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ONE-SIDED COMPLEMENTS AND SOLUTIONS OF THE EQUATION $aXb = c$ IN SEMIRINGS

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Given multiplicatively-regular elements a and b in a semiring R, and given an element c of R, we find a complete set of solutions to the equation $aXb = c$. This result is then extended to equations over matrix semirings.

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1. Semirings. We follow the notation and terminology of [5], to which the reader is referred for all undefined notions and unproven assertions. Let R be a semiring. An element a is *multiplicatively regular* if and only if there exists an element a^- of R, called a *generalized inverse* of a, satisfying $aa^-a = a$. If such an element exists then the element $a^\times = a^-aa^-$ satisfies the conditions $aa^\times a = a$ and $a^\times aa^\times = a^\times$. We call the element $a^\times$ of R a *Thierrin-Vagner inverse* of a. The details are given in [5].

If a is multiplicatively idempotent then it has a Thierrin-Vagner inverse and, indeed, we can choose $a^\times = a$. Thus we can always assume that $0^\times = 0$ and $1^\times = 1$. If a has a multiplicative inverse, we can choose $a^\times = a^{-1}$. If R is a semifield we see that every element is multiplicatively regular. This happens, for example, in such important and applicable semirings as the schedule algebra $(\mathbb{R} \cup \{-\infty\}, \max, +)$.

Regularity in fuzzy matrix rings is studied in [2]. For algorithms to calculate Moore-Penrose pseudoinverses of matrices over additively-idempotent semirings, which are special cases of Thierrin-Vagner inverses, refer to [7]. Also refer to [3] for calculation of generalized inverses for semirings of matrices over bounded distributive lattices.

We note too that if $a \in R$ is multiplicatively regular then so is $a^\times$ and so are $a^\times a$ and $aa^\times$, and indeed $(a^\times a)^\times = a^\times a$ and $(aa^\times)^\times = aa^\times$. Moreover, both of these elements are multiplicatively idempotent. Thus we have two functions from the set of all multiplicatively-regular elements of R to the set $I^\times(R)$ of all multiplicatively-idempotent elements of R given by $\lambda : a \mapsto a^\times a$ and $\rho : a \mapsto aa^\times$ and these functions satisfy $\lambda^2 = \lambda$ and $\rho^2 = \rho$. Moreover, for each $a \in R$ we have

$$a\lambda(a) = a = \rho(a)a,$$
$$\lambda(a^\times)a^\times = a^\times = a^\times\rho(a^\times). \tag{1.1}$$

We are interested in the following problem: *given multiplicatively-regular elements $a, b \in R$ and given an element $c \in R$, find a complete set of solutions to the equation $aXb = c$ in R.* Such problems arise in various contexts—for example in the theory of formal codes [1] or in the context of rewriting systems and similar problems in formal language theory. Also see [9]. They also appear in the consideration of fuzzy and semiring-valued relations [4] and fuzzy bilinear equations [8], and arise naturally in control theory with coefficients taken from the $(\max, +)$ algebra or from the semiring of fuzzy numbers. For certain noncommutative rings, such as rings of matrices or rings of operators over a linear space, they have an extensive literature, and the results there can often be extended to matrix semirings over semirings, for example.

Note that if there exists a solution x to the equation

$$aXb = c, \tag{1.2}$$

then

$$c = axb = \rho(a)(axb)\lambda(b) = \rho(a)c\lambda(b). \tag{1.3}$$

Conversely, if $c \in R$ satisfies $\rho(a)c\lambda(b) = c$, then $a^{\times}cb^{\times}$ is a solution for (1.2). Thus (1.2) has a nonempty set of solutions if and only if c satisfies this condition. This allows us to rephrase our problem as follows: *given multiplicatively regular elements $a, b \in R$ and given an element $c \in R$ satisfying $\rho(a)c\lambda(b) = c$, find a complete set of solutions of (1.2) in R.*

Let a be an element of a semiring R. An element $a^{[r]}$ of R is called a *right complement* of a if and only if $aa^{[r]} = 0$ and $a + a^{[r]} = 1$. An element $a^{[l]}$ of R is a *left complement* of a if and only if $a^{[l]}a = 0$ and $a^{[l]} + a = 1$. If a has both a right complement $a^{[r]}$ and a left complement $a^{[l]}$, then these must be equal. Indeed, we note that in this case

$$\begin{aligned} a^{[l]} &= a^{[l]}(a + a^{[r]}) = a^{[l]}a + a^{[l]}a^{[r]} = a^{[l]}a^{[r]} \\ &= aa^{[r]} + a^{[l]}a^{[r]} = (a + a^{[l]})a^{[r]} = a^{[r]}. \end{aligned} \tag{1.4}$$

Such an element is called a *complement* of a and is denoted by $a^{\perp}$. Complements, when they exist, are necessarily unique.

Example 1.1. Right and left complements need not be the same. For example, let S be the ring of all upper-triangular matrices over the ring $\mathbb{Z}$ of integers, and let R be the semiring $\text{ideal}(S)$ consisting of S and of all (two-sided) ideals of S. The operations on R are the usual addition and multiplication of ideals. If $I = \left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right]$ and $H = \left[\begin{smallmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{smallmatrix}\right]$ then it is easy to verify that $H = I^{[l]}$ but $H \neq I^{[r]}$.

Complements of elements of a semiring are studied in [5, Chapter 5]; they play a very important role in the theory and applications of semirings. Since the inspiration for complements came from lattice theory, they were assumed to be two-sided. However, here we have to look at the notion of a one-sided complement.

Note that if $a \in R$ has a right complement then $a \in I^{\times}(R)$ since

$$a = a1 = a\left(a + a^{[r]}\right) = a^2 + aa^{[r]} = a^2 \tag{1.5}$$

and the same is, of course, true if a has a left complement. Thus, if we denote the set of all elements of R having a right (resp., left) complement by $\text{rcomp}(R)$ (resp., $\text{lcomp}(R)$), and if we denote the set of all elements of R having a complement by $\text{comp}(R)$, we see that

$$\text{rcomp}(R) \cap \text{lcomp}(R) = \text{comp}(R), \tag{1.6}$$

and if we denote the set of all elements of R having a one-sided complement by $\text{ocomp}(R)$, that is, $\text{ocomp}(R) = \text{rcomp}(R) \cup \text{lcomp}(R)$, then we see that

$$\text{ocomp}(R) \subseteq I^{\times}(R). \tag{1.7}$$

Also, we note that if $a \in \text{rcomp}(R)$ then any right complement $a^{[r]}$ of a belongs to $\text{lcomp}(R)$ and, indeed, a itself is a left complement of $a^{[r]}$. Similarly, if $a \in \text{lcomp}(R)$ then any left complement of a belongs to $\text{rcomp}(R)$. Thus we see that $\text{ocomp}(R)$ is closed under taking left and right complements.

Note that if $\gamma : R \to S$ is a morphism of semirings, then $\gamma(\text{ocomp}(R)) \subseteq \text{ocomp}(S)$. Indeed, if $a \in R$ has a right complement $a^{[r]}$ then $0_S = \gamma(0_R) = \gamma(aa^{[r]}) = \gamma(a)\gamma(a^{[r]})$ and $1_S = \gamma(1_R) = \gamma(a + a^{[r]}) = \gamma(a) + \gamma(a^{[r]})$ so $\gamma(a^{[r]})$ is a right complement of $\gamma(a)$. Similarly, if a has a left complement $a^{[l]}$ then $\gamma(a^{[l]})$ is a left complement of $\gamma(a)$.

Assume that a and b are multiplicatively-regular elements of R such that $\lambda(a)$ has a right complement $\lambda(a)^{[r]}$ and that $\rho(b)$ has a left complement $\rho(b)^{[l]}$. Then we note that $a\lambda(a)^{[r]} = \rho(a)a\lambda(a)^{[r]} = a\lambda(a)\lambda(a)^{[r]} = 0$ and $\rho(b)^{[l]}b = \rho(b)^{[l]}b\lambda(b) = \rho(b)^{[l]}\rho(b)b = 0$.

Given an element c of R, define a function $\alpha_c : R \to R$ by setting

$$\alpha_c : y \longmapsto a^{\times}cb^{\times} + \lambda(a)y\rho(b)^{[l]} + \lambda(a)^{[r]}y. \tag{1.8}$$

Then the foregoing discussion leads us to the following result.

Proposition 1.2. *If a and b are multiplicatively-regular elements of a semiring R satisfying the condition that $\lambda(a) \in \text{rcomp}(R)$ and $\rho(b) \in \text{lcomp}(R)$, and if c is an element of R satisfying $\rho(a)c\lambda(b) = c$, then a complete set of solutions of (1.2) is given by $\{\alpha_c(y) \mid y \in R\}$. If c does not satisfy this condition then (1.2) has no solutions in R.*

Proof. If c does not satisfy the given condition then we have already seen that (1.2) has no solutions in R. Assume therefore that it does. From the hypothesis of the theorem we then see that

$$\begin{aligned} a\alpha_c(y)b &= \rho(a)c\lambda(b) + \rho(a)ay\rho(b)^{[l]}b + a\lambda(a)^{[r]}yb \\ &= \rho(a)c\lambda(b) \\ &= c, \end{aligned} \tag{1.9}$$

so $\alpha_c(y)$ is a solution to (1.2) for any $y \in R$. Moreover, we note that if $x \in R$ is a solution of (1.2) then $\alpha_c(x) = x$. Indeed, if $axb = c$ then

$$\begin{aligned} \alpha_c(x) &= a^{\times}cb^{\times} + \lambda(a)x\rho(b)^{[l]} + \lambda(a)^{[r]}x \\ &= \lambda(a)x\rho(b) + \lambda(a)x\rho(b)^{[l]} + \lambda(a)^{[r]}x \\ &= \lambda(a)x\left[\rho(b) + \rho(b)^{[l]}\right] + \lambda(a)^{[r]}x \\ &= \lambda(a)x + \lambda(a)^{[r]}x \\ &= \left[\lambda(a) + \lambda(a)^{[r]}\right]x \\ &= x \end{aligned} \tag{1.10}$$

and the proof is complete. □

In particular, we have the following examples.

Example 1.3. Suppose that R is a semiring. If a and b are multiplicatively-regular elements of R satisfying the condition that both $\lambda(a)$ and $\rho(b)$ have additive inverses, then we can set $\lambda(a)^{[r]} = 1 - \lambda(a)$ and $\rho(b)^{[l]} = 1 - \rho(b)$. In this case, both $\lambda(a)$ and $\rho(b)$ in fact belong to $\mathrm{comp}(R)$. This surely happens if R is a ring.

Example 1.4. Suppose that R is a Boolean algebra. If a and b are multiplicatively-regular elements of R, we can set $\lambda(a)^{[r]} = a'$ and $\rho(b)^{[l]} = \rho(b)'$.

Example 1.5. Following the terminology of [5], we say that a semiring R is *plain* if and only if $a + b = b$ for $a, b \in R$ implies that $a = 0$. It is *simple* if and only if $a + 1 = 1$ for all $a \in R$, and it is *yoked* if for each pair a, b of elements of R there exists an element c of R satisfying $a + c = b$ or $b + c = a$. By [5, Example 5.6] we see that every multiplicatively-idempotent element of a plain simple yoked semiring has a complement and so, for such semirings, $\lambda(a)^{[r]}$ and $\rho(b)^{[l]}$ exist for all multiplicatively-regular elements a and b of R.

Among the most applicable families of semirings which are not rings are *zerosumfree* semirings, namely semirings which satisfy the condition that $a + b = 0$ when and only when $a = b = 0$. Bounded distributive lattices are examples of such semirings, as are semirings of (two-sided) ideals of rings and information algebras in the sense of [6]. We make some remarks concerning the behavior of one-sided complements in such semirings.

Proposition 1.6. *If R is a zerosumfree semiring and if $a \in \mathrm{rcomp}(R)$ while $b \in \mathrm{ocomp}(R)$ then $aba^{[r]} = 0$.*

Proof. Indeed, if b' is a one-sided complement of b then

$$aba^{[r]} + ab'a^{[r]} = a(b + b')a^{[r]} = aa^{[r]} = 0, \tag{1.11}$$

and so $aba^{[r]} = 0$ since R is zerosumfree. □

Similarly, if $a \in \mathrm{lcomp}(R)$ while $b \in \mathrm{ocomp}(R)$ then $a^{[l]}ba = 0$.

Proposition 1.7. *If R is a zerosumfree semiring and if $a, b \in \mathrm{rcomp}(R)$ then $a + a^{[r]}b \in \mathrm{rcomp}(R)$.*

Proof. Indeed, we note that $a + a^{[r]}b + a^{[r]}b^{[r]} = a + a^{[r]}(b + b^{[r]}) = a + a^{[r]} = 1$ while $(a + a^{[r]}b)a^{[r]}b^{[r]} = a^{[r]}ba^{[r]}b^{[r]}$. But we have already seen that $a^{[r]} \in \mathrm{ocomp}(R)$ so, by Proposition 1.6, $ba^{[r]}b^{[r]} = 0$. Thus $a^{[r]}b^{[r]}$ is a right complement of $a + a^{[r]}b$. □

Similarly, we note that if $a, b \in \mathrm{lcomp}(R)$ then $a + ba^{[l]} \in \mathrm{rcomp}(R)$.

Proposition 1.8. *If R is a zerosumfree semiring and if $a, b \in \mathrm{rcomp}(R)$ then $ab \in \mathrm{rcomp}(R)$. Moreover, if $\mathrm{rcomp}(R)$ is closed under sums then every element of $\mathrm{rcomp}(R)$ is additively idempotent.*

Proof. Indeed, we note that $ab + (a^{[r]} + ab^{[r]}) = a(b + b^{[r]}) + a^{[r]} = a + a^{[r]} = 1$ and $(ab)(a^{[r]} + ab^{[r]}) = aba^{[r]} + a(bab^{[r]})$ and this equals 0, as we have already noted.

Now assume that $\mathrm{rcomp}(R)$ is closed under sums. Then, in particular, $1+1 \in \mathrm{rcomp}(R)$ so, if $a \in \mathrm{rcomp}(R)$ we see that $a+a = a(1+1) \in \mathrm{rcomp}(R)$. Let b be a right complement of $a+a$. Then $ab+ab = (a+a)b = 0$ and, by zerosumfreeness, we deduce that $ab = 0$. Therefore $a = a1 = (a+a+b) = a^2+a^2 = a+a$, showing that a is additively idempotent. □

Similarly, we note that if $a,b \in \mathrm{lcomp}(R)$ then $ab \in \mathrm{lcomp}(R)$ and if $\mathrm{lcomp}(R)$ is closed under sums then each of its members is additively idempotent.

2. Semimodules over matrix semirings. If R is a semiring then so is the set $\mathcal{M}_{n\times n}(R)$ of all $n\times n$ matrices over R, with addition and multiplication defined in the standard manner. We denote the additive identity in $\mathcal{M}_{n\times n}(R)$ by $O_{n\times n}$ and the multiplicative identity in $\mathcal{M}_{n\times n}(R)$ by $I_{n\times n}$. Moreover, if k and n are positive integers then the set $\mathcal{M}_{k\times n}(R)$ of all $k\times n$ matrices over R is canonically a left semimodule over $\mathcal{M}_{k\times k}(R)$ and a right semimodule over $\mathcal{M}_{n\times n}(R)$. We denote the additive identity in $\mathcal{M}_{k\times n}(R)$ by $O_{k\times n}$. Furthermore, if $A \in \mathcal{M}_{k\times n}(R)$ and $B \in \mathcal{M}_{n\times k}(R)$, then the products $AB \in \mathcal{M}_{k\times k}(R)$ and $BA \in \mathcal{M}_{n\times n}(R)$ are defined in the usual manner. A *generalized inverse* of $A \in \mathcal{M}_{k\times n}(R)$ is a matrix $A^- \in \mathcal{M}_{n\times k}(R)$ satisfying $AA^-A = A$. If such a generalized inverse exists, then A is multiplicatively regular. Again, if A is multiplicatively regular then the *Thierrin-Vagner inverse* of A is defined to be $A^\times = A^-AA^- \in \mathcal{M}_{n\times k}(R)$ and this matrix satisfies $AA^\times A = A$ and $A^\times AA^\times = A^\times$. If $A \in \mathcal{M}_{k\times n}(R)$ is regular then, as before, we define the matrices $\lambda(A) = A^\times A \in \mathcal{M}_{n\times n}(R)$ and $\rho(A) = AA^\times \in \mathcal{M}_{k\times k}(R)$.

Example 2.1. Consider the special case of $A = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \in \mathcal{M}_{k\times 1}(R)$. Then A has a generalized inverse $A^- = [b_1,\dots,b_k]$ if and only if the element $e = \sum_{i=1}^{k} b_i a_i$ of R satisfies $a_i e = a_i$ for all $1 \le i \le k$.

Given $A \in \mathcal{M}_{k\times n}(R)$ and $B \in \mathcal{M}_{n\times k}(R)$ having generalized inverses, and given $C \in \mathcal{M}_{k\times k}(R)$, we then note, as above, that whenever there exists a matrix $T \in \mathcal{M}_{n\times n}(R)$ satisfying $ATB = C$ we have

$$C = ATB = AA^\times ATBB^\times B = (AA^\times)C(B^\times B) = \rho(A)C\lambda(B). \tag{2.1}$$

A matrix $A \in \mathcal{M}_{k\times n}(R)$ is *right regularly complemented* if and only if it has a generalized inverse $A^- \in \mathcal{M}_{n\times k}(R)$ and there exists a multiplicatively-regular matrix $A^{[r]} \in \mathcal{M}_{n\times n}(R)$ satisfying the conditions $AA^{[r]} = O_{k\times n}$ and $A^\times A + A^{[r]} = I_{n\times n}$. Similarly, $B \in \mathcal{M}_{n\times k}(R)$ is *left regularly complemented* if and only if it has a generalized inverse $B^- \in \mathcal{M}_{k\times k}(R)$ and there exists a multiplicatively-regular matrix $B^{[l]}\mathcal{M}_{n\times n}(R)$ satisfying the conditions $B^{[l]}B = O_{n\times k}$ and $BB^\times + B^{[l]} = I_{n\times n}$.

Example 2.2. Again, consider the special case of $A = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \in \mathcal{M}_{k\times 1}(R)$. Then A is right regularly complemented if and only if it has a generalized inverse $A^- = [b_1,\dots,b_k]$ and if there exists a multiplicatively-regular element $c = A^{[r]} \in R$ satisfying $a_i c = 0$ for all $1 \le i \le n$ and $\sum_{i=1}^{k} b_i a_i + c = 1$. Note that, in this case, c is a right complement of $\sum_{i=1}^{k} b_i a_i$. Similarly, A is left regularly complemented if and only if it has a generalized inverse $A^- = [b_1,\dots,b_k]$ and there exists a multiplicatively-regular matrix $A^{[l]} = [d_{ij}] \in \mathcal{M}_{k\times k}(R)$ satisfying $\sum_{i=1}^{k} b_i a_i = 0$ and

$$a_i b_j + d_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.2}$$

Suppose that $A \in \mathcal{M}_{k\times n}(R)$ and $B \in \mathcal{M}_{n\times k}(R)$ are matrices having generalized inverses and satisfying the condition that A is right regularly complemented while B is left regularly complemented. Then each matrix $C \in \mathcal{M}_{k\times k}(R)$ defines a function $\alpha_C : \mathcal{M}_{n\times n}(R) \to \mathcal{M}_{n\times n}(R)$ by setting

$$\alpha_C : Y \longmapsto A^{\times} C B^{\times} + \lambda(A) Y B^{[l]} + \lambda(A)^{[r]} Y. \tag{2.3}$$

We can now generalize Proposition 1.2 as follows.

Proposition 2.3. *Let R be a semiring. Let $A \in \mathcal{M}_{k\times n}(R)$ and $B \in \mathcal{M}_{n\times k}(R)$ be matrices having generalized inverses and satisfying the condition that A is right regularly complemented while B is left regularly complemented. Furthermore, let $C \in \mathcal{M}_{k\times k}(R)$ be such that there exists a matrix $T \in \mathcal{M}_{n\times n}(R)$ that satisfies $ATB = C$. Then a complete set of solutions of (1.2) is given by*

$$\{\alpha_C(Y) \mid Y \in \mathcal{M}_{n\times n}(R)\}. \tag{2.4}$$

If T does not satisfy this equation then (1.2) has no solutions in $\mathcal{M}_{n\times n}(R)$.

The proof is essentially the same as that of Proposition 1.2.

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