Also,

$$\int_0^\infty |t^{-p/2} \mathcal{L}_j^{a+p}(t)|^r dt \leq C \bar{j}^{1-r/2-pr/2} \tag{2.3}$$

for  $r/2 + pr \leq 2$ ,  $r \neq 4$ ,  $a > -2/r$ , and all  $j \geq 0$ . In particular,  $\|\mathcal{L}_j^a\|_{L^r(\mathbb{R}^+)}^r \leq C_j \bar{j}^{1-r/2}$  for  $0 < r < 4$ ,  $a > -2/r$ , and all  $j \geq 0$ . As to  $r \geq 4$ , it follows from [6, Lemma 1] that, for  $a \geq 0$ ,

$$\|\mathcal{L}_j^a\|_{L^r(\mathbb{R}^+)} = \begin{cases} O(j^{1/r-1/2}), & \text{for } 1 \leq r < 4; \\ O(j^{1/r-1/2}(\log j)^{1/r}), & \text{for } r = 4; \\ O(j^{-1/r}), & \text{for } r > 4. \end{cases} \tag{2.4}$$

Let  $s_n(t)$  denote the partial sums of Laguerre series defined by

$$s_n(t) = \sum_{j=0}^n c_j \mathcal{L}_j^a(t). \tag{2.5}$$

Set  $b_j = c_j \sqrt{j!/\Gamma(j+a+1)}$ . Then

$$s_n(t) = e^{-t/2} t^{a/2} \sum_{j=0}^n b_j L_j^a(t). \tag{2.6}$$

For  $t > 0$  and  $n \in \mathbb{N}$ , the well-known equation

$$L_j^{a+1}(t) - L_{j+1}^{a+1}(t) = -L_{j+1}^a(t) \tag{2.7}$$

and the summation by parts yield

$$s_n(t) = e^{-t/2} t^{a/2} \left( \sum_{j=0}^n (\Delta b_j) L_j^{a+1}(t) + b_{n+1} L_n^{a+1}(t) \right). \tag{2.8}$$

Repeating the same process, we get

$$\begin{aligned} s_n(t) &= e^{-t/2} t^{a/2} \sum_{j=0}^n (\Delta^p b_j) L_j^{a+p}(t) + e^{-t/2} t^{a/2} \sum_{j=0}^{p-1} (\Delta^j b_{n+1}) L_n^{a+j+1}(t) \\ &\equiv I_1(t) + I_2(t). \end{aligned} \tag{2.9}$$

Using the inequality  $1 - \sqrt{1-y} \leq y$  for  $y \in [0, 1]$ , we have

$$\left| \Delta^k \left( \sqrt{\frac{j!}{\Gamma(j+a+1)}} \right) \right| \leq C_p \sqrt{\frac{j!}{\Gamma(j+a+1)}} \frac{a}{j+a+1} \leq a C_p \bar{j}^{-a/2-1} \quad \text{for } 1 \leq k \leq p, \tag{2.10}$$

which with Leibniz's rule implies

$$\begin{aligned} |\Delta^p b_j| &= \left| \Delta^p \left( c_j \sqrt{\frac{j!}{\Gamma(j+a+1)}} \right) \right| \\ &\leq \sqrt{\frac{(j+p)!}{\Gamma(j+p+a+1)}} |\Delta^p c_j| + C_{ap} \bar{j}^{-a/2-1} \sum_{i=0}^{p-1} \binom{p}{i} |\Delta^i c_j| \\ &\leq C_{ap} \left\{ \bar{j}^{-a/2} |\Delta^p c_j| + \bar{j}^{-a/2-1} \left( \max_{j \leq k \leq j+p-1} |c_k| \right) \right\}. \end{aligned} \tag{2.11}$$

Thus,

$$|I_1(t)| \leq C_{ap} e^{-t/2} t^{a/2} \sum_{j=0}^n \left\{ \bar{j}^{-a/2} |\Delta^p c_j| + \bar{j}^{-a/2-1} \left( \max_{j \leq k \leq j+p-1} |c_k| \right) \right\} |L_j^{a+p}(t)|. \tag{2.12}$$

Condition (1.8) says that the inequality  $|c_j| \bar{j}^{p/2-1/4} \leq C(\overline{\log \bar{j}})^{-1-\epsilon}$  holds for all  $j$ 's with sufficiently large  $C$ . Hence, conditions (1.7), (1.8), and estimate (2.2) yield

$$\begin{aligned} |I_1(t)| &\leq C_t \sum_{j=0}^n \left\{ \bar{j}^{-a/2} |\Delta^p c_j| + \bar{j}^{-a/2-1} \left( \max_{j \leq k \leq j+p-1} |c_k| \right) \right\} \bar{j}^{(a+p)/2-1/4} \\ &\leq C_t \sum_{j=0}^n \bar{j}^{p/2-1/4} |\Delta^p c_j| + \sum_{j=0}^{\infty} \bar{j}^{-1} \max_{j \leq k \leq j+p-1} \bar{k}^{p/2-1/4} |c_k| \\ &\leq C_t \left( \sum_{j=0}^n \bar{j}^{p/2-1/4} |\Delta^p c_j| + \sum_{j=0}^{\infty} \frac{1}{\bar{j}(\overline{\log \bar{j}})^{1+\epsilon}} \right) \\ &< \infty \quad \forall n \in \mathbb{N}. \end{aligned} \tag{2.13}$$

On the other hand, (1.8), (2.2), and the equality

$$\Delta^j b_n = \sum_{i=0}^j \binom{j}{i} (-1)^i b_{n+i} \tag{2.14}$$

imply

$$\begin{aligned} |I_2(t)| &\leq e^{-t/2} t^{a/2} \sum_{j=0}^{p-1} \left\{ \sum_{i=0}^j \binom{j}{i} (n+1+i)^{-a/2} |c_{n+1+i}| \right\} |L_n^{a+j+1}(t)| \\ &\leq C_{ap} \sum_{j=0}^{p-1} \left\{ \sum_{i=0}^j \binom{j}{i} (n+1+i)^{(j+1)/2-1/4} |c_{n+1+i}| t^{-(j+1)/2-1/4} (1+t)^{1/6} \right\} \\ &\leq C_{ap} \sup_{k>n} \{k^{p/2-1/4} |c_k|\} \sum_{j=0}^{p-1} t^{-(j+1)/2-1/4} (1+t)^{1/6} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{2.15}$$

Hence,  $s_n(t)$  converges pointwise to

$$f(t) \equiv e^{-t/2} t^{a/2} \sum_{j=0}^{\infty} (\Delta^p b_j) L_j^{a+p}(t) \tag{2.16}$$

provided (1.7) and (1.8) hold. Hence, we have the following lemma.

**LEMMA 2.1.** *Let  $a \geq 0$ . Assume that  $\{c_j : j \geq 0\}$  satisfies conditions (1.7) and (1.8). Then, the Laguerre series  $\sum c_j \mathcal{L}_j^a(t)$  converges pointwise to the function  $f(t)$  in (2.16),  $t \in \mathbb{R}^+$ .*

Now we are ready to prove [Theorem 1.1](#). Since  $r \leq 1$  and  $r/2 + pr \leq 2$ ,  $p/2 - 1/4 \leq 2/r - 3/2$  which says that [\(1.11\)](#) is stronger than [\(1.8\)](#). Also, we have

$$\sum_{j=0}^{\infty} (|\Delta^p c_j| j^{p/2-1/4})^r \leq \sum_{j=0}^{\infty} |\Delta^p c_j|^r j^{1-r/2}. \tag{2.17}$$

Thus, condition [\(1.10\)](#) yields

$$\sum_{j=0}^{\infty} (|\Delta^p c_j| j^{p/2-1/4})^r < \infty, \tag{2.18}$$

which implies the validity of [\(1.7\)](#) since  $\ell^r \subseteq \ell^1$ . By [Lemma 2.1](#), we get the pointwise convergence.

To finish the proof of [Theorem 1.1](#), we still need to check its  $L^r$ -convergence. From [\(2.9\)](#) and inequality

$$\|g + h\|_r^r \leq \|g\|_r^r + \|h\|_r^r \quad \text{for } 0 < r \leq 1, \tag{2.19}$$

we have

$$\begin{aligned} \int_0^{\infty} |s_n(t) - f(t)|^r dt &\leq \sum_{j=n+1}^{\infty} \int_0^{\infty} \{e^{-t/2} t^{a/2} |(\Delta^p b_j) L_j^{a+p}(t)|\}^r dt \\ &\quad + \sum_{j=0}^{p-1} \int_0^{\infty} \{e^{-t/2} t^{a/2} |(\Delta^j b_{n+1}) L_n^{a+j+1}(t)|\}^r dt \\ &\equiv I_3 + I_4. \end{aligned} \tag{2.20}$$

The definition of  $\mathcal{L}_j^a$ , [\(1.10\)](#), [\(1.11\)](#), [\(2.3\)](#), and [\(2.11\)](#) give us

$$\begin{aligned} I_3 &\leq C_{ap} \sum_{j=n+1}^{\infty} |j^{a/2} \Delta^p b_j|^r j^{pr/2} \int_0^{\infty} |t^{-p/2} \mathcal{L}_j^{a+p}(t)|^r dt \\ &\leq C_{ap} \sum_{j=n+1}^{\infty} \left( |\Delta^p c_j| + j^{-1} \max_{j \leq k \leq j+p-1} |c_k| \right)^r j^{1-r/2} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \tag{2.21}$$

$$\begin{aligned} I_4 &\leq C_{ap} \sum_{j=0}^{p-1} |n^{a/2} \Delta^j b_{n+1}|^r n^{(j+1)r/2} \int_0^{\infty} |t^{-(j+1)/2} \mathcal{L}_n^{a+j+1}(t)|^r dt \\ &\leq C_{ap} \max_{n \leq k \leq n+p} |c_k|^r n^{1-r/2} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence, [Theorem 1.1](#) follows immediately.

**3. Proofs of Hardy-Littlewood type inequalities.** From the previous arguments, conditions [\(1.10\)](#) and [\(1.11\)](#) imply that the series  $\sum c_j \mathcal{L}_j^a(t)$  converges pointwise and in  $L^r$ -metric to  $f(t)$ . We show the Hardy-Littlewood type inequalities as follows.

**PROOF OF COROLLARY 1.2.** The hypotheses of Corollary 1.2, the monotone convergence theorem, (2.4), and (2.19) can be used to show that

$$\begin{aligned} \int_0^\infty |f(t)|^r dt &= \int_0^\infty \left| \sum_{j=0}^\infty c_j \mathcal{L}_j^a(t) \right|^r dt \\ &\leq \sum_{j=0}^\infty |c_j|^r \int_0^\infty |\mathcal{L}_j^a(t)|^r dt \\ &\leq C \sum_{j=0}^\infty |c_j|^r \bar{j}^{1-r/2}. \end{aligned} \tag{3.1}$$

□

**PROOF OF THEOREM 1.3.** Let  $f \in L^r(\mathbb{R}^+)$ ,  $r \geq 1$ , and  $c_j = \int_0^\infty f(t) \mathcal{L}_j^a(t) dt$ . Hölder's inequality and (2.4) yield

$$\begin{aligned} |c_j|^r \bar{j}^\beta &= \left| \int_0^\infty f(t) \mathcal{L}_j^a(t) dt \right|^r \bar{j}^\beta \\ &\leq \|f\|_r^r \left( \int_0^\infty |\mathcal{L}_j^a(t)|^{r'} dt \right)^{r/r'} \bar{j}^\beta \\ &\leq \begin{cases} C \|f\|_{L^r(\mathbb{R}^+)}^r \bar{j}^{r/r' - r/2 + \beta}, & \text{for } r > \frac{4}{3}, \\ C \|f\|_{L^r(\mathbb{R}^+)}^r \bar{j}^{-1/3 + \beta} (\log \bar{j})^{1/3}, & \text{for } r = \frac{4}{3}, \\ C \|f\|_{L^r(\mathbb{R}^+)}^r \bar{j}^{-r/r' + \beta}, & \text{for } r < \frac{4}{3}, \end{cases} \end{aligned} \tag{3.2}$$

where  $1/r + 1/r' = 1$ . Since  $\beta < -r/2$  implies  $r/r' - r/2 + \beta < -1$ ,

$$\sum_{j=0}^\infty |c_j|^r \bar{j}^\beta \leq C \|f\|_r^r. \tag{3.3}$$

□

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