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GLOBAL ASYMPTOTIC STABILITY OF INHOMOGENEOUS ITERATES

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Let (M,d) be a finite-dimensional complete metric space, and $\{T_n\}$ a sequence of uniformly convergent operators on M. We study the non-autonomous discrete dynamical system $x_{n+1} = T_n x_n$ and the globally asymptotic stability of the inhomogeneous iterates of $\{T_n\}$. Then we apply the results to investigate the stability of equilibrium of T when it satisfies certain type of sublinear conditions with respect to the partial order defined by a closed convex cone. The examples of application to nonlinear difference equations are also given.

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1. Introduction. Let (M,d) be a finite-dimensional complete metric space. Kruse and Nesemunn [7] discussed the global asymptotic stability of the equilibrium of a discrete dynamic system $x_{n+1} = Tx_n$ in (M,d). In particular, they chose for d the Thompson's metric to generalize the strong negative feedback property of the nonlinear difference equations and proved the global stability of the equilibrium of a Putnam difference equation.

In this paper, we study the non-autonomous discrete dynamical system $x_{n+1} = T_n x_n$ in (M,d). When $\{T_n\}$ has "the bounded orbit property" (to be explained in Section 2) and uniformly converges to a mapping T which satisfies certain contractive condition with respect to its equilibrium, we show that the equilibrium is globally asymptotic stable under the inhomogeneous iterates of $\{T_n\}$, which is our main result of Section 2. In Section 3, we apply that result to investigate the stability of equilibrium of T when it satisfies certain type of sublinear conditions with respect to the partial order defined by a closed convex cone. The examples of application to nonlinear difference equations are given in Section 4.

2. Stability of the inhomogeneous iterates in metric space. In this section, (M,d) stands for a finite-dimensional complete metric space.

Lemma 2.1. *Let $T : M \to M$ be continuous and x^* a fixed point of T in M. Suppose that there exists some integer $k > 1$ such that*

$$d(T^k x, x^*) < d(x, x^*), \quad x \neq x^*. \tag{2.1}$$

Then for any bounded subset $B \subset M$ with $x^* \notin \bar{B}$, there exists $r = r(B,k) \in (0,1)$ such that

$$d(T^k x, x^*) \le r d(x, x^*) \tag{2.2}$$

for any $x \in B$.

Proof. Without loss of generality, we assume that B is closed (otherwise, we can take the closure of B since $x^* \notin \bar{B}$). For each $x \neq x^*$, let

$$r(x) = \inf\{\alpha : d(T^k x, x^*) \le \alpha d(x, x^*)\}. \tag{2.3}$$

By (2.1), $r(x) < 1$. We claim that $r(x)$ is continuous on B. For if not, there exist $y_0 \in B$ and $\{y_n\} \subset B$ such that $y_n \to y_0$ while $\|r(y_0) - r(y_n)\| \ge \delta$ for some $\delta > 0$ and all n. Then there exists an infinite subsequence of $\{y_n\}$, we still denote it by $\{y_n\}$ for simplicity, such that either

(i) $r(y_0) - r(y_n) \ge \delta$, or

(ii) $r(y_0) - r(y_n) \le -\delta$.

In the case of (i), $r(y_n) \le r(y_0) - \delta$, then

$$d(T^k y_n, x^*) \le r(y_n) d(y_n, x^*) \le (r(y_0) - \delta) d(y_n, x^*). \tag{2.4}$$

Letting $n \to \infty$, we have

$$d(T^k y_0, x^*) \le (r(y_0) - \delta) d(y_0, x^*) \tag{2.5}$$

which contradicts with (2.3). In the case of (ii), $r(y_n) \ge r(y_0) + \delta$. Hence

$$d(T^k y_n, x^*) \ge \left(r(y_0) + \frac{\delta}{2}\right) d(y_n, x^*). \tag{2.6}$$

Letting $n \to \infty$, we have

$$d(T^k y_0, x^*) \ge \left(r(y_0) + \frac{\delta}{2}\right) d(y_0, x^*) \tag{2.7}$$

which contradicts with (2.3) again. The claim is proved.

Since B is compact, we have $r = \max\{r(x) : x \in B\} < 1$. Equation (2.2) is proved. □

Let $\phi : [0, \infty) \to [0, \infty)$ satisfy

$$\phi(0) = 0, \qquad \phi(t) \le L(a,b)t \quad \forall 0 < a \le t \le b < \infty, \tag{2.8}$$

where $L(a,b) \in (0,1)$ is a constant depending on a and b.

We need the following lemma [1, Lemma 2.1] (cf. [4, Lemma 1] for related results).

Lemma 2.2. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers that satisfy*

$$a_{n+1} \le \phi(a_n) + b_n, \quad n = 1, 2, \ldots. \tag{2.9}$$

If $\{a_n\}$ is bounded and $b_n \to 0$, then $a_n \to 0$.

Let $\{T_n\}$ be a sequence of operators on M. We say $\{T_n\}$ has "the bounded orbit property" [8] if for each $x \in M$, there exist $y \in X$ and $R(x) > 0$ such that $d(T_n \circ \cdots \circ T_1 x, y) \le R(x)$ for $n \ge 1$. The lumped operator $T(m,k)$ (cf. [2, 3]) is defined by $T(m,k) = T_{m+(k-1)} \circ T_{m+(k-2)} \circ \cdots \circ T_m$.

Theorem 2.3. *Let x^* be a fixed point of T in M. Suppose that there exists some integer $k > 1$ such that*

$$d(T^k x, x^*) < d(x, x^*), \quad x \neq x^*. \tag{2.10}$$

Let $\{T_n\}$ be a sequence of continuous operators on M. If the sequence of lumped operator $\{T(m,k)\}$ converges to T^k uniformly on M as $m \to \infty$ (this assumption is particularly fulfilled if $\{T_n\}$ converges to T uniformly on M), and $\{T_n\}$ has the bounded orbit property, then $\lim_{n\to\infty} T_n \circ \cdots \circ T_1 x_0 = x^*$ *for any $x_0 \in M$.*

Proof. We proceed the proof in two steps.

Step 1. For any given $x \in M$ and $0 \le i < k$, denote $S_m(i) = T(mk+i,k)$, $S = T^k$, $y_0 = S_0(i)x$, and $y_{m+1} = S_m(i)y_m$, $m = 0,1,2,\ldots$. Since $\{T_n\}$ has the bounded orbit property, $\{y_m\}$ is bounded. In view of Lemma 2.1, there exists a ϕ which satisfies (2.8) such that

$$\begin{aligned} d(y_{m+1}, x^*) &\le d(Sy_m, x^*) + d(Sy_m, S_m(i)y_m) \\ &\le \phi(d(y_m, x^*)) + d(Sy_m, S_m(i)y_m). \end{aligned} \tag{2.11}$$

Put $a_m = d(y_m, x^*)$ and $b_m = d(Sy_m, S_m(i)y_m)$. The sequence $\{a_m\}$ is bounded and $b_m \to \infty$, due to the uniform convergence. By Lemma 2.2, we get $\lim_{m\to\infty} d(y_m, x^*) = 0$.

Step 2. Denote $x_1 = T_1 x_0$ and $x_{n+1} = T_n x_n$, $n = 1,2,\ldots$. For any natural number n, there exist nonnegative integers $m(n)$ and $i(n)$ with $0 \le i(n) < k$ such that $n = m(n)k + i(n)$, and $m(n) \to \infty$ as $n \to \infty$. Observing $x_n = S_{m(n)}(i(n)) \circ S_{m(n)-1}(i(n)) \circ \cdots \circ S_0(i(n)) \circ T_{i(n)-1} \circ \cdots \circ T_1 x_0$ and using Step 1, $\lim_{n\to\infty} d(x_n, x^*) = 0$. □

We get [7, Theorem 1] as a corollary to Theorem 2.3.

Corollary 2.4. *Let x^* be a fixed point of a continuous operator T in M. Suppose that there exists some integer $k > 1$ such that*

$$d(T^k x, x^*) < d(x, x^*), \quad x \neq x^*. \tag{2.12}$$

Then $\lim_{n\to\infty} T^n x_0 = x^*$ *for any $x_0 \in M$.*

Proof. We only need to let $T_n = T$ for all n in Theorem 2.3, and observe that in this case, $\{T_n\}$ automatically has the bounded orbit property due to the contractive assumption on T. □

3. Stability in ordered Banach spaces.

In this section, B stands for a real finite-dimensional Banach space partially ordered by a closed convex cone P having nonempty interior $\mathring{P}$. We write $x \le y$ if $y - x \in P$, and $x \ll y$ if $y - x \in \mathring{P}$. Two points $x, y \in P - \{0\}$ are called comparable if there exist positive numbers λ and μ such that $\lambda x \le y \le \mu x$. This defines an equivalent relationship, and splits $P - \{0\}$ into disjoint components of P. The interior $\mathring{P}$ is a component of P if $\mathring{P} \neq \emptyset$. Since any finite-dimensional cone is normal (see [6], [7, Proposition 1.1]), we can assume the norm is monotone, that is, $x \le y$ implies $\|x\| \le \|y\|$ (otherwise, we can take an equivalent norm which is monotone). Let $B_r(x)$ denote the ball $\{y \in B : \|y - x\| < r\}$.

Let $x,y \in C$, where C is a component of P, and

$$M\left(\frac{x}{y}\right) = \inf\{\lambda : x \le \lambda y\}, \qquad M\left(\frac{y}{x}\right) = \inf\{\mu : y \le \mu x\}. \tag{3.1}$$

Thompson's metric is defined by

$$\bar{d}(x,y) = \ln\left\{\max\left[M\left(\frac{x}{y}\right), M\left(\frac{y}{x}\right)\right]\right\}, \tag{3.2}$$

where $\bar{d}(x,y)$ is a metric on C, C is complete with respect to $\bar{d}$ due to [9, Lemma 3].

We need the following lemma proved by Krause and Nussbaum [5, Lemma 2.3].

Lemma 3.1. (i) *Let $x,y \in \mathring{P}$ and $r > 0$ be a number such that the closed norm ball of radius r and center x and y, respectively, is contained in P. Then*

$$\bar{d}(x,y) \le \ln\left(1 + \frac{\|x-y\|}{r}\right). \tag{3.3}$$

(ii) *If P is a normal cone and the norm is monotone on P, then for $x,y \in P - \{0\}$,*

$$\|x-y\| \le \left(2e^{\bar{d}(x,y)} - e^{-\bar{d}(x,y)} - 1\right)\min\{\|x\|,\|y\|\}. \tag{3.4}$$

Theorem 3.2. *Let $x^* \in \mathring{P}$ be a fixed point of a mapping f on $\mathring{P}$, and $f_n : \mathring{P} \to \mathring{P}$, $n \ge 1$, be a sequence of mappings. There exists an integer $k > 0$ such that the sequence of lumped mappings $\{F(m,k)\}$ converges uniformly to f^k on $\mathring{P}$ as $m \to \infty$, where $F(m,k) = f_{m+(k-1)} \circ f_{m+(k-2)} \circ \cdots \circ f_m$ (this assumption is particularly fulfilled if $\{f_n\}$ converges to f uniformly on $\mathring{P}$). Suppose that*

(i) *for all $t \in (0,1)$,*

$$y \ge tx^* \quad \textit{implies} \quad f^k(y) \gg tx^* \tag{3.5}$$

and for all $s > 1$,

$$y \le sx^* \quad \textit{implies} \quad f^k(y) \ll sx^*, \tag{3.6}$$

where $y \in \mathring{P}$,

(ii) *there exist real numbers $a > 0$, $b > 0$, and $e \in \mathring{P}$ such that $ae \le f^k(x) \le be$ for all $x \in \mathring{P}$.*

Then $\lim_{n\to\infty} f_n \circ f_{n-1} \circ \cdots \circ f_1(x) = x^*$ for any initial point $x \in \mathring{P}$.

Proof. Since $ae \in \mathring{P}$, there exists $r > 0$ such that $B_r(ae) \subset P$. Let $x \in \mathring{P}$ and $w \in B_r(f^k(x))$, that is, $w = f^k(x) + z$ for some z with $\|z\| < r$. Now $w = f^k(x) + z \ge ae + z \in P$ due to $B_r(ae) \subset P$. Hence $w \in P$, so that $B_r(f^k(x)) \subset P$. For this $r > 0$, there exists $N > 0$, such that $F(m,k)(x) \in B_r(f^k(x))$ for all $m > N$ and $x \in \mathring{P}$ by the uniform convergence. In view of (3.3), the uniform convergence of $\{F(m,k)\}$ to f^k in norm implies the uniform convergence of $\{F(m,k)\}$ to f^k in $\bar{d}$ as $m \to \infty$.

We can choose $N > 0$ large enough such that $\bar{d}(F(m,k)(x), f^k(x)) < \ln(b/a)$ for all $m > N$ and $x \in \mathring{P}$. Noting that $ae \le f^k(x) \le be$ for all $x \in \mathring{P}$ implies $\bar{d}(f^k(x),e) \le$

$\ln(b/a)$ for all $x \in \overset{\circ}{P}$, we have $\bar{d}(F(m,k)(x),e) \le 2\ln(b/a)$ for all $x \in \overset{\circ}{P}$ and $m > N$. So that for $n > N+k$,

$$\bar{d}(f_n \circ f_{n-1} \circ \cdots \circ f_1(x), e) \le 2\ln\frac{b}{a}. \tag{3.7}$$

For a given $x \in B$, let

$$R(x) = \max\left\{\bar{d}(f_1(x),e), \bar{d}(f_2 \circ f_1(x),e), \ldots, \bar{d}(f_{N+k} \circ \cdots \circ f_2 \circ f_1(x),e), 2\ln\frac{b}{a}\right\}. \tag{3.8}$$

Therefore $\{f_n\}$ has the bounded orbit property with respect to $\bar{d}$.

Let $x \in \overset{\circ}{P}$ and $x \ne x^*$. We discuss it in two cases:

Case 1 ($M(x/x^*) \ge M(x^*/x)$). Then $M(x/x^*) > 1$ ($M(x/x^*) \ne x^*$ since $x \ne x^*$). Note that $x \le M(x/x^*)x^*$ implies that $f^k(x) \ll M(x/x^*)x^*$ by (3.6). Hence $M(f^k(x)/x^*) < M(x/x^*)$. On the other hand, $x \ge M(x^*/x)^{-1}x^* \ge M(x/x^*)^{-1}x^*$ implies that $f^k(x) \gg M(x/x^*)^{-1}x^*$ by (3.5). It follows that $M(x^*/f^k(x)) < M(x/x^*)$.

Case 2 ($M(x/x^*) < M(x^*/x)$). Then $M(x^*/x) > 1$. Now $x \ge M(x^*/x)^{-1}x^*$ implies that $f^k(x) \gg M(x^*/x)^{-1}x^*$ by (3.5). So that $M(x^*/f^k(x)) < M(x/x^*)$. On the other hand, $x \le M(x/x^*)x^* < M(x^*/x)x^*$ implies that $f^k(x) \ll M(x^*/x)x^*$. Therefore $M(f^k(x)/x^*) < M(x^*/x)$.

Combining the above, we have $\bar{d}(f^k(x)/x^*) < \bar{d}(x/x^*)$. Using Theorem 2.3, $\lim_{n\to\infty} \bar{d}(f_n \circ f_{n-1} \circ \cdots \circ f_1(x), x^*) = 0$ for any initial point $x \in \overset{\circ}{P}$. An application of Lemma 3.1(ii) concludes the proof. □

From the proof of Theorem 3.2, it is clear that the assumption (ii) in that theorem only used to guarantee the uniform convergence of $\{f_n\}$ to f in $\bar{d}$ and the bounded orbit property of $\{f_n\}$. Hence in the case of considering only a single mapping f instead of a sequence of convergent mappings, we can dispense that assumption and have the following corollary.

Corollary 3.3. *Let $f : \overset{\circ}{P} \to \overset{\circ}{P}$ be a mapping with a fixed point $x^* \in \overset{\circ}{P}$. Suppose that there exists an integer $k > 0$ such that*

$$y \ge tx^* \quad \text{implies} \quad f^k(y) \gg tx^* \tag{3.9}$$

for all $t \in (0,1)$, and

$$y \le sx^* \quad \text{implies} \quad f^k(y) \ll sx^* \tag{3.10}$$

for all $s > 1$, where $y \in \overset{\circ}{P}$. Then $\lim_{n\to\infty} f^n x = x^*$ for any $x \in \overset{\circ}{P}$.

4. Examples.

For a positive integer k, let $\mathbb{R}^k$ be partially ordered by

$$\mathbb{R}^k_+ = \left\{ \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix} : u_i \ge 0, i = 1, \ldots, k \right\}. \tag{4.1}$$

We denote

$$\operatorname{int}\mathbb{R}_+^k=\left\{\begin{pmatrix}u_1\\ \vdots\\ u_k\end{pmatrix}:u_i>0,\ i=1,\dots,\right\}. \tag{4.2}$$

Example 4.1. Suppose that $f:\operatorname{int}\mathbb{R}_+^2\to(0,\infty)$ is a nondecreasing function such that

$$f(tx,tx)\ge tf(x,x)\quad \forall t\in(0,1) \tag{4.3}$$

with

$$\lim_{x\to\infty}f(x,x)=\infty,\qquad \lim_{x\to 0}f(x,x)=0. \tag{4.4}$$

Some simple examples of such functions can be $f(x,y)=\sqrt{xy}$, or $f(x,y)=x+\sqrt{y}$, and so forth.

Consider the nonlinear difference equation

$$x_{n+1}=\frac{af(x_n,x_{n-1})+b}{cf(x_n,x_{n-1})+d},\quad n=0,1,2,\dots, \tag{4.5}$$

with positive initial conditions x_{-1} and x_0, where a, b, c, and d are positive numbers and $ad-bc>0$. Denoting

$$F(x)=\frac{af(x,x)+b}{cf(x,x)+d},\quad x>0, \tag{4.6}$$

we have the following lemma.

Lemma 4.2. (i) *F is nondecreasing on* $(0,\infty)$.

(ii) $\lim_{x\to\infty}F(x)=a/c$ *and* $\lim_{x\to 0}F(x)=b/d$.

(iii) *For* $t\in(0,1)$, $F(tx)>tF(x)$.

Proof. The proof of (i) follows from the nondecreasing assumption on f and the fact that $(au+b)/(cu+d)$ is increasing when $ad-bc>0$.

The proof of (ii) follows from (4.4).

To prove (iii), observe that

$$\begin{aligned} F(tx)&=\frac{af(tx,tx)+b}{cf(tx,tx)+d}\ge\frac{atf(x,x)+b}{ctf(x,x)+d}>\frac{atf(x,x)+b}{cf(x,x)+d}\\ &=t\frac{af(x,x)+b/t}{cf(x,x)+d}>t\frac{af(x,x)+b}{cf(x,x)+d}\\ &=tF(x),\end{aligned} \tag{4.7}$$

for $t\in(0,1)$. □

Equation (4.5) can be associated with the mapping $T:\operatorname{int}\mathbb{R}_+^2\to\operatorname{int}\mathbb{R}_+^2$ by

$$T\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}u_2\\ \dfrac{af(u_2,u_1)+b}{cf(u_2,u_1)+d}\end{pmatrix}. \tag{4.8}$$

It is clear that $\bar{x}$ is an equilibrium of (4.5) if and only if $\binom{\bar{x}}{\bar{x}}$ is a fixed point of T, and $\bar{x}$ is globally asymptotically stable if and only if $\binom{\bar{x}}{\bar{x}}$ is an attracting fixed point of T. In the following, suppose that $\bar{x}$ is an equilibrium of (4.5), that is, $\bar{x} = F(\bar{x})$.

For $\binom{y_1}{y_2} \geq t\binom{\bar{x}}{\bar{x}}$, $t \in (0,1)$,

$$\begin{aligned}
T\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &\geq T\begin{pmatrix} t\bar{x} \\ t\bar{x} \end{pmatrix} = \begin{pmatrix} t\bar{x} \\ \dfrac{af(t\bar{x},t\bar{x})+b}{cf(t\bar{x},t\bar{x})+d} \end{pmatrix} = \begin{pmatrix} t\bar{x} \\ F(t\bar{x}) \end{pmatrix}, \\
T^2\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &\geq \begin{pmatrix} F(t\bar{x}) \\ \dfrac{af(F(t\bar{x}),t\bar{x})+b}{cf(F(t\bar{x}),t\bar{x})+d} \end{pmatrix} \geq \begin{pmatrix} F(t\bar{x}) \\ \dfrac{af(tF(\bar{x}),t\bar{x})+b}{cf(tF(\bar{x}),t\bar{x})+d} \end{pmatrix} \\
&= \begin{pmatrix} F(t\bar{x}) \\ F(t\bar{x}) \end{pmatrix} \gg t\begin{pmatrix} F(\bar{x}) \\ F(\bar{x}) \end{pmatrix} = t\begin{pmatrix} \bar{x} \\ \bar{x} \end{pmatrix}.
\end{aligned} \tag{4.9}$$

Noting that Lemma 4.2(iii) implies $F(sx) < sF(x)$, for $s > 1$. Then for $\binom{y_1}{y_2} \leq s\binom{\bar{x}}{\bar{x}}$, $s > 1$,

$$\begin{aligned}
T\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &\leq T\begin{pmatrix} s\bar{x} \\ s\bar{x} \end{pmatrix} = \begin{pmatrix} s\bar{x} \\ \dfrac{af(s\bar{x},s\bar{x})+b}{cf(s\bar{x},s\bar{x})+d} \end{pmatrix} = \begin{pmatrix} s\bar{x} \\ F(s\bar{x}) \end{pmatrix}, \\
T^2\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &\leq \begin{pmatrix} F(s\bar{x}) \\ \dfrac{af(F(s\bar{x}),s\bar{x})+b}{cf(F(s\bar{x}),s\bar{x})+d} \end{pmatrix} \leq \begin{pmatrix} F(s\bar{x}) \\ \dfrac{af(sF(\bar{x}),s\bar{x})+b}{cf(sF(\bar{x}),s\bar{x})+d} \end{pmatrix} \\
&= \begin{pmatrix} F(s\bar{x}) \\ F(s\bar{x}) \end{pmatrix} \ll s\begin{pmatrix} F(\bar{x}) \\ F(\bar{x}) \end{pmatrix} = s\begin{pmatrix} \bar{x} \\ \bar{x} \end{pmatrix}.
\end{aligned} \tag{4.10}$$

Hence we can apply Corollary 3.3 to conclude that $\bar{x}$ is globally asymptotically stable.

In the numerical calculations, there will be some round-off errors. Hence we may consider the following perturbed equations of (4.5):

$$x_{n+1} = \frac{af(x_n,x_{n-1})+b}{cf(x_n,x_{n-1})+d} + \epsilon_n, \quad n = 0,1,2,\ldots, \tag{4.11}$$

with positive initial conditions x_0 and x_1, where a, b, c, and d are positive numbers, $ad - bc > 0$, and $\epsilon_n \geq 0$. Equation (4.11) is associated with the mappings $T_n : \operatorname{int}\mathbb{R}^2_+ \to \operatorname{int}\mathbb{R}^2_+$ by

$$T_n\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ \dfrac{af(u_2,u_1)+b}{cf(u_2,u_1)+d} + \epsilon_n \end{pmatrix}. \tag{4.12}$$

Suppose that $\epsilon_n \to 0$. Then T_n converges to T uniformly on $\operatorname{int}\mathbb{R}^2_+$ as $n \to \infty$. The arguments of Lemma 4.2(i) and (ii) tell us that

$$\frac{b}{d}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq T^2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \leq \frac{a}{c}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.13}$$

Thus, Theorem 3.2 can be applied and

$$\lim_{n\to\infty} T_n \circ T_{n-1} \circ \cdots \circ T_1 \begin{pmatrix} x_{-1} \\ x_0 \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{x} \end{pmatrix} \tag{4.14}$$

for any initial $\binom{x_{-1}}{x_0} \in \operatorname{int}\mathbb{R}_+^2$, that is, the recursive sequence $\{x_n\}$ defined by (4.5) also converges to $\bar{x}$ for any positive initial conditions x_{-1} and x_0.

Example 4.3. As the second example of application, we will discuss the following rational recursive sequence which is investigated in [5, pages 59–64]:

$$x_{n+1} = \frac{a + \sum_{i=0}^{k} a_i x_{n-i}}{b + \sum_{i=0}^{k} b_i x_{n-i}}, \quad n = 0, 1, \ldots, \tag{4.15}$$

where k is a nonnegative integer,

$$\begin{gathered} a_0, \ldots, a_k, b_0, \ldots, b_k \in [0, \infty), \quad a, b \in (0, \infty), \\ \sum_{i=0}^{k} a_i = 1, \qquad B = \sum_{i=0}^{k} b_i > 0, \end{gathered} \tag{4.16}$$

and where the initial conditions $x_{-k}, \ldots, x_0$ are positive numbers.

Kocić and Ladas, in [5], proved that the unique positive equilibrium

$$\bar{x} = \frac{1 - b + \sqrt{(1-b)^2 + 4aB}}{2B} \tag{4.17}$$

of (4.15) is globally asymptotically stable if $b > 1$. We are going to show that it is globally asymptotically stable also if $b \geq aB$ and $a_i \geq b_i/B$, $i = 0, \ldots, k$, by using Corollary 3.3.

Equation (4.15) is associated with the mapping $T : \operatorname{int}\mathbb{R}_+^{k+1} \to \operatorname{int}\mathbb{R}_+^{k+1}$ by

$$T \begin{pmatrix} u_0 \\ \vdots \\ u_{k-1} \\ u_k \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_k \\ \dfrac{a + \sum_{i=0}^{k} a_i u_{k-i}}{b + \sum_{i=0}^{k} b_i u_{k-i}} \end{pmatrix}. \tag{4.18}$$

For $\begin{pmatrix} y_0 \\ \vdots \\ y_k \end{pmatrix} \geq t \begin{pmatrix} \bar{x} \\ \vdots \\ \bar{x} \end{pmatrix}$, $t \in (0,1)$, let $y^* = \sum_{i=0}^{k} a_i y_{k-i}$ and $y_* = 1/B \sum_{i=0}^{k} b_i y_{k-i}$. It is clear that $y^* \geq y_* \geq t\bar{x}$ since $a_i \geq b_i/B$ and $y_i \geq t\bar{x}$. Now

$$T \begin{pmatrix} y_0 \\ \vdots \\ y_{k-1} \\ y_k \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_k \\ \dfrac{a + \sum_{i=0}^{k} a_i y_{k-i}}{b + \sum_{i=0}^{k} b_i y_{k-i}} \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_k \\ \dfrac{a + y^*}{b + B y_*} \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_k \\ y_{k+1} \end{pmatrix}, \tag{4.19}$$

where $y_{k+1} = (a + y^*)/(b + B y_*) = (a + \sum_{i=0}^{k} a_i y_{k-i})/(b + \sum_{i=0}^{k} b_i y_{k-i})$.

Let $h(x) = (a+x)/(b+Bx)$. Similar to Lemma 4.2(i) and (iii), we can prove that $h(x)$ is nondecreasing since $b \geq aB$, and $h(tx) = t(a/t+x)/(b+Btx) > t(a+x)/(b+Bx) = th(x)$ for $t \in (0,1)$ and $x > 0$. Then $y_{k+1} \geq (a+y_*)/(b+By_*) = h(y_*) \geq h(t\bar{x}) > th(\bar{x}) = t\bar{x}$.

Repeating the above argument for $y_1,\ldots,y_{k+1}$ yields

$$y_{k+2} = \frac{a+\sum_{i=0}^{k} a_i y_{(k+1)-i}}{b+\sum_{i=0}^{k} b_i y_{(k+1)-i}} \geq h(t\bar{x}) > t\bar{x},$$

$$T^2\begin{pmatrix} y_0 \\ \vdots \\ y_{k-1} \\ y_k \end{pmatrix} = T\begin{pmatrix} y_1 \\ \vdots \\ y_k \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} y_2 \\ \vdots \\ y_{k+1} \\ y_{k+2} \end{pmatrix}. \tag{4.20}$$

Continuing the above procedure, we can prove that

$$T^{k+1}\begin{pmatrix} y_0 \\ \vdots \\ y_k \end{pmatrix} = \begin{pmatrix} y_{k+1} \\ \vdots \\ y_{2k+1} \end{pmatrix} \geq \begin{pmatrix} h(t\bar{x}) \\ \vdots \\ h(t\bar{x}) \end{pmatrix} \gg t\begin{pmatrix} \bar{x} \\ \vdots \\ \bar{x} \end{pmatrix}, \tag{4.21}$$

where

$$y_{k+j} = \frac{a+\sum_{i=0}^{k} a_i y_{(k+j-1)-i}}{b+\sum_{i=0}^{k} b_i y_{(k+j-1)-i}} \geq h(t\bar{x}) > t\bar{x}, \quad j = 1,\ldots,k+1. \tag{4.22}$$

By a similar argument as above and using $h(sx) < sh(x)$ for $s > 1$ and $x > 0$, we have

$$T^{k}\begin{pmatrix} y_0 \\ \vdots \\ y_k \end{pmatrix} \leq \begin{pmatrix} h(s\bar{x}) \\ \vdots \\ h(s\bar{x}) \end{pmatrix} \ll s\begin{pmatrix} \bar{x} \\ \vdots \\ \bar{x} \end{pmatrix} \tag{4.23}$$

for $s > 1$.

Therefore, $\bar{x}$ is globally asymptotically stable by Corollary 3.3. In the same spirit of Example 4.1, this stability will be preserved for sufficiently small perturbations, due to Theorem 3.2.

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