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COINCIDENCES AND FIXED POINTS OF RECIPROCALLY CONTINUOUS AND COMPATIBLE HYBRID MAPS

S. L. SINGH and S. N. MISHRA

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It is proved that a pair of reciprocally continuous and nonvacuously compatible single-valued and multivalued maps on a metric space possesses a coincidence. Besides addressing two historical problems in fixed point theory, this result is applied to obtain new general coincidence and fixed point theorems for single-valued and multivalued maps on metric spaces under tight minimal conditions.

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1. Introduction. The concept of compatible maps has proven useful for generalizing results in the context of metric fixed point theory for continuous single-valued and multivalued maps (cf. [1, 2, 4, 11, 15, 16, 20, 21, 22, 23, 30, 31]). Recently, reciprocal continuity for a pair of (discontinuous) single-valued maps has been introduced in [23] and promoted as a means to comprehensive results.

First, we introduce reciprocal continuity for a hybrid pair of single-valued and multivalued maps, and emulate the joint merits of reciprocal continuity and compatibility of a hybrid pair in the setting of metric spaces. We give a general principle (Theorem 2.8) stating that nonvacuously compatible and reciprocally continuous hybrid pair on a metric space has a coincidence. This seems to be of vital interest in view of a historically significant and negatively settled problem that a pair of continuous and commuting self-maps on the closed interval $[0,1]$ has a common fixed point (see [3, 13, 17]) and that continuous and commuting maps on a complete metric space need not have a coincidence even (see Remark 2.7(v)). This principle presents another view of a significant result of Jungck [17, Theorem 3.6] on a metric space as well. We apply Theorem 2.8 to obtain a coincidence and fixed point theorem for a hybrid quadruple of maps on a metric space satisfying a very general contractive type condition which includes several general conditions studied by Beg and Azam [1], Ćirić [6], Das and Naik [10], Jungck [16], Kaneko [19, 20], Rhoades et al. [27], Singh et al. [29], Tan and Minh [33], and others. One of our results presents another view of recent resolutions to a still open fixed point problem of Simon Reich (see [4, 5, 8, 9, 25, 26]). Our final result on a compact metric space extends and generalizes fixed point theorems from [7, 17, 32, 33].

In this paper, consistent with [22, page 620], (X,d) denotes a metric space, id the identity map on X, $\mathrm{CL}(X)$ (resp., $\mathrm{CB}(X)$) the nonempty closed (resp., closed and bounded) subsets of X and H for the Hausdorff (resp., generalized Hausdorff) metric on $\mathrm{CB}(X)$ (resp., $\mathrm{CL}(X)$). Further, $d(A,B)$ denotes the ordinary distance between

nonempty subsets A and B of X while $d(x,B)$ stands for $d(A,B)$ when $A=\{x\}$. The set of natural numbers is denoted by $\mathbb{N}$.

2. Reciprocal continuity

Definition 2.1. The maps $T:X\to \mathrm{CL}(X)$ and $f:X\to X$ are reciprocally continuous on X (resp., at $t\in X$) if and only if $fTx\in \mathrm{CL}(X)$ for each $x\in X$ (resp., $fTt\in \mathrm{CL}(X)$) and $\lim_n fTx_n=fM$, $\lim_n Tfx_n=Tt$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_n Tx_n=M\in \mathrm{CL}(X),\qquad \lim_n fx_n=t\in M. \tag{2.1}$$

We may use r.c. for "reciprocal continuity" or "reciprocal continuous" as the situation demands. For self-maps $f,g:X\to X$, this definition due to Pant [23] reads: f and g are r.c. if and only if $\lim_n gfx_n=gt$ and $\lim_n fgx_n=ft$ whenever $\{x_n\}\subset X$ is such that $\lim_n gx_n=\lim_n fx_n=t\in X$. Clearly, any continuous pair is reciprocally continuous but, as the following examples show, the converse is not true.

Example 2.2 (see [24, Example 2.1])**.** Let $X=[0,\infty)$ with the usual metric. Define $g,f:X\to X$ by

$$gx=\begin{cases}\dfrac{1}{2}+x & \text{if } 0\le x<\dfrac{1}{2},\\ 1 & \text{if } x=\dfrac{1}{2},\\ 0 & \text{if } x>\dfrac{1}{2},\end{cases}\qquad fx=\begin{cases}\dfrac{1}{2}-x & \text{if } 0\le x<\dfrac{1}{2},\\ \dfrac{1}{2} & \text{if } x=\dfrac{1}{2},\\ 1 & \text{if } x>\dfrac{1}{2}.\end{cases} \tag{2.2}$$

These maps are discontinuous at $x=1/2$. However, they are r.c. (take a decreasing sequence $\{x_n\}$ converging to 0).

Example 2.3. Let $X=R^1$,

$$TX=\begin{cases}\left[\dfrac{1}{2},x+1\right] & \text{if } x>0,\\ \{0\} & \text{if } x=0,\\ \left[x-1,-\dfrac{1}{2}\right] & \text{if } x<0,\end{cases}\qquad fx=\begin{cases}2x+1 & \text{if } x>0,\\ 0 & \text{if } x=0,\\ 2x-1 & \text{if } x<0.\end{cases} \tag{2.3}$$

Then T and f are r.c. at $x=0$ (take $x_n=0$, $n\in\mathbb{N}$). Notice that there is a discontinuity at their common fixed point ($x=0$).

For continuity of multivalued maps at their fixed and common fixed points, refer to [12]. The following definition is due to Kaneko and Sessa [20] and Beg and Azam [1] when $T:X\to \mathrm{CB}(X)$.

Definition 2.4. The maps $T:X\to \mathrm{CL}(X)$ and $f:X\to X$ are compatible if and only if $fTx\in \mathrm{CL}(X)$ for each $x\in X$ and $\lim_n H(Tfx_n,fTx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n Tx_n=M\in \mathrm{CL}(X)$ and $\lim_n fx_n=t\in M$.

Evidently commuting maps T, f (i.e., when $fTx=Tfx$, $x\in X$) are weakly commuting (i.e., $H(Tfx,fTx)\le d(fx,Tx)$, $x\in X$, [19, 29]), weakly commuting T, f are compatible, and compatible T, f are weakly compatible (i.e., when $fTx=Tfx$

whenever $fx \in Tx$, [18]) but the reverse implication is not true. For an excellent discussion on the role of weak compatibility in fixed point considerations, refer to Jungck and Rhoades [18]. For self-maps $f, g : X \to X$, Definition 2.4 due to Jungck [15, 16] reads: f and g are compatible if and only if $\lim_n d(gfx_n, fgx_n) = 0$ wherever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t \in X$. Notice that the maps g, f of Example 2.2 are not compatible (take $\{x_n\}$ as in Example 2.2). So r.c. need not imply compatibility. Taking X as in Example 2.2 and defining $fx = 3x^2/(x^2+8)$ and $gx = x$, if $x < 2$, $gx = 2$ if $x \geq 2$, one may conclude that f, g are r.c. but not compatible (take $x_n = 4$, $n \in \mathbb{N}$). Hence we assert that: *the r.c. and compatibility are independent concepts*.

Following Itoh and Takahashi [14], $T : X \to \mathrm{CL}(X)$ and $f : X \to X$ are IT-commuting (commuting in the sense of Itoh-Takahashi [14]) at a point $v \in X$ if $Tfv \subset fTv$. Further, T and f are IT-commuting on X if they are IT-commuting at each point $v \in X$. We remark that the IT-commutativity of a hybrid pair (T, f) at a point v is more general than its compatibility (cf. [31, Example 1]) and weak compatibility at the point v.

Example 2.5. Let $X = [0, \infty)$ be endowed with the usual metric and

$$Tx = \begin{cases} [0, x] & \text{if } x < 2, \\ [4, 2+x] & \text{if } x \geq 2, \end{cases} \qquad fx = \begin{cases} x & \text{if } x < 2, \\ 4 & \text{if } x \geq 2. \end{cases} \tag{2.4}$$

We see that T and f are compatible for $x < 2$ but not for $x \geq 2$ (e.g., take $x_n = 2 + n$, $n \in \mathbb{N}$). Further, T and f are not r.c. (e.g., take $x_n = 2 - 1/n$, $n \in \mathbb{N}$).

Example 2.6. Let $X = [2, \infty)$ with the usual metric and $Tx = \{1 + x\}$ and $fx = 2x + 1$. We see that there does not exist a sequence $\{x_n\} \subset X$ such that $\{fx_n\}$ and $\{Tx_n\}$ both converge to the same element in X. Thus requirements of compatibility are vacuously satisfied .

Remark 2.7. (i) If a compatible pair (T, f) is such that $Tt = fM$, then it is evident from Definitions 2.1 and 2.4 that the continuity of one of T or f is sufficient to ensure the r.c. of the pair (T, f).

(ii) If T and f are r.c. and nonvacuously compatible, then $Tt = fM$. See the proof of Theorem 2.8.

(iii) The r.c. at a point $t \in X$ may be verified by considering all sequences $\{x_n\} \subset X$ such that $\lim_n fx_n = t \in M = \lim_n Tx_n$. If there does not exist such a sequence then the definition holds vacuously, and the maps are r.c. (see Example 2.6). This observation applies to the compatibility of T and f as well. Hence nonvacuous compatibility of T and f implies the existence of at least a sequence $\{x_n\}$ in X such that $\{fx_n\}$ and $\{Tx_n\}$ both converge as per requirements of Definition 2.4.

(iv) If the pair (T, f) is compatible at a point $v \in X$ and v is a coincidence point of T and f, that is, $fv \in Tv$, then $fTv = Tfv$ (see [20, page 260]). Indeed, commutativity, weak commutativity, and compatibility of T and f are equivalent at a coincidence point v of T and f.

(v) A pair of continuous and commuting selfmaps of a complete metric space need not have a coincidence; for example, $gx = 1 + x$ and $fx = x$, $x \in [0, \infty)$. (Notice the vacuous compatibility of g and f.)

The following is the main result of this section. In all that follows, $C(T,f)$ stands for the collection of coincidence points of T and f, that is, $C(T,f)=\{v : fv \in Tv\}$.

Theorem 2.8. *Let (X,d) be a metric space and $T : X \to \mathrm{CL}(X)$ and $f : X \to X$. If T and f are reciprocally continuous and nonvacuously compatible on X then $C(T,f)$ is nonempty. Further, T and f have a common fixed point ft, provided $fft = ft$ for some $t \in C(T,f)$.*

Proof. Since T and f are nonvacuously compatible, there exits a sequence $\{x_n\}$ in X such that $\{fx_n\}$ and $\{Tx_n\}$ converge, respectively, to $t \in X$ and $M \in \mathrm{CL}(X)$ such that $t \in M$ and $\lim_n H(Tfx_n, fTx_n) = 0$. This, in view of the r.c. of T and f, yields $H(Tt,fM) = 0$, and $Tt = fM$. Now, $t \in M$ implies $ft \in fM$. Therefore $ft \in Tt$, and $C(T,f)$ is nonempty. Further, $ft = fft$ implies $ft \in fTt = Tft$ (cf. Remark 2.7(i)). This completes the proof. □

For a better appreciation of Theorem 2.8 and the relative roles of r.c. and compatibility, consider the following result of Mizoguchi and Takahashi [21, Theorem 3].

Theorem 2.9. *Let K be a closed convex subset of a uniformly convex Banach space. Let $T : K \to \mathrm{CB}(X)$ and $f : K \to K$ be such that Tx is convex for each $x \in K$, $H(Tx,Ty) \le q\|x-y\|$, $x,y \in K$, $0 \le q < 1$, and $\|fx-fp\| \le \|x-p\|$, $x \in K$, $p \in F(f)$, where $F(f)$ denotes the set of fixed points of f. If T and f are IT-commuting on K, then T and f have a common fixed point, that is, there exists $z \in F(f)$ with $fz \in Tz$.*

Further, Theorem 2.8 applies to discontinuous maps and a common fixed point may be a point of discontinuity as well (see Example 2.3). Notice that, besides several stronger conditions on the space, T of Theorem 2.9 is a multivalued contraction and f is nonexpansive about fixed points.

3. Results on metric spaces and Reich's problem. First, we give a very general coincidence and fixed point theorem under very tight conditions. Let ψ denote the family of maps ϕ from the set $\mathbb{R}^+$ of nonnegative reals to itself such that $\phi(t) < t$ for all $t > 0$.

Theorem 3.1. *Let (X,d) be a metric space and $S,T : X \to \mathrm{CL}(X)$ and $f,g : X \to X$ such that*

(1) *$S(X) \subset g(X)$ and the pair (S,f) is reciprocally continuous and nonvacuously compatible.*

If there exists $\phi \in \psi$ such that

(2) *$H(Sx,Ty) \le \phi(M(x,y))$ for x, y in X, where*

$$M(x,y) = \max\{d(fx,gy), d(fx,Sy), d(gy,Ty), d(fx,Ty), d(gy,Sx)\}, \tag{3.1}$$

then $C(S,f)$ and $C(T,g)$ are nonempty. Further,

(Ia) *S and f have a common fixed point ft, provided $fft = ft$ for some $t \in C(S,t)$;*

(Ib) *T and g have a common fixed point gu, provided $ggu = gu$ and T, g are IT-commuting at $u \in C(T,g)$;*

(Ic) *S, T, f, and g have a common fixed point, provided (Ia) and (Ib) both are true.*

Proof. By Theorem 2.8, (1) implies that $C(S,f)$ is nonempty, that is, $ft \in St$ for some $t \in X$. Since $S(X) \subset g(X)$, there is a point $u \in X$ such that $ft = gu \in St$. So by (2),

$$\begin{aligned} d(gu,Tu) &\le H(St,Tu) \\ &\le \phi(\max\{d(ft,gu),d(ft,St),d(gu,Tu),d(ft,Tu),d(gu,St)\}) \\ &= \phi(d(gu,Tu)) < d(gu,Tu) \quad \text{if } gu \notin Tu. \end{aligned} \tag{3.2}$$

So $gu \in Tu$ and $C(T,g)$ is nonempty.

(Ia) and (Ib) may be shown following the last part of the proof of Theorem 2.8. Now (Ic) is immediate. □

Several contractive conditions studied in [1, 4, 6, 10, 11, 12, 16, 18, 19, 20, 27, 28, 29, 33] are special cases of (2). For example, if we take $\phi(t) = qt$ $(0 \le q < 1)$, $f = g$, $S = T$, and $T : X \to X$, then we obtain the condition studied in [10]. Notice that the contractive condition studied in [6] is a special case of that of [10].

Reich [25, 26] posed the following question: let (X,d) be a complete metric space and $T : X \to \mathrm{CB}(X)$ such that $H(Tx,Ty) \le k(d(x,y))d(x,y)$ for all distinct $x,y \in X$, where $k : (0:\infty) \to (0,1)$ with $\lim_{r\to t^+} \sup k(r) < 1$ for each $t > 0$. Then, does T have a fixed point? Mizoguchi and Takahashi [21] have shown that T has a fixed point when $\lim_{r\to t^+} k(r) < 1$ for each $t \ge 0$. Chang [4] has generalized this result, and Theorem 3.1 presents an extension of Chang's main result [4, Theorem 1]. However, Reich's problem remains open and needs further resolution.

The following example shows that the nonvacuous compatibility of one of the pairs (S,f) or (T,g) is essential even if $f = g = \mathrm{id}$.

Example 3.2 (see [28])**.** Let $X = \{1,2,3,4\}$ with metric d defined by

$$d(1,2) = d(3,4) = 2, \qquad d(1,3) = d(2,4) = 1, \qquad d(1,4) = d(2,3) = \frac{3}{2}. \tag{3.3}$$

Define S and T by $S1 = S4 = \{2\}$, $S2 = S3 = \{1\}$ and $T1 = T3 = \{4\}$, $T2 = T4 = \{3\}$. Take $f = g = \mathrm{id}$. Then Definition 2.4 is satisfied with $\phi(t) = 3t/4$. All conditions of Theorem 3.1 are satisfied except (1). Indeed the pairs (S,id) and (T,id) are vacuously compatible.

The following result generalizes several main results from [1, 11, 16, 18, 19, 20] and others.

Corollary 3.3. *Let (X,d) be a complete metric space and $S,T : X \to \mathrm{CL}(X)$, $f,g : X \to X$ such that*

(3) *$S(X) \subset g(X)$, $T(X) \subset f(X)$, and the pair (S,f) is compatible and reciprocally continuous.*

If there exists $q \in (0,1)$ such that

(4) $H(Sx,Ty) \le qm(x,y)$ *for* x, y *in* X, *where*

$$m(x,y) = \max\left\{d(fx,gy), d(fx,Sx), d(gy,Ty), \frac{1}{2}[d(fx,Ty)+d(gy,Sx)]\right\}, \quad (3.4)$$

then $C(S,f)$ *and* $C(T,g)$ *are nonempty. Further, conclusions (Ia), (Ib), and (Ic) are also true.*

Proof. Since $S(X) \subset g(X)$ and $T(X) \subset f(X)$, we construct the sequences $\{x_n\}$, $\{y_n\} \subset X$ as in [11, 31] such that, for each $n \in \mathbb{N}$,

$$y_{2n-1} = gx_{2n-1} \in Sx_{2n-2}, \qquad y_{2n} = fx_{2n} \in Tx_{2n-1}. \quad (3.5)$$

Then as in [11],

$$\begin{aligned} d(y_{2n-1},y_{2n}) &\le q^{-1/2} d(y_{2n-2},y_{2n-1}), \\ d(y_{2n},y_{2n+1}) &\le q^{-1/2} d(y_{2n-1},y_{2n}). \end{aligned} \quad (3.6)$$

So $\{y_n\}$ converges to a point $t \in X$, [11, Theorem 2]. By (4),

$$\begin{aligned} H(Sx_{2n},Tx_{2n-1}) &\le qm(x_{2n},x_{2n-1}) \\ &\le q\max\{d(y_{2n},y_{2n-1}), d(y_{2n},y_{2n+1})\} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (3.7)$$

Therefore, $\lim_{k,n\to\infty} H(Sx_{2n},Sx_{2k}) \le \lim_{k,n\to\infty} d(y_{2n-1},y_{2k-1}) = 0$, and $\{Sx_{2n}\}$ is Cauchy in $\mathrm{CL}(X)$. The space X being complete, the hyper space $\mathrm{CL}(X)$ is also complete, and $\{Sx_{2n}\}$ converges to an M in $\mathrm{CL}(X)$. So

$$d(t,M) \le d(t,gx_{2n-1}) + H(Sx_{2n-2},M) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\ t \in M. \quad (3.8)$$

Thus, for the sequence $\{x_n\}$ in X, we have $\{Sx_{2n}\}$ and $\{gx_{2n-1}\}$ converging, respectively, to M and $t \in M$. Therefore the compatibility of the pair (S,g) is nonvacuous. Hence the proof is immediate from Theorem 3.1 by observing that (4) implies (2). □

Remark 3.4. (i) It is evident from the proof of Theorem 3.1 that $ft \in St$ and $ft = gu \in Tu$, that is, (S,t) and (T,g) may have different coincidence points with $ft = gu$. See Example 3.5 in support of this observation, which applies to Corollary 3.3 as well.

(ii) The power of Corollary 3.3 is appreciated by observing that the main result of [11, Theorem 2] is obtained under condition (4) when all the maps S, T, f, g are continuous and the pairs (S,f) and (T,g) are compatible. Moreover, Corollary 3.3 generalizes several other interesting results from [1, 16, 19, 20] and the references therein.

(iii) If $f = \mathrm{id}$ under the condition (4) of Corollary 3.3, then the pair (S,f) is automatically compatible and r.c. Thus if $f = g = \mathrm{id}$ then Corollary 3.3 states that $S,T : X \to \mathrm{CL}(X)$, satisfying condition (4), have a common fixed point in complete X.

Therefore, several general results surveyed in [21] are contained in Corollary 3.3 (e.g., Theorems 12, 15, and 16).

Example 3.5. Let $X=[0,\infty)$ be endowed with the usual metric. Let $f,g,S,T:X\to X$ be such that $fx=8x^4$, $gx=8x^8$, $Sx=x^4+7/16$, and $Tx=x^8+7/16$. Evidently, for any $x,y\in X$, $d(Sx,Ty)=1/8d(fx,gy)$, that is, condition (4) is satisfied with $q=1/8$. Further, $S(X)=T(X)=[7/16,\infty]\subset X=f(X)=g(X)$, the pair (S,f) is compatible and continuous. So all the hypotheses of Corollary 3.3 are satisfied and $f(1/2)=S(1/2)$ and $g(2^{-1/2})=T(2^{-1/2})$, that is, f, S have a coincidence at $x=1/2$, and g, T have a (different) coincidence at $x=(2^{-1/2})$. Notice that $f(1/2)=g(2^{-1/2})$ and the pair (T,g) is not compatible.

Corollary 3.6. *Let f, g, S, T be self-maps of a complete metric space (X,d) such that $S(X)\subset g(X)$, $T(X)\subset f(X)$, and the pair (S,f) is compatible and reciprocally continuous. If there exists $q\in(0,1)$ such that $d(Sx,Ty)\le qm(x,y)$ for all x, y in X, then*

(IIa) *S and f have a common fixed point;*

(IIb) *T and g have a coincidence at $x=u\in X$;*

(IIc) *f, g, S, and T have a common fixed point provided that T and g are weakly compatible.*

The tightness of the conditions in this result is evident from the fact that Jungck's [16, Theorem 3.1] is Corollary 3.6(IIc) with the r.c. replaced by continuity of f, g, and the weak compatibility of T, g (at $u\in X$) replaced by the compatibility of T, g on X.

4. Results on compact spaces. We apply Corollary 3.3 to obtain a new coincidence theorem for a hybrid of multivalued and single-valued maps on the setting of a compact metric space generalizing and extending [7, Theorem 1], [16, Theorem 3.2], and relevant results of Smithson [32] and Tan and Minh [33].

Theorem 4.1. *Let f, g be continuous self-maps of a compact metric space (X,d) and let $S,T:X\to \mathrm{CB}(X)$ be continuous such that $S(X)\subset g(X)$, $T(X)\subset f(X)$, and the pair (S,f) is compatible. If $H(Sx,Ty)<m(x,y)$ (see condition (4)) when $m(x,y)>0$, then $C(S,f)$ and $C(T,g)$ are nonempty. Further, (Ia)–(Ic) are also true.*

Proof. In view of the conclusions of Corollary 3.3, it is enough to show that $C(S,t)$ and $C(T,g)$ are nonempty. We claim that $m(x,y)=0$ for some $x,y\in X$. Otherwise the function $w(x,y)=H(Sx,Ty)/m(x,y)$ is continuous and satisfies $w(x,y)<1$ for $(x,y)\in X\times X$. Since $X\times X$ is compact, there exist $v,z\in X$ such that $w(x,y)\le w(v,z)=q<1$ for $x,y\in X$. Consequently, $H(Sx,Ty)\le qm(x,y)$ for $x,y\in X$ and some $q\in(0,1)$. So, by Corollary 3.3 (see also Remark 3.4(i)), there exist $u,t\in X$ such that $ft\in St$ and $ft=gu\in Tu$, and we have $m(t,u)=0$, contradicting $m(t,u)>0$. Thus $m(x,y)=0$ for some $x,y\in X$. Consequently, $fx=gy$, $fx\in Sx$, $gy\in Ty$, and this completes the proof. □

Corollary 4.2. *Let f, g, S, and T be continuous self-maps of a compact metric space (X,d) such that $S(X)\subset g(X)$, $T(X)\subset f(X)$, and the pair (S,f) is compatible. If $d(Sx,Ty)<m(x,y)$ when $m(x,y)>0$, then conclusions (IIa)–(IIc) are true.*

Its proof may also be completed using Corollary 3.6 and following the proof of Theorem 4.1. Jungck [16, Theorem 3.2] is Corollary 3.6(IIc) when the pair (T, g) is also compatible on X.

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S. L. Singh: Department of Mathematics, Gurukula Kangri Vishwavidyalaya, Hardwar 249404, India
E-mail address: mathgkv@tatpar.com

S. N. Mishra: Department of Mathematics, University of Transkei, Umtata 5100, South Africa
E-mail address: mishra@getafix.utr.ac.za