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CONDITION R AND FEFFERMAN'S MAPPING THEOREM

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It is shown that Fefferman's mapping theorem extends to the case of manifolds, that is a biholomorphic map between two strictly pseudoconvex manifolds extends smoothly to their boundaries.

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1. Introduction. A central question in complex analysis is "does every proper holomorphic mapping $f : D \to D'$ of bounded domains D, D' with smooth boundaries in $\mathbb{C}^n$ extend smoothly to the boundary of D?"

The answer has been known to be "yes" in dimension one for a long time. In higher dimensions, in case D and D' are strictly pseudoconvex and f is biholomorphic, Fefferman's famous mapping theorem [8] answers the question in the affirmative.

Bell and Ligocka [4] simplified the proof of Fefferman's mapping theorem and extended the theorem to a wide class of pseudoconvex domains.

In [3], Fefferman's mapping theorem was extended to smoothly bounded pseudoconvex subdomains of Stein manifolds that satisfy condition R. Thereafter, the question was asked whether all smoothly bounded pseudoconvex domains satisfy condition R. Recently, Barrett [2] and Christ [5] have shown that this question has an answer in the negative. But that was not the end of condition R, because the case of strictly pseudoconvex manifolds that are not Stein had not been determined. At first it was thought (because of the work of Barrett [1]) that one could not do without the assumption of Steinness.

In this note, we show that a strictly pseudoconvex manifold need not be Stein before it satisfies condition R; and following the work of Bedford et al. [3], we extend Fefferman's mapping theorem to all strictly pseudoconvex manifolds.

2. Preliminaries. Let Ω be a relatively compact domain in an n-dimensional complex manifold X. The space $L^2_{(n,0)}(\Omega)$ is defined to be the set of $(n,0)$ forms ω such that

$$\|\omega\|^2 = \left(\sqrt{-1}\right)^{n^2} \int_\Omega \omega \wedge \bar{\omega} \tag{2.1}$$

is finite. The space $L^2_{(n,0)}(\Omega)$ is a Hilbert space with inner product given by

$$(\omega, \eta) = \left(\sqrt{-1}\right)^{n^2} \int_\Omega \omega \wedge \bar{\eta}. \tag{2.2}$$

The Bergman-Kobayashi projection P_Ω associated to Ω is the orthogonal projection of $L^2_{(n,0)}(\Omega)$ onto $H_{(n,0)}(\Omega)$, the closed subspace of $L^2_{(n,0)}(\Omega)$ consisting of holomorphic $(n,0)$ forms. If Ω has a smooth boundary, Ω satisfies condition R if the Bergman-Kobayashi projection associated to Ω maps $C^\infty_{(n,0)}(\bar{\Omega})$ into $C^\infty_{(n,0)}(\bar{\Omega})$.

To make use of the proof in [3], we show that if Ω above has smooth boundary and it is strictly pseudoconvex, then Ω satisfies condition R; and, in addition, if p_0 is a point in X near the boundary $\partial\Omega$ of Ω, then there are n functions $g_1,\dots,g_n$ that are holomorphic in a neighborhood of $\bar{\Omega}$ and that form a coordinate system at p_0.

Our main result is the following theorem.

Theorem 2.1. *Let X_1 and X_2 be n-dimensional complex manifolds and $\Omega_1 \Subset X_1$, $\Omega_2 \Subset X_2$ strictly pseudoconvex subdomains with smooth boundaries. Let $f:\Omega_1 \to \Omega_2$ be a biholomorphic mapping between Ω_1 and Ω_2. Then f extends smoothly to a C^∞ diffeomorphism of $\bar{\Omega}_1$ and $\bar{\Omega}_2$.*

3. Condition R. To establish condition R for smoothly bounded strictly pseudoconvex subdomains of complex manifolds, we need a result of Gunning and Rossi [9] which we met on the way to proving theorems in [6, 7]. Their result is the following theorem.

Theorem 3.1. *Let Ω be a strictly pseudoconvex domain in a complex manifold Y. There are a Stein manifold X and a proper holomorphic mapping $\pi:\Omega\to X$ with the following properties:*

(i) $\pi:\mathbb{O}_X \cong \mathbb{O}_\Omega$;

(ii) *there are finitely many points $x_1,\dots,x_z$ in X such that $\pi^{-1}(x_j)$ is a compact subvariety of Ω of positive dimension, and $\pi:\Omega\setminus\bigcup\pi^{-1}(x_j)\cong X\setminus\{x_1,\dots,x_z\}$.*

The first statement means that the rings of holomorphic functions $\mathbb{O}_X$ and $\mathbb{O}_\Omega$ on X and Ω, respectively, are isomorphic under the map induced by π. The second means that $\Omega\setminus\cup\pi^{-1}(x_j)$ and $X\setminus\{x_1,\dots,x_z\}$ are biholomorphic.

Now from the proof of Theorem 3.1 as given in [9], it is clear that there is a strictly pseudoconvex neighborhood Ω' of $\bar{\Omega}$ such that Ω' can replace Ω in Theorem 3.1 so that the compact set $\cup\pi^{-1}(x_j)$ corresponding to Ω' is contained in Ω.

If X' corresponds to Ω' in Theorem 3.1 and $X=\pi(\Omega)$, then clearly if Ω has a smooth boundary then X is a Stein strictly pseudoconvex manifold with a smooth boundary, and therefore, as is well-known, X satisfies condition R.

We can regard Ω as imbedded in X. Then it is clear that $L^2_{(n,0)}(\Omega)=L^2_{(n,0)}(X)$ and $H_{(n,0)}(\Omega)=H_{(n,0)}(X)$. Therefore the Bergman-Kobayashi projections P_X and P_Ω are equal, and it is not difficult to see (using Sobolev spaces) that Ω satisfies condition R.

4. Local coordinates near the boundary. Again from Theorem 3.1 we get the last theorem that we need in the proof of Theorem 2.1.

Theorem 4.1. *Let Ω be a strictly pseudoconvex subdomain of a complex manifold Y. Then near the boundary $\partial\Omega$ of Ω, local coordinates are given by holomorphic functions in a neighborhood of $\bar{\Omega}$.*

Proof. As indicated in Section 3, from the proof of Theorem 3.1 as given in [9] it is clear that there is a strictly pseudoconvex neighborhood Ω' of $\bar{\Omega}$ such that Ω' can replace Ω in Theorem 3.1 so that the compact set $\cup\pi^{-1}(x_j)$ corresponding to Ω' is contained in Ω. Now if p_0 is a point in Ω' near the boundary $\partial\Omega$ of Ω, let $\pi(p_0)$ have holomorphic functions $g_1,\ldots,g_n$ on the Stein manifold X that form local coordinates at $\pi(p_0)$. Then $g_1\circ\pi,\ldots,g_n\circ\pi$ form local coordinates at p_0, which are holomorphic in a neighborhood of $\bar{\Omega}$. □

5. Proof of Theorem 2.1. The proof of Theorem 2.1 relies on the following two lemmas whose proofs are in [3].

Lemma 5.1. *If ω is a holomorphic $(n,0)$ form in $C^\infty_{(n,0)}(\bar{\Omega}_2)$, then $f^*\omega$ is in $C^\infty_{(n,0)}(\bar{\Omega}_1)$.*

Lemma 5.2. *If ω is a holomorphic $(n,0)$ form in $C^\infty_{(n,0)}(\bar{\Omega}_2)$ that vanishes to at most finite order at any boundary point of Ω_2, then $f^*\omega$ vanishes to at most finite order at any boundary point of Ω_1.*

Now to prove Theorem 2.1, we initiate the proof of Theorem 2.1 in [3]:

Let p_0 be a boundary point of Ω_1 and let $z_1,\ldots,z_n$ be holomorphic coordinates near p_0. We show that f extends smoothly to $\partial\Omega_1$ near p_0. Let $\{p_i\}$ be a sequence of points in Ω_1 that converges to p_0. Then $\{f(p_i)\}$ converges to a point q_0 in $\partial\Omega_2$. Let $g_1,\ldots,g_n$ be n functions on Ω_2 that extend to be holomorphic in a neighborhood of $\bar{\Omega}_2$ in X_2 and that form a coordinate chart at q_0. Define a holomorphic function u near p_0 via

$$u\,dz_1\wedge dz_2\wedge\cdots\wedge dz_n = f^*(dg_1\wedge dg_2\wedge\cdots\wedge dg_n). \tag{5.1}$$

By Lemmas 5.1 and 5.2 u extends smoothly to $\partial\Omega_1$ near p_0 and u vanishes to a finite order near p_0.

If $\alpha=(\alpha_1,\ldots,\alpha_n)$ is a multi-index, then we define $g^\alpha=\prod_{i=1}^n g_i^{\alpha_i}$. Lemma 5.1 implies that the form $f^*(g^\alpha dg_1\wedge\cdots\wedge dg_n)$ extends smoothly to $\partial\Omega_1$ near p_0 for each α. Hence, u and $u(g^\alpha\circ f)$ extend smoothly to $\partial\Omega_1$ near p_0 for each α, and u vanishes to at most finite order at p_0. By the division theorem cited in [3], $g_i\circ f$ extends smoothly to $\partial\Omega_1$ near p_0 for each i. Hence f extends smoothly to $\partial\Omega_1$ near p_0. Since p_0 was arbitrarily chosen, we conclude that f extends smoothly to all of $\partial\Omega_1$. Now we can replace f by f^{-1} and then the theorem follows.

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