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RESEARCH NOTES

ALTERNATIVE INTEGRATION PROCEDURE FOR SCALE-INVARIANT ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. For an ordinary differential equation invariant under a one-parameter group of scale transformations $x \to \lambda x$, $y \to \lambda^{\alpha} y$, $y' \to \lambda^{\alpha-1} y'$, $y'' \to \lambda^{\alpha-2} y''$, etc., it is shown by example that an explicit analytical general solution may be obtainable in parametric form in terms of the scale-invariant variable $\zeta \equiv \int^{x} y^{-1/\alpha}\, dx$. This alternative integration may go through, as it does for the example equation $y'' = kxy^{-2}y'$, in cases for which the customary dependent and independent variables $(x^{-\alpha}y)$ and $(\ell n\ x)$ do not yield an analytically integrable transformed equation.

KEY WORDS AND PHRASES. Ordinary differential equations, Integration method, Scale-invariance.

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1. INTRODUCTION.

Consider an ordinary differential equation

$$F(x,y,y',y'',\dots) = 0 \qquad (y' \equiv dy/dx,\ y'' \equiv d^2y/dx^2,\dots) \tag{1.1}$$

that is invariant under a one-parameter group of scale transformations with $F(\lambda x,\lambda^{\alpha}y,\ \lambda^{\alpha-1}y',\ \lambda^{\alpha-2}y'',\dots) \equiv \lambda^{\beta}\,F(x,y,y',y'',\dots)$ for certain prescribed real constants α, β and all real $\lambda > 0$. The customary procedure [1] for reducing the order of such an equation, and thus commonly facilitating its solution, involves the introduction of new dependent and independent variables $\eta \equiv x^{-\alpha}y$ and $\xi \equiv \ell n\ x$. For example, in the case of the scale-invariant ($\alpha = 1$, $\beta = 1$) nonlinear second-order equation.

$$y^2y'' - kxy' = 0 \qquad (k = \text{const}) \tag{1.2}$$

or, equivalently, its inversed form $y^2(d^2x/dy^2) = -\,kx(dx/dy)^2$ with $x = x(y)$, introduction of the variables $\xi \equiv \ell n\ x$, $\eta \equiv x^{-1}y = \eta(\xi)$ yields [1]

$$\eta^2\,\frac{d^2\eta}{d\xi^2} + (\eta^2 - k)\,\frac{d\eta}{d\xi} - k\eta = 0\ , \tag{1.3}$$

which is equivalent to a first-order equation in $p \equiv d\eta/d\xi = p(\eta)$; however, there is no analytical method for obtaining an integrating factor for the resulting first-order equation, and thus the customary procedure is ineffectual for solving (1.2) by analytical means.

In such cases for which the scale-invariant equation is not amenable to solution by the customary method, an analytical general solution may possibly be obtainable in parametric form in terms of the scale-invariant variable

$$\zeta \equiv \int_{\text{const}}^{x} y^{-1/\alpha}\,dx\ . \tag{1.4}$$

To see how this works, consider the $\alpha = 1$ equation (1.2) with $x = x(\zeta)$ and

$y = dx/d\zeta \equiv \dot{x}$, according to (1.4). Then $y' = (dx/d\zeta)^{-1}(dy/d\zeta) \equiv y^{-1}\dot{y}$ and (1.2) becomes

$$y\,\dot{y}^{-1}\,\ddot{y} - \dot{y} - kx = 0\ . \tag{1.5}$$

Differentiating (1.5) with respect to ζ and dividing out a common factor of y yields

$$\dot{y}^{-1}\,\dddot{y} - \dot{y}^{-2}(\ddot{y})^2 - k = 0 \tag{1.6}$$

with the immediate integral

$$\dot{y}^{-1}\,\ddot{y} = k\zeta\ . \tag{1.7}$$

A trivial constant of integration, associated with the arbitrariness in the constant lower limit of integration in (1.4), is omitted from the right side of (1.7). Integrating the latter equation twice yields

$$y = A\int_0^\zeta (\exp\tfrac{1}{2}kv^2)dv + B \qquad (A,\ B = \text{const})\ . \tag{1.8}$$

Finally, by putting (1.8) into the integral formula $x = \int^\zeta y\,d\zeta$, one obtains

$$x = A\left[\zeta\int_0^\zeta (\exp\tfrac{1}{2}kv^2)dv - k^{-1}(\exp\tfrac{1}{2}k\zeta^2)\right] + B\zeta\,. \tag{1.9}$$

Since there is no way of getting $y = y(x)$ explicitly by eliminating ζ from the general solution to (1.2) given parametrically by (1.8) and (1.9), it is clear why the customary solutional procedure based on (1.3) is ineffectual. The potency of this alternative integration procedure stems in part from the greater generality of parametric representations.

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