Internat. J. Math. & Math. Sci.
Vol. 2 #3 (1979) 503-522

ON AN INITIAL-BOUNDARY VALUE PROBLEM FOR THE NONLINEAR SCHRÖDINGER EQUATION

HERBERT GAJEWSKI

Akademie der Wissenschaften der DDR
Zentralinstitut für Mathematik und Mechanik
DDR - 108 Berlin
Mohrenstrabe 39 , Germany

(Received August 22, 1978)

ABSTRACT. We study an initial-boundary value problem for the nonlinear Schrödinger equation, a simple mathematical model for the interaction between electromagnetic waves and a plasma layer. We prove a global existence and uniqueness theorem and establish a Galerkin method for solving numerically the problem.

KEY WORDS AND PHRASES. Conserved integrals, a priori estimates, unique global solution, convergence of Fourier's method.

AMS(MOS) SUBJECT CLASSIFICATION (1970) CODES. 35B45, 35D05, 35D10, 35Q99, 65M99.

1. INTRODUCTION.

This paper is concerned mainly with the initial-boundary value problem

$$i\,u_t + u_{xx} + k|u|^2u = 0,\quad u(0,x) = u_0(x)\ , \tag{1.1}$$

$$u_x(t,0)=i\alpha_0(2a_0-u(t,0))\ ,\ u_x(t,1)=i\alpha_1(u(t,1)-2a_1) \tag{1.2}$$

where $i^2=-1$, the subscripts t and x denote partial differentiating with respect to the time coordinate $t\in[0,T]$, $T>0$, and the spatial coordinate $x\in[0,1]$, respectively, k and α_j , $j=0,1$, are real constants, the a_j's are (in general) complex constants.

(1.1) is the standard form of the nonlinear Schrödinger equation. Only technical modifications are necessary to extend our results to somewhat more general equations like $iu_t+u_{xx}+k|u|^2u+a(x)u=f(t,x)$.

The boundary conditions (1.2) can be written in the more suggestive form

$$\frac{\partial^l}{\partial x^l}\,(a_0\exp(i\alpha_0x) + U_0\exp(-i\alpha_0x) - u)\Big|_{x=0} = 0\ ,\ l=0,1\ ,$$

$$\frac{\partial^l}{\partial x^l}\,(a_1\exp(-i\alpha_1(x-1)) + U_1\exp(i\alpha_1(x-1)) - u)\Big|_{x=1} = 0.$$

The problem (1.1), (1.2) may be considered as a simple mathematical model for the interaction of stationary electromagnetic waves $a_0\exp(i\alpha_0x)$ for $x<0$ and $a_1\exp(-i\alpha_1(x-1))$ for $x>1$ with a plasma layer localized in the interval $[0,1]$. The functions U_j , $j=0,1$, defined by $U_j(t)=u(t,j)-a_j$ represent the reflection and transmission properties of the plasma layer [1].

Recently, the initial value problem (1.1) has been studied extensively for solutions which vanish at $|x|=\infty$ [2,10] or which are periodic in x [3] . The nonlinear Schrödinger equation connected with these boundary conditions has such distinguished properties as an associated inverse scattering problem and an infinite set of conserved functionals F_n. Unfortunately, the boundary conditions (1.2) do not imply such properties. Especially, the functionals F_n are not conserved. Nevertheless, we shall use the functionals F_1 and F_5 (cf. [10]) to prove important a priori estimates.

The paper consists of five sections. In the second section we introduce notations and state some results concerning a linear ordinary differential operator. This operator turns out to be self-adjoint with respect to the homogeneous boundary conditions corresponding to (1.2) (i. e., $a_0=a_1=0$). In the third section we prove an existence and uniqueness result for a regularized problem originating from (1.1), (1.2) by addition of a regularization term which may be interpreted physically as damping [7]. The fourth section contains our main result, a global existence and uniqueness theorem for problem (1.1), (1.2). Our proof bases on the approximation of (1.1), (1.2) by the regularized problems mentioned above. In the last section we establish Galerkin's method as a procedure to solve (1.1), (1.2) numerically. The eigenfunctions of the self-adjoint operator studied in Section 2 serve us as appropriate base functions.

2. PRELIMINARIES

Throughout this paper c denotes various constants. For a complex number z we denote by $\bar{z}$, z, Re z and Im z conjugate complex number, modulus, real and imaginary part, respectively. C^1, H^1 and L^q are the usual spaces of complex-valued functions defined on the interval $(0,1)$ provided with the norms

$$\|v\|_{C^1} = \sum_{j=0}^{1} \max_{x\in[0,1]} \left|\frac{d^j v(x)}{dx^j}\right| , \quad \|v\|_{H^1} = \left(\sum_{j=0}^{1} \int_0^1 \left|\frac{d^j v}{dx^j}\right|^2 dx \right)^{1/2} ,$$

$$\|v\|_q = \left(\int_0^1 |v|^q \, dx \right)^{1/q} , \quad 1\le q<\infty , \ \|v\|_\infty = \operatorname*{ess\,sup}_{x\in[0,1]} |v(x)| .$$

We write

$$C = C^0, \ \|v\|_C = \|v\|_{C^0} , \quad H=H^0=L^2 , \quad \|v\| = \|v\|_2 , \ (v,w) = \int_0^1 v\,\bar{w}\,dx.$$

The space H^1 is continuously embedded into C and it holds (cf. [3])

$$\|v\|_C^2 \le \|v\| (\|v\| + 2\,\|v_x\|) , \quad v\in H^1. \tag{2.1}$$

In what follows the operator A defined by

$$A\,v = -v_{xx} + 2ipv_x + (ip'+p^2)v , \quad p' = \frac{dp}{dx} , \tag{2.2}$$
$$D(A) = \left\{ v \in H^2 \;\middle|\; v_x(0)=-i\alpha_0 v(0) , \quad v_x(1)=i\alpha_1 v(1) \right\}$$

plays an important role. Here $p=p(x)$ is a real function such that

$$p\in H^3 , \quad p(0) = -\alpha_0 , \quad p(1) = \alpha_1. \tag{2.3}$$

REMARK 2.1 The function $p=(\alpha_0+\alpha_1)x-\alpha_0$ may serve as an example for p.

LEMMA 2.1 The operator $A\in(D(A)\to H)$ is self-adjoint and nonnegative. Its energetic space is H^1. A has a pure point spectrum. Its eigenvalues are $\lambda_n = n^2\pi^2$, $n=0,1,2,\ldots$ Each eigenvalue is single. The corresponding orthonormal eigenfunctions are

$$h_n=r_n e^{iP(x)}\cos n\pi x, \quad P(x)= \int_0^x p(s)ds , \quad r_n=\begin{cases}1 & \text{if } n=0,\\ \sqrt{2} & \text{if } n=1,2,\ldots\end{cases} \tag{2.4}$$

PROOF. The operator A is closely related to the Laplacian with Neumann's conditions. Indeed, it is easy to check that v is solution of the problem

$$A v = f \quad , \quad f \in H \quad , \quad v \in D(A)$$

if and only if $w = e^{-iP} v \in H^2$ is solution of Neumann's problem

$$-w_{xx} = e^{-iP} f \quad , \quad w_x(0) = w_x(1) = 0 \ .$$

From this fact and from the well-known properties of Neumann's problem (cf. [9] the lemma follows.

Provided with the scalar product

$$((v,w)) = (v+Av, w+Aw)$$

and the corresponding norm

$$\| v \|_V^2 = \| v+Av \|^2$$

$D(A)$ becomes a Hilbert space V. We denote by $\langle \cdot , \cdot \rangle$ the pairing between V and its dual space V'. Because of Riesz' representation theorem the mapping $E \in (H \to V')$ defined by

$$\langle E f , v \rangle = (f, v+Av) \quad , \quad \forall v \in V$$

is one-to-one and isometric. Thus we can identify V' and H.

LEMMA 2.2 The V-norm and the H^2-norm are equivalent on V.

PROOF. Evidently we have $\|v\|_V \leq c(p) \|v\|_{H^2}$. On the other hand it holds for $v \in V$

$$\begin{aligned}
(Av,v) &= (-v_{xx} + 2ipv_x + (ip' + p^2)v, v) \\
&= \Big[(-v_x + ipv)\bar{v}\Big]_0^1 + \int_0^1 (|v_x|^2 + i(pv_x\bar{v} - p'v\bar{v} - pv\bar{v}_x + p'v\bar{v}) + p^2|v|^2)dx \\
&= \int_0^1 (|v_x|^2 + p^2|v|^2 - 2p\,\mathrm{Im}(v_x\bar{v}))\,dx \geq \tfrac{1}{2}\|v_x\|^2 - c\|v\|^2
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
\|Av\|^2 = (Av,Av) &= \|v_{xx}\|^2 + \int_0^1 \{ 4p^2|v_x|^2 + (p^4 + (p')^2)|v|^2 \\
&\quad + 2\mathrm{Re}[v_{xx} 2ip\bar{v}_x + v_{xx}(ip' - p^2)\bar{v} - 2ipv_x(ip' - p^2)\bar{v}]\}dx \\
&\geq \tfrac{1}{2}\|v_{xx}\|^2 - c\|v\|_{H^1}^2 \ .
\end{aligned}$$

Hence we get

$$\|v\|_{H^2}^2 = \|v\|^2 + \|v_x\|^2 + \|v_{xx}\|^2 \leq c(\|v\|^2 + 2(Av,v) + \|Av\|^2) = c\,\|v\|_V^2$$

and the lemma is proved.

LEMMA 2.3 For $g \in H$ let $g_n = \sum_{l=0}^{n} (g,h_l)h_l$. Then $g_n \to g$ (strongly) in H . Moreover, if $g \in V$, then $g_n \to g$ in H^2 .

PROOF. The first statement follows from Lemma 2.1 (cf. [9], Satz 21.1). Let now $g \in V$. On account of Lemma 2.1 we have the representation $A\, g = \sum_{l=0}^{\infty} \lambda_l (g,h_l)h_l$, that is $g_n \to g$ in V . Because of Lemma 2.2 this implies $g_n \to g$ in H^2 .

In view of Section 4 we still note that for arbitrarily small $\delta > 0$ the following estimate is valid

$$\| v_x \|^2 = -2\, \mathrm{Re}(v,v_{xx}) \leq 2\|v\|\|v_{xx}\| \leq \delta\|v_{xx}\|^2 + \tfrac{1}{\delta}\|v\|^2 \quad \forall\, v \in V . \qquad (2.6)$$

In what follows $S = [0,T]$ denotes a bounded time interval. For a Banach space B we denote by

- $C(S;B)$ the Banach space of continuous (B-valued) functions provided with the norm $\|u\|_{C(S;B)} = \max_{t\in S}\|u(t)\|_B$,
- $C_w(S;B)$ the space of weakly continuous functions,
- $L^2(S;B)$ the Banach space of Bochner-integrable functions provided with the norm $\|u\|^2_{L^2(S;B)} = \int_S \|u(t)\|^2_B \, ds$,
- $H^1(S;B)$ the Banach space of functions $u \in L^2(S;B)$ having a derivative $u' = \frac{du}{dt} \in L^2(S;B)$ taken in the sense of distributions on $(0,T)$ with values in B .

REMARK 2.3 Clearly, the relation $L^2(S;H)=L^2((0,T)\times(0,1))$ holds. Accordingly, we shall occasionally consider "abstract" functions as "ordinary" ones and vice-versa.

3. THE NONLINEAR SCHRÖDINGER EQUATION WITH DAMPING

In this section we consider the problem

$$i\, u_t + k_1 u_{xx} + (k_2 + k_3|u|^2)u = 0 \ , \ u(0,x)=u_0(x), \ u \in H^1(S;H^2) \ , \qquad (3.1)$$

$$u_x(t,0) = i\alpha_0(2a_0 - u(t,0)) \ , \quad u_x(t,1) = i\alpha_1(u(t,1) - 2a_1) \qquad (3.2)$$

with real constants α_0, α_1 and (in general) complex constants a_0, a_1, k_1, k_2 and k_3 satisfying the assumptions

$$\alpha_0 \mathrm{Re}\, k_1 \geq 0, \quad \alpha_1 \mathrm{Re}\, k_1 \geq 0, \quad \mathrm{Im}\, k_1 < 0, \quad \mathrm{Im}\, k_3 \geq 0 . \tag{3.3}$$

REMARK 3.1 Under the assumptions (3.3) the term $\mathrm{Im}\, k_1 u_{xx} + \mathrm{Im}\, k_3 |u|^2 u$ may be interpreted physically as damping (cf. [7]).

REMARK 3.2 It requires only technical modifications to treat (3.1), (3.2) if a right hand side or functions $a_j = a_j(t)$ are admitted.

In order to get homogeneous boundary conditions we make the ansatz

$$u = v + u_a , \quad u_o = v_o + u_a \tag{3.4}$$

with a function $u_a \in H^3$ satisfying (3.2).

REMARK 3.3 For instance we can choose

$$u_a = -i(\alpha_0 a_0 (1-x)^2 \exp(iP) + \alpha_1 a_1 x^2 \exp(i(P-P(1)))),$$

where $P=P(x)$ is the function from (2.4).

Now we can rewrite (3.1), (3.2) as follows

$$i\, v_t + k_1(v+u_a)_{xx} + B\, v = 0 , \quad v(0) = v_o , \quad v \in H^1(S;V), \tag{3.5}$$

where $B\, v = (k_2 + k_3 |v + u_a|^2)(v + u_a)$.

THEOREM 3.1 Suppose (2.3) and $v_o = u_o - u_a \in V \cap H^3$. Then the problem (3.1), (3.2) has a unique solution.

PROOF. For real parameters $r > 0$ we define by

$$(P_r v((x) = \begin{cases} v(x) & \text{if } |v(x)| \leq r , \\ r \dfrac{v(x)}{|v(x)|} & \text{if } |v(x)| > r \end{cases} \tag{3.6}$$

operators $P_r \in (C \rightarrow C)$. It is easy to check that for v, v_1, $v_2 \in V$

$$\| P_r v \|_C \leq r \quad , \quad \| P_r v_1 - P_r v_2 \|_C \leq \| v_1 - v_2 \|_C .$$

Thus the operator $B_r \in (H^1 \rightarrow H^1)$ defined by

$$B_r\, v = (k_2 + k_3 |P_r\, (v+u_a)|^2) P_r (v+u_a) \tag{3.7}$$

satisfies for $v_j \in H^1$, $w_j = u_a + v_j$, $j=1,2$, the estimate

$$\begin{aligned}\|B_r v_1 - B_r v_2\| &= \|(k_2+k_3|P_r w_1|^2)(P_r w_1 - P_r w_2) + k_3(|P_r w_1|^2 - |P_r w_2|^2)P_r w_2\| \\ &\leq \|(|k_2| + |k_3| r^2)\,|v_1-v_2| + |k_3|\,|P_r w_1 - P_r w_2|\,|P_r w_1 + P_r w_2|\,r\,\| \\ &\leq (|k_2| + 3|k_3|\,r^2)\|v_1-v_2\| = c(r)\|v_1-v_2\| .\end{aligned} \tag{3.8}$$

Moreover, for $v \in H^1$ we have with $w=v+u_a$

$$\begin{aligned}\|B_r v\| &= \|(k_2+k_3|P_r w|^2)P_r w\| \leq |k_2|\,\|w\| + |k_3|\,\|w\|_6^3 \\ &\leq |k_2|\,\|w\| + |k_3|\,\|w\|^2(\|w\| + 2\|w_x\|) = c(\|v\|)(1+\|v_x\|) .\end{aligned} \tag{3.9}$$

For the time being we replace (3.5) by the problem

$$i\, v_t + k_1(v+u_a)_{xx} + B_r v = 0 , \quad v(0) = v_o , \quad v \in H^1(S;V) , \tag{3.10}$$

which we can write also as a standard evolution equation

$$v_t + C_r v = 0, \quad v(0) = v_o, \quad v \in H^1(S;V) , \tag{3.11}$$

where the operator $C_r \in (V \longrightarrow V')$ ist given by

$$C_r v = -i(k_1(v+u_a)_{xx} + B_r v) .$$

In order to apply results on evolution equations we now verify some properties of C_r. Using (3.8) we obtain for v_1, $v_2 \in V$ with $v=v_1-v_2$

$$\begin{aligned}\|C_r v_1 - C_r v_2\|_{V'} &= \|C_r v_1 - C_r v_2\| = \|k_1 v_{xx} + B_r v_1 - B_r v_2\| \\ &\leq |k_1|\,\|v_{xx}\| + c(r)\|v\| \leq c(r)\|v\|_V ,\end{aligned} \tag{3.12}$$

that is the Lipschitz-continuity of C_r . Next we note that C_r possesses the following monotonicity property

$$\begin{aligned}2\mathrm{Re}\langle C_r v_1 - C_r v_2, v_1 - v_2\rangle &= 2\mathrm{Im}(k_1(v_{xx}+Av+v-(v+Av)) + B_r v_1 - B_r v_2, v+Av) \\ &\geq -\mathrm{Im}\, k_1 \|v\|_V^2 - c(r)\|v\|_{H^1}^2 .\end{aligned} \tag{3.13}$$

Finally, $v_o \in V \cap H^3$ implies $C_r v_o \in H^1$. Using these facts we can conclude the existence of a unique solution v_r of (3.11) from Satz 3.1 and Bemerkung 5 in [5].

Now we want to show that for sufficiently large chosen r the function $u_r = v_r + u_a$ is the (unique) solution of (3.1), (3.2). Clearly, it suffices to find a r-independent a priori estimate for u_r in $C(S;H^1)$. We proceed in two steps. Setting $v=v_r$, $u=u_r$,

$U_j(t)=u_r(t,j)-a_j$, $j=0,1$, we get from (3.10)

$$0=2\mathrm{Im}(iu_t+k_1u_{xx}+(k_2+k_3|P_ru|^2)P_ru,u)=$$
$$= (\|u\|^2)_t+2\mathrm{Re}(k_1\sum_{j=0}^{1}\alpha_j(U_j-a_j)(\bar U_j+\bar a_j)) - 2\mathrm{Im}(k_1\|u_x\|^2 -$$
$$- ((k_2+k_3|P_ru|^2)P_ru,u)) = (\|u\|^2)_t+2\mathrm{Re}k_1\sum_{j=0}^{1}\alpha_j(|U_j|^2-|a_j|^2)$$
$$-2\mathrm{Im}(2k_1\sum_{j=0}^{1}\alpha_j\mathrm{Im}(\bar a_jU_j)+k_1\|u_x\|^2-((k_2+k_3\|P_ru\|^2)P_ru,u))$$
$$\geq (\|u\|^2)_t+\mathrm{Re}k_1\sum_{j=0}^{1}\alpha_j|U_j|^2-2\mathrm{Im}k_1\|u_x\|^2-2|k_2|\,\|u\|^2+2\mathrm{Im}k_3\|P_ru\|_4^4 - c ,$$

where the constant c is independent of r . Hence by Gronwall's lemma we conclude

$$\|u_r\|^2_{C(S;H)} - \mathrm{Im}\ k_1\|u_r\|^2_{L^2(S;H^1)} +\mathrm{Re}\ k_1\sum_{j=0}^{1}\alpha_j\|U_{rj}\|^2_{L^2(S)} \leq c. \quad (3.14)$$

In the second step we multiply (3.10) by $Av=-v_{xx}+2ipv_x+(ip'+p^2)v$ and obtain, using the symmetry of A ,

$$0 = 2\mathrm{Im}(iv_t+k_1u_{xx}+B_rv,Av)$$
$$= (v,Av)_t - 2\mathrm{Im}(k_1(Av-u_{axx}-2ipv_x-(ip'+p^2)v)-B_rv,Av)$$
$$\geq (v,Av)_t - 2\mathrm{Im}\ k_1\|Av\|^2 - 2\|k_1(u_{axx}+2ipv_x+(ip'+p^2)v-B_rv\|\,\|Av\| .$$

Taking into account (2.5), (3.9) and (3.14), we get by Gronwall's lemma the desired a priori estimate

$$\|u_r\|_{C(S;H^1)} \leq c \quad (3.15)$$

which ends the proof.

THEOREM 3.1 has been stated mainly in view of its application in the next section. For the sake of completeness we still formulate an existence and uniqueness result for the damped problem holding for arbitrary initial values $u_0\in H$. For this purpose we start from the following weak formulation of (3.1), (3.2)

$$\int_S ((iu_t+(k_2+k_3|u|^2)u,h)+k_1(i\sum_{j=0}^{1}\alpha_j(u(t,j)-2a_j)\bar{h}(t,j)-(u_x,h_x))dt=0 \tag{3.16}$$

$$u(0) = u_0 \ , \quad \forall\ h\in L^2(S;H^1) \ , \quad u\in L^2(S;H^1)\cap H^1(S;(H^1)') \ .$$

Then, using Theorem 3.1, (3.14) and the fact that $V\subset H^3$ lies densely in H (cf. Lemma 2.3), the following result is easily to prove.

PROPOSITION 3.1. Suppose (3.3) and $u_0\in H$. Then the problem (3.16) has a unique solution.

4. THE NONLINEAR SCHRÖDINGER EQUATION

We return now to the problem

$$i\ u_t + u_{xx} + k|u|^2u = 0 \ , \quad u(0,x) = u_0(x) \ , \tag{4.1}$$

$$u_x(t,0) = i\alpha_0(2a_0-u(t,0)), \quad u_x(t,1) = i\alpha_1(u(t,1)-2a_1) \ . \tag{4.2}$$

Our main result is

THEOREM 4.1. Suppose $\alpha_j \geq 0$, $j=0,1$, $v_0=u_0-u_a\in V$. Then the problem (4.1), (4.2) has a unique solution $u\in C(S;H^2)$ with $u_t\in C(S;H)$. Moreover, it holds $U_j=u(\cdot,j)-a_j\in H^1(S)$, $j=0,1$.

PROOF. (Uniqueness) Let u_1, u_2 be appropriate solutions of (4.1) (4.2). Setting $u=u_1-u_2$ and $U_j=u(\cdot,j)$ we obtain from (4.1)

$$\begin{aligned} 0 &= 2\mathrm{Im}(iu_t+u_{xx}+k(|u_1|^2u_1-|u_2|^2u_2),u) \\ &= (\|u\|^2)_t + 2k\mathrm{Im}(|u_1|^2u+(|u_1|^2-|u_2|^2)u_2,u) + 2\sum_{j=0}^{1}\alpha_j|U|_j^2 \geq \\ &\geq (\|u\|^2)_t - 3|k|(\|u_1\|^2_{C(S;C)} + \|u_2\|^2_{C(S;C)})\|u\|^2 \ . \end{aligned}$$

Integration with respect to t yields

$$\|u(t)\|^2 \leq c(u_1,u_2)\int_0^t \|u(s)\|^2 ds \ .$$

Applying Gronwall's lemma, we conclude from this $u=0$, that is $u_1=u_2$.

(Existence) We approximate (4.1) by equations of the form (3.1). To this end let $\varepsilon > 0$ be a regularization parameter and $(v_{0\varepsilon})$ a corresponding set of functions such that $v_{0\varepsilon}\in V\cap H^3$ and $v_{0\varepsilon}\to v_0$ in

V as $\varepsilon \to 0$. (The existence of such a set is guaranteed by Lemma 2.3.) We consider now the problem

$$i\, u_t + (1-i\varepsilon)u_{xx}+k|u|^2u = 0,\ u\in H^1(S;H^2),\ u(0)=u_{0\varepsilon}=v_{0\varepsilon}+u_a \tag{4.3}$$

under the boundary conditions (4.2). By Theorem 3.1 for each $\varepsilon>0$ there exists a unique solution u of (4.2), (4.3). In order to be able to pass to the limit $\varepsilon\to 0$ we need two a priori estimates. The first one is

$$\|u_\varepsilon\|^2_{C(S;H)}+\varepsilon\|u\|^2_{L^2(S;H^1)}+\sum_{j=0}^{n}\alpha_j|U_{\varepsilon j}|^2_{L^2(S)}\le c\ , \tag{4.4}$$

which can be proved exactly as (3.14). The crucial part of this proof is the second a priori estimate which we are going to prove now. For the time being we drop the subscript ε setting $u=u_\varepsilon$, $U_j=u_\varepsilon(\cdot,j)-a_j$, $j=0,1$. From (4.3) it follows

$$0 = 2\mathrm{Re}\int_0^t (iu_t+(1-i\varepsilon)u_{xx}+k|u|^2u,(1-i\varepsilon)u_{xxt}+\tfrac{3}{2}k|u|^2u_t)ds$$

$$= 2\mathrm{Re}\int_0^t\{\left[i(1+i\varepsilon)u_t\bar{u}_{xt}\right]_0^1-i(1+i\varepsilon)\|u_{xt}\|^2+(1+\varepsilon^2)(u_{xx},u_{xxt})+$$

$$+k(1+i\varepsilon)(|u|^2u,u_{xxt})+\tfrac{3}{2}ki\|u\,u_t\|^2+\tfrac{3}{2}k(1-i\varepsilon)(u_{xx}|u|^2,u_t)+$$

$$+\tfrac{3}{2}k^2(|u|^4u,u_t)\}ds$$

and thus

$$\int_0^t(2\sum_{j=0}^{1}\alpha_j|U_{jt}|^2+2\varepsilon\|u_{xt}\|^2)ds+(1+\varepsilon^2)\|u_{xx}(t)\|^2+\tfrac{k^2}{2}\|u(t)\|_6^6$$

$$=(1+\varepsilon^2)\|u_{xx}(0)\|^2+\tfrac{k^2}{2}\|u(0)\|_6^6-k\int_0^t\mathrm{Re}(2(|u|^2u,u_{xxt})+3(u_{xx}|u|^2,u_t))ds+$$

$$+\varepsilon k\int_0^t\mathrm{Im}((2(|u|^2u,u_{xxt})-3(u_{xx}|u|^2,u_t))ds \tag{4.5}$$

$$=(1+\varepsilon^2)\|u_{xx}(0)\|^2+\tfrac{k^2}{2}\|u(0)\|_6^6+k\int_0^t I_1\,ds+\varepsilon k\int_0^t I_2\,ds.$$

Let us now reform the integrands I_1 and I_2. We have

$$I_1 = -\mathrm{Re}(2(|u|^2u,u_{xxt})+3(u_{xx}|u|^2,u_t))= 2\mathrm{Re}\{((|u|^2)_xu+|u|^2u_x,u_{xt}) -$$

$$[|u|^2u\,\bar{u}_{xt}]_0^1 + 2(u_x,(|u|^2)_xu_t+|u|^2u_{tx}) - 2[u_x|u|^2\bar{u}_t]_0^1 + \mathrm{Re}(u_{xx}|u|^2,u_t)$$
$$= 3(|u|^2,(\|u_x\|^2)_t) + \tfrac{1}{2}(\|(|u|^2)_x\|^2)_t + \mathrm{Re}(2(|u|^2)_xu_x + |u|^2u_{xx},u_t) -$$
$$2\mathrm{Re}[|u|^2(u\,\bar{u}_{xt} + 2u_x\bar{u}_t)]_0^1 ,$$

where

$$\mathrm{Re}(2(|u|^2)_xu_x+|u|^2u_{xx},u_t) = \mathrm{Im}(2(|u|^2)_xu_x+|u|^2u_{xx},(1-i\varepsilon)u_{xx}+k|u|^2u)$$
$$= \varepsilon(2\mathrm{Re}((|u|^2)_xu_x,u_{xx})+\|uu_{xx}\|^2)+2\mathrm{Im}[((|u|^2)_xu_x,u_{xx})+$$
$$+((|u|^2)_xu_x+\tfrac{1}{2}|u|^2u_{xx},\ k|u|^2u)$$
$$= \varepsilon(2\mathrm{Re}((|u|^2)_xu_x,u_{xx})+\|uu_{xx}\|^2)+2\mathrm{Im}((|u|^2)_xu_x,u_{xx})+k\mathrm{Im}[|u|^4u_x\bar{u}]_0^1$$

and

$$2\mathrm{Im}((|u|^2)_xu_x,u_{xx}) = 2\mathrm{Im}(u|u_x|^2 + u_x^2\bar{u},u_{xx})$$
$$= -2\mathrm{Im}((2u_xu_{xx}\bar{u}+u_x^2\bar{u}_x,\ u_x) - (u|u_x|^2,u_{xx})) + 2\mathrm{Im}[|u_x|^2\bar{u}\,u_x]_0^1$$
$$= 6\mathrm{Im}(u|u_x|^2,u_{xx}) + 2\mathrm{Im}[|u_x|^2\bar{u}\,u_x]_0^1$$
$$= 6\mathrm{Im}(u|u_x|^2,(1-i\varepsilon)u_{xx}+k|u|^2u)-6\varepsilon\mathrm{Re}(u|u_x|^2,u_{xx})+2\mathrm{Im}[|u_x|^2\bar{u}\,u_x]_0^1$$
$$= 3((|u|^2)_t,|u_x|^2) - 6\varepsilon\mathrm{Re}(u|u_x|^2,u_{xx}) + 2\mathrm{Im}[|u_x|^2\bar{u}\,u_x]_0^1 .$$

Hence we obtain

$$I_1 = 3(\|uu_x\|^2)_t + \tfrac{1}{2}(\|(|u|^2)_x\|^2)_t+\varepsilon(2\mathrm{Re}((|u|^2)_xu_x-3u|u_x|^2,u_{xx})+\|uu_{xx}\|^2)$$
$$-[2\mathrm{Re}(|u|^2(u\bar{u}_{xt}+2u_x\bar{u}_t)) - k\mathrm{Im}(|u|^4u_x\bar{u}) - 2\mathrm{Im}(|u_x|^2\bar{u}u_x)]_0^1 .$$

Next we have

$$I_2 = \mathrm{Im}(2(|u|^2u,u_{xxt}) - 3(u_{xx}|u|^2,u_t)) =$$
$$= \mathrm{Im}\{2[|u|^2u\bar{u}_{xt}]_0^1 - 2((|u|^2u)_x,u_{xt}) - 3(u_{xx}|u|^2,i((1-i\varepsilon)u_{xx}+k|u|^2u))\}$$
$$= \mathrm{Im}\{2[|u|^2u\,\bar{u}_{xt}]_0^1 - 2((|u|^2u)_x,u_{xt})\} + 3\|uu_{xx}\|^2 + 3k\mathrm{Re}(u_{xx}|u|^4,u) .$$

Combining these expressions we get

$$k\int_0^t (I_1+I_2)ds = k[3\|uu_x\|^2 + \tfrac{1}{2}\|(|u|^2)_x\|^2]_0^t + 2\varepsilon k\mathrm{Im}\int_0^t[|u|^2u\,\bar{u}_{xt}]_0^1 ds +$$

$$+ k\varepsilon\int_0^t\{ \mathrm{Re}(2((|u|^2)_x u_x - 3u(2|u_x|^2 - k|u|^4), u_{xx}) + 2(2\|uu_{xx}\|^2 - \mathrm{Im}((|u|^2u)_x, u_{xt}))\}\, ds -$$

$$k\int_0^t[\, 2\mathrm{Re}(|u|^2(u\bar{u}_{xt} + 2u_x\bar{u}_t)) - k\mathrm{Im}(|u|^4 u_x\bar{u}) - 2\mathrm{Im}(|u_x|^2\bar{u}u_x)\,]_0^1\, ds \tag{4.6}$$

$$= k[3\|uu_x\|^2 + \tfrac{1}{2}\|(|u|^2)_x\|^2]_0^t - \frac{\varepsilon k}{2}\sum_{j=0}^{1}\alpha_j|U_j + a_j|^4\,]_0^t + k\varepsilon\int_0^t\{\mathrm{Re}(2((|u|^2)_x u_x$$
$$- 3u(2|u_x|^2 - k|u|^4), u_{xx}) + 2(2\|uu_{xx}\|^2 - \mathrm{Im}((|u|^2)_x u + |u|^2 u_x, u_{xt}))\}\, ds$$
$$- k\sum_{j=0}^{1}\int_0^t\{2\mathrm{Im}\,\alpha_j|U_j + a_j|^2(3a_j - U_j)\bar{U}_{jt} - k\alpha_j|U_j + a_j|^4(|U_j|^2 - |a_j|^2)$$
$$- 2\alpha_j^3|U_j - a_j|^2(|U_j|^2 - |a_j|^2)\}\, ds\ .$$

Now we want to estimate this expression term by term. Firstly, it follows from $v_{0\varepsilon} \longrightarrow v_0$ in V that

$$\|u(0)\|_{H^2} = \|u_\varepsilon(0)\|_{H^2} = \|u_{0\varepsilon}\|_{H^2} = \|u_a + v_{0\varepsilon}\|_{H^2} \leqq \|u_a\|_{H^2} + c\|v_{0\varepsilon}\|_V \leqq c\ .$$

Hence we have

$$k(3\|uu_x\|^2 + \tfrac{1}{2}\|(|u|^2)_x\|^2 - \frac{\varepsilon}{2}\sum_{j=0}^{1}\alpha_j|U_j + a_j|^4)(0) \leqq c\ . \tag{4.7}$$

Since, because of (1.6) and (3.4),

$$\|u_x\| \leqq \|u_x - u_{ax}\| + \|u_{ax}\| + \|u_{ax}\| \leqq \delta\|u_{xx} - u_{axx}\| + \tfrac{1}{\delta}\|u - u_a\| + \|u_{ax}\| \leqq \delta\|u_{xx}\| + c, \tag{4.8}$$

we find that

$$\begin{aligned} k(3\|u_x u\|^2 + \tfrac{1}{2}\|(|u^2|)_x\|^2) &\leqq 5|k|\,\|uu_x\|^2 \leqq 5|k|\,\|u_x\|_\infty^2\|u\|^2 \\ &\leqq 5|k|\,\|u_x\|(\|u_x\| + 2\|u_{xx}\|)\,\|u\|^2 \leqq \tfrac{1}{8}\|u_{xx}\|^2 + c\ . \end{aligned} \tag{4.9}$$

Next, applying (4.4) and (4.8), we obtain for sufficiently small ε

$$k\varepsilon\int_0^t \mathrm{Re}(2(|u|^2)_x u_x - 3u(2|u_x|^2 - k|u|^4), u_{xx})\, ds$$
$$\leqq |k|\varepsilon\int_0^t (4\|u\|\,\|u_x\|_\infty^2 + 3\|u\|(2\|u_x\|_\infty^2 + |k|\,\|u\|_\infty^4))\,\|u_{xx}\|\, ds \tag{4.10}$$
$$\leqq |k|\varepsilon\int_0^t\|u\|(10\|u_x\|(\|u_x\| + 2\|u_{xx}\|) + 3|k|\,\|u\|^2(\|u\| + 2\|u_x\|)^2)\|u_{xx}\|\, ds$$
$$\leqq c(1 + \sqrt{\varepsilon}\|u_{xx}\|_{C(S;H)}^2) \leqq c + \tfrac{1}{8}\|u_{xx}\|_{C(S;H)}^2$$

and

$$\begin{aligned}
&2\,k\varepsilon\int_0^t (2\|uu_{xx}\|^2-\mathrm{Im}((|u|^2)_x u + |u|^2u_x,u_{xt}))\,ds \\
&\leq 2|k|\varepsilon\int_0^t (2\|u\|_\infty^2\|u_{xx}\|^2+3\|u\|_\infty^2\|u_x\|\|u_{xt}\|)\,ds \qquad (4.11)\\
&\leq 2|k|\,\varepsilon\int_0^t (2\|u\|(\|u\|+2\|u_x\|)\|u_{xx}\|^2+3\|u\|(\|u\|+2\|u_x\|)\|u_{xt}\|)\,ds
\end{aligned}$$

$$\leq c(1+\sqrt{\varepsilon}\|u_{xx}\|^2_{C(C;H)}) + 2\varepsilon\int_0^t\|u_{xt}\|^2ds \leq c + \tfrac{1}{8}\|u_{xx}\|^2_{C(S;H)}+2\varepsilon\int_0^t\|u_{xt}\|^2ds\ .$$

It remains to estimate the boundary terms. To this end we deduce from (2.1), (4.4) and (4.8) that for arbitrarily small $\delta>0$

$$\begin{aligned}
\|U_j\|^4_{C(S)} &\leq \|u-a_j\|^2_{C(S;H)}(\|u-a_j\|_{C(S;H)} + 2\|u_x\|_{C(S;H)}\ )^2\\
&\leq c(1 +\delta\|u_{xx}\|^2\ )
\end{aligned}$$

and

$$\alpha_j\int_0^t|U_j|^6ds \leq \alpha_j\|U_j\|^2_{L^2(S)}\|U_j\|^4_{C(S)} \leq c(1 +\delta\|u_{xx}\|^2_{C(S;H)})\ .$$

Thus we find

$$-\tfrac{\varepsilon}{2}k\sum_{j=0}^1\alpha_j|U_j(t)+a_j|^4\leq\varepsilon c(1+\sum_{j=0}^1\alpha_j|U_j(t)|^4)\leq c + \tfrac{1}{16}\|u_{xx}\|^2_{C(S;H)} \qquad (4.12)$$

and

$$\begin{aligned}
&-k\sum_{j=0}^1\int_0^1\{2\mathrm{Im}[\alpha_j|U_j+a_j|^2\ (3a_j-U_j)\bar{U}_{jt}]-k\alpha_j|U_j+a_j|^4(|U_j|^2-|a_j|^2)-\\
&2\alpha_j^3|U_j-a_j|^2(|U_j|^2-|a_j|^2)\}ds\leq c_1+\sum_{j=0}^1\int_0^t(c_2\alpha_j|U_j|^6+\alpha_j|U_{jt}|^2)ds \qquad (4.13)\\
&\leq c + \tfrac{1}{16}\|u_{xx}\|^2_{C(S;H)}+\sum_{j=0}^1\alpha_j\int_0^t|U_{jt}|^2\ ds\ .
\end{aligned}$$

Now from (4.5)-(4.7) and (4.9)-(4.13) we obtain

$$\sum_{j=0}^1\alpha_j\int_0^t|U_{jt}|^2ds+\|u_{xx}(t)\|^2 + \|u(t)\|_6^6 \leq \tfrac{1}{2}\|u_{xx}\|^2_{C(S;H)} + c.$$

Hence the desired second a priori estimate

$$\|u_\varepsilon\|^2_{C(S;H^2)} + \|u_\varepsilon\|^6_{C(S;L^6)} + \sum_{j=0}^1\alpha_j\|U_{\varepsilon jt}\|^2_{L^2(S)} \leq c \qquad (4.14)$$

follows. Via (4.3) we still get

$$\|u_{\varepsilon t}\|_{C(S;H)} \leq c\ . \qquad (4.15)$$

According to a well known compactness lemma (cf. [6], Chap. I, Th. 1.5) (4.14) and (4.15) imply the precompactness of the set (u_ε) in $L^2(S;H^1)$. Consequently, there exist a sequence (ε_n) tending to zero as $n \to \infty$ and a function $u \in L^2(S;H^2)$ with $u_t \in L^2(S;H)$ and $U_j = u(\cdot,j) - a_j \in H^1(S)$, $j=0,1$, such that the sequence $(u_n) = (u_{\varepsilon_n})$ satisfies

$$\begin{aligned} &u_n \to u \quad \text{(strongly) in} \quad L^2(S;H^1), \\ &u_n \rightharpoonup u \quad \text{(weakly) in} \quad L^2(S;H^2), \\ &u_{nt} \rightharpoonup u_t \text{ in } L^2(S;H), \quad U_{nj} \rightharpoonup U_j \quad \text{in} \quad L^2(S). \end{aligned} \tag{4.16}$$

Now we want to show that u is solution of (4.1), (4.2). From the first relation in (4.16) it follows that $|u_n|^2 u_n \to |u|^2 u$ in $L^2(S;H)$. Thus we can pass to the limit $\varepsilon_n \to 0$ in (4.3) and obtain (4.1). Further, $u \in L^2(S;H^2)$ and $u_t \in L^2(S;H)$ imply $u \in C(S;H^1)$. Therefore by (4.14) we see (cf.[8]) that u belongs to $C_w(S;H^2)$ and satisfies the boundary conditions (4.2). Then the inclusion $u_t \in C_w(S;H)$ is a consequence of (4.1).

In order to show that even $u \in C(S;H^2)$ and $u_t \in C(S;H)$ we adapt an idea of the paper [8]. We extend u by setting $u(t)=u(0)=u_o$ for $t<0$. Let $r=r_\varepsilon=r_\varepsilon(t)$ be an appropriate even smoothing kernel (cf. [8,9]) and

$$(r_\varepsilon * u)(t) = \int r_\varepsilon(t-s)u(s)\, ds \quad , \quad \int = \int_{-\infty}^{\infty}$$

Further let $h = h_\delta = h_\delta(s)$ be $\{1$ for $s \in [\delta, t-\delta]$, 0 for $s \notin [0,t]$ and linear in the intervals $[0,\delta]$ and $[t-\delta,t]\}$.

We set $q=q_\gamma = r_\gamma$ and $v=r*(h(q*u)_t)=r*(h(q'*u))=r*(h(q*u_t))$.

From the evident relations

$$0 = i\int (v,v)_t \, ds = 2\mathrm{Im}\int i(v_t,v)\, ds ,$$

$$0 = -2\mathrm{Im}\int (v_x,v_x)\, ds = 2\mathrm{Im}\int \left((v_{xx},v) - \left[v_x\bar{v}\right]_0^1\right) ds$$

we deduce

$$0=2\mathrm{Im}\int((iv_t+v_{xx},v)-\left[v_x\bar v\right]_0^1)ds=2\mathrm{Im}\int\{(r'*(h(q_\gamma*(iu_t+u_{xx})))-$$

$$r*(h'(q_\gamma*u_{xx})),\ r*(h(q_\gamma*u_t)))-i\sum_{j=0}^{1}\alpha_j|r*(h(q_\gamma*U_{jt}))|^2\}ds\ .$$

Letting $\gamma\to 0$ and using $u_t,u_{xx}\in C_w(S;H)$, $U_{jt}\in L^2(S)$, we find

$$0=2\mathrm{Im}\int\{(r'*(h(iu_t+u_{xx}))-r*(h'u_{xx}),\ r*(hu_t))-i\sum_{j=0}^{1}\alpha_j|r*(hU_{jt})|^2\}ds$$

$$=2\mathrm{Im}\int\{(r*(h'(iu_t+u_{xx}))+r*(h(iu_t+u_{xx})_t)-r*(h'u_{xx})\ ,\ r*(hu_t))-$$

$$i\sum_{j=0}^{1}\alpha_j|r*(hU_{jt})|^2\}ds\ =\ 2\mathrm{Im}\int\{(ir*(h_\delta' u_t)\ -\ r*(h_\delta(k|u|^2u)_t)\ ,$$

$$r*(h_\delta u_t))\ -\ i\sum_{j=0}^{1}\alpha_j|r*(h_\delta U_{jt})|^2\}ds\ .$$

Next we let $\delta\to 0$. Since $|u|^2u\in H^1(S;H)$ it follows (cf. [8])

$$0=2\mathrm{Im}\left[(iu_t,r_\varepsilon*r_\varepsilon*(h_ou_t))\right]_t^0\ -\ 2\mathrm{Im}\int\{(r_\varepsilon*(h_o(k|u|^2u)_t),\ r*(h_ou_t))+$$

$$+\ i\sum_{j=0}^{1}\alpha_j|r_\varepsilon*(h_oU_{jt})|^2\}ds.$$

Finally, letting $\varepsilon\to 0$, we see that (cf. [8])

$$\|u_t(t)\|^2\ =\ \|u_t(0)\|^2-2\int_0^t\{\mathrm{Im}(k(|u|^2u)_t,\ u_t)\ +\ \sum_{j=0}^{1}\alpha_j|U_{jt}|^2\}ds.$$

Because of $u_t\in C_w(S;H)$ this equation implies $u_t\in C(S;H)$. Now the remaining inclusion $u_{xx}\in C(S;H)$ is a consequence of (4.1).

Theorem 4.1 is proved.

5. GALERKIN'S METHOD

In this section we establish Galerkin's method as a procedure to solve the problems (3.1), (3.2) and (4.1), (4.2) numerically. We look for approximative solutions of the form

$$u_n=u_n(t)=u_a+\sum_{l=0}^{n}b_l(t)h_l,\ u_n(0)=u_{no}=u_a+\sum_{l=0}^{n}\beta_lh_l,\ \beta_l=(v_o,h_l)\ . \tag{5.1}$$

Here h_l, u_a and v_o are the functions given by (2.4) and (3.4). A function u_n having the form (5.1) is said to be the n-th Galerkin approximation of the solution u of (3.1), (3.2) if the (complex-valued) coefficient functions b_l solve the initial value problem

$$(iu_{nt}+k_1u_{nxx}+(k_2+k_3|P_ru_n|^2P_ru_n,h_l)=,\ b_l(0)=\beta_l,\ l=0,\dots,n. \tag{5.2}$$

Here P_r ist the operator defined by (3.6) and r is an arbitrary bound for $\max|u(t,x)|$, $t\in S$, $x\in[0,1]$.

REMARK 5.1. We can get a suitable bound r by calculating explicitly the constant c in (3.15).

REMARK 5.2. By introducing the operator P_r in (5.2) we have slightly modified the usual Galerkin rule. We had to do so because we could not find $C(S;H^1)$-a priori estimates for the classical Galerkin approximations.

REMARK 5.3. In order to solve (5.2) numerically one can introduce the functions $v_n=e^{-iP}u_n=e^{-iP}u_a+\sum_{l=0}^{n} b_l\cos l\pi x$ and rewrite (5.2) as follows

$$\begin{aligned}\dot b_l+ik_1\lambda_lb_l=(i(k_2+k_3|P_rv_n|^2)P_rv_n+i(e^{-iP}u_a)_{xx}-k_1(2pv_{nx}+\\ +(p'+ip^2)v_n),\ \cos l\pi x)\ ,\quad \dot b_l=\tfrac{d}{dt}b_l,\quad l=0,\dots,n\ ,\\ b_l(0)=\beta_l\ .\end{aligned} \tag{5.2'}$$

THEOREM 5.1. Let the assumptions of Theorem 3.1 be satisfied. Let (u_n) be the Galerkin sequence given by (5.1), (5.2) and let u be the solution of (3.1), (3.2). Then

$$u_n\to u \text{ in } C(S;H^2),\quad u_{nt}\to u_t \text{ in } L^2(S;H^1) \text{ and } C(S;H)\ . \tag{5.3}$$

PROOF. We can regard the function $v_n=u_n-u_a$ as n-th Galerkin approximation of the solution of problem (3.11). Therefore, taking into account (3.12), (3.13) and the relation $C_rv_o\in H^1$, the theorem follows from [4], Satz 2.3.

COROLLARY 5.1. Let $U_{nj}(t)=u_n(t,j)-a_j$ and $U_j(t)=u(t,j)-a_j$, $j=0,1$. Then

$$U_{nj} \to U_j \quad \text{in} \quad C(S) \;, \quad (U_{nj})_t \to (U_j)_t \quad \text{in} \quad L^2(S) \;.$$

PROOF. The assertions follow immediately from (5.3) and (2.1).

Now we turn to Galerkin's method for the undamped problem (4.1), (4.2). A function u_n of the form (5.1) is said to be the n-th Galerkin approximation of the problem (4.1), (4.2) if the functions b_l solve the following initial value problem

$$(iu_{nt}+u_{nxx}+k|P_r u_n|^2 P_r u_n, h_l)=0, \quad b_l(0)=\beta_l \;, \quad l=0,1,\dots,n. \tag{5.4}$$

Here again P_r is the operator from (3.6), r is an arbitrary bound for $\max|u(t,x)|$, $t\in S$, $x\in[0,1]$. (The existence of such a bound is guaranteed by (4.14).)

REMARK 5.4. Introducing $v_n=e^{-iP}u_a+\sum_{l=0}^{n} b_l \cos l\pi x$, we can write (5.4) in the form

$$\dot b_l+i\lambda_l b_l=(ik|P_r v_n|^2 P_r v_n+i(e^{-iP}u_a)_{xx}-2pv_{nx}-(p'+ip^2)v_n, \cos l\pi x) \;,$$
$$b_l(0)=\beta_l \;, \quad l=0,1,\dots,n \;,$$

which is more convenient for numerical purpose.

THEOREM 5.2. Suppose $\alpha_j\geq 0$, $j=0,1$, $v_o=u_o-u_a\in V$. Let (u_n) be the Galerkin sequence given by (5.1), (5.4) and let u be the solution of (4.1), (4.2). Set $U_{nj}=u_n(t,j)-a_j$, $U_j(t)=u(t,j)-a_j$, $j=0,1$. Then

$$u_n \to u \quad \text{in} \quad C(S;H) \;, \quad \sqrt{\alpha_j}\, U_{nj} \to \sqrt{\alpha_j}\, U_j \quad \text{in} \quad L^2(S) \;.$$

PROOF. We write $w_n=u_a+\sum_{l=0}^{n}(u-u_a,h_l)h_l$. Now from Lemma 2.3, $u_o-u_a\in V$ and Theorem 4.1 it follows that

$$u_{no} \to u_o \quad \text{in} \quad H^2, \quad w_n \to u \quad \text{in } L^2(S;H^2) \text{ and } \; C(S;H^1) \;,$$

$$w_{nt} \to u_t \quad \text{in} \quad L^2(S;H) . \tag{5.5}$$

Setting $q_n = u - u_n$, $Q_{nj} = (\cdot,j) - u_n(\cdot,j)$, $z_n = w_n - u$, $Z_{nj} = w_n(\cdot,j) - u(\cdot,j)$, we conclude from (5.1) and (5.4) that

$$\begin{aligned} 0 &= 2\mathrm{Im}\int_0^t (iq_{nt} + q_{nxx} + k(|u|^2u - |P_ru_n|^2 P_ru_n),\ q_n + z_n)\,ds \\ &= \|q_n(t)\|^2 - \|q_n(0)\|^2 + \sum_{j=0}^{1}\int_0^t \alpha_j |Q_{nj}|^2 + 2\mathrm{Im}\int_0^t \{k(|P_ru|^2 P_r u - \\ &\quad |P_ru_n|^2 P_ru_n\ ,\ q_n + z_n) - i(q_n, z_{nt}) + 2i\sum_{j=0}^{1}\alpha_j Q_{nj}\bar{Z}_{nj} + (q_n z_{nxx})\}\,ds + \\ &\quad 2\mathrm{Re}\left[(q_n(t), z_n(t)) - (q_n(0), z_n(0))\right]. \end{aligned}$$

Using (3.8) (for $k_2=0$, $k_3=k$) and (5.5) we deduce from this equation the theorem.

REMARK 5.5. The proved convergence of the boundary values $u_n(t,j)$ is of some physical interest because they represent the reflexion and transmission properties of the plasma layer described by (4.1), (4.2).

REFERENCES

1. Andreev, N. E. and K. Sauer, Zeitliche Entwicklung des Reflexionskoeffizienten einer Plasmaschicht bei starker elektromagnetischer Einstrahlung, Akad. d. Wiss. d. DDR, Zentralinstitut f. Elektronenphysik, Preprint 77-4.
2. Baillon, J. and T. Caneve, M. Figueira, Equation de Schrödinger non linéaire, C. r. Acad. Sc. Paris, Ser. A, 284, 869-872(1977).
3. Gajewski, H., Über Näherungsmethoden zur Lösung der nichtlinearen Schrödinger,Gleichung, Math. Nachr. (to appear).

4. Gajewski, H. and K. Gröger, Ein Projektions-Iterationsverfahren für Evolutionsgleichungen, Math. Nachr. 72, 119-136(1976).

5. Gröger, K., Regularitätsaussagen für Evolutionsgleichungen mit stark monotonen Operatoren, Math. Nachr. 67, 21-34(1975).

6. Lions, J. L., Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod Gauthier-Villars, Paris 1969.

7. Pereira, N. R., L. Stenflo, Nonlinear Schrödinger equation including growth and damping, The Physics of Fluids 20, 1733-1735(1977).

8. Strauss, W. A., On continuity of functions with values in various Banach spaces, Pacific J. Math. 19, 543-551(1966).

9. Triebel, H., Höhere Analysis, Dt. Verl. d. Wiss. Berlin 1972.

10. Zakharov, V. E., A. B. Shabat, Exact theory of two-dimensional self-focussing and one-dimensional self-modulation of waves in nonlinear media, Jnl. Exp. Theor. Phys. (JETP) 61, 118-134(1971) (uss.).