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SLIGHTLY β-CONTINUOUS FUNCTIONS

TAKASHI NOIRI

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ABSTRACT. We define a function $f: X \to Y$ to be slightly β-continuous if for every clopen set V of Y, $f^{-1}(V) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(V))))$. We obtain several properties of such a function. Especially, we define the notion of ultra-regularizations of a topology and obtain interesting characterizations of slightly β-continuous functions by using it.

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1. Introduction. Semi-open sets, preopen sets, α-sets, and β-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets many authors introduced and studied various types of generalizations of continuity. In 1980 Jain [15] introduced the notion of slightly continuous functions. Recently, Nour [24] defined slightly semi-continuous functions as a weak form of slight continuity and investigated the functions. Quite recently, Noiri and Chae [23] have further investigated slightly semi-continuous functions. On the other hand, Pal and Bhattacharyya [7] defined a function to be faintly precontinuous if the preimages of each clopen set of the codomain is preopen and obtained many properties of such functions. Slight continuity implies both slight semi-continuity and faint precontinuity but not conversely.

In this paper, we introduce the notion of slight β-continuity which is implied by both slight semi-continuity and faint precontinuity. We establish several properties of such functions. Especially, we define the notion of ultra-regularization of a topology and obtain interesting characterizations of slight β-continuity, slight semi-continuity, faint precontinuity and slight continuity. Moreover, we investigate the relationships between slight β-continuity, contra-β-continuity [13], and β-continuity [1].

2. Preliminaries. Let (X,τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *β-open* [1] or *semi-preopen* [5] (resp., *semi-open* [17], *preopen* [19], *α-open* [21]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ (resp., $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$). The family of all semi-open (resp., preopen, α-open, β-open) sets in (X,τ) is denoted by $\mathrm{SO}(X)$ (resp., $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, or $\mathrm{SPO}(X)$). The complement of a semi-open (resp., preopen, α-open, β-open) set is said to be *semi-closed* (resp., *preclosed*, *α-closed*, *β-closed*, or semi-preclosed). If A is both semi-open and semi-closed, then it is said to be *semi-regular* [9]. If A is both β-open and β-closed, then it is said to be *semi-pre-regular* or *β-clopen*. The family of all semi-regular (resp., semi-preopen, semi-pre-regular, clopen) sets of X is denoted by $\mathrm{SR}(X)$ (resp., $\mathrm{SPO}(X)$, $\mathrm{SPR}(X)$, $\mathrm{CO}(X)$). The family of all clopen (resp., semi-preopen, semi-pre-regular) sets of X containing $x \in$

X is denoted by $\mathrm{CO}(X,x)$ (resp., $\mathrm{SPO}(X,x)$, $\mathrm{SPR}(X,x)$). The intersection of all semi-closed (resp., preclosed, β-closed) sets of X containing A is called the *semi-closure* [8] (resp., *preclosure* [11], *semi-preclosure* [5] or *β-closure* [3]) of A and is denoted by $\mathrm{sCl}(A)$ (resp., $\mathrm{pCl}(A)$, $\mathrm{spCl}(A)$, or ${}_\beta\mathrm{Cl}(A)$).

The following basic properties of the semi-preclosure are useful in the sequel.

Lemma 2.1 (see Abd El-Monsef et al. [3] and Andrijević [5]). *The following statements hold for a subset A of a topological space (X,τ):*
(a) $\mathrm{spCl}(A) = A\cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(b) $x\in \mathrm{spCl}(A)$ *if and only if* $A\cap U \neq \emptyset$ *for every* $U\in \mathrm{SPO}(X,x)$,
(c) *A is β-closed if and only if* $A = \mathrm{spCl}(A)$.

Lemma 2.2 (see Jafari and Noiri [14]). *If A is a β-open set of a topological space (X,τ), then $\mathrm{spCl}(A)$ is β-open in (X,τ).*

Throughout the present paper, (X,τ) and (Y,σ) (or simply X and Y) denote topological spaces and $f:(X,\tau)\to(Y,\sigma)$ (or simply $f:X\to Y$) presents a (single-valued) function.

Definition 2.3. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be *slightly continuous* [15] (resp., *slightly semi-continuous* [24], *faintly precontinuous* [7]) if for each point $x\in X$ and each clopen set V containing $f(x)$ there exists an open set U (resp., $U\in\mathrm{SO}(X)$, $U\in\mathrm{PO}(X)$) containing x such that $f(U)\subset V$.

Definition 2.4. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be *β-continuous* [1] (resp., *semi-continuous* [17], *precontinuous* [19]) if for each point $x\in X$ and each open set V containing $f(x)$ there exists $U\in\mathrm{SPO}(X)$ (resp., $U\in\mathrm{SO}(X)$, $U\in\mathrm{PO}(X)$) containing x such that $f(U)\subset V$.

3. Characterizations

Definition 3.1. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be *slightly β-continuous* (briefly *sl.β.c.*) if for each point $x\in X$ and each clopen set V containing $f(x)$ there exists a β-open set U of X containing x such that $f(U)\subset V$.

Theorem 3.2. *For a function $f:(X,\tau)\to(Y,\sigma)$, the following statements are equivalent:*
(a) *f is slightly β-continuous;*
(b) $f^{-1}(V)\in\mathrm{SPO}(X)$ *for each* $V\in\mathrm{CO}(Y)$;
(c) $f^{-1}(V)\in\mathrm{SPR}(X)$ *for each* $V\in\mathrm{CO}(Y)$;
(d) *for each $x\in X$ and each $V\in\mathrm{CO}(Y,f(x))$, there exists $U\in\mathrm{SPR}(X,x)$ such that $f(U)\subset V$;*
(e) *for each $x\in X$ and each $V\in\mathrm{CO}(Y,f(x))$, there exists $U\in\mathrm{SPO}(X,x)$ such that $f(\mathrm{spCl}(U))\subset V$.*

Proof. The proof is easily obtained by using Lemma 2.2. □

Let (X,τ) be a topological space. Since the intersection of two clopen sets of (X,τ) is clopen, the clopen subsets of (X,τ) may be used as a base for a topology on X. The topology is called the *ultra-regularization* of τ and is denoted by τ_u. A topological space (X,τ) is said to be *ultra regular* [12] if $\tau = \tau_u$. Each element of τ_u is said to be *δ^*-open* [29]. Note that ultra-regular spaces are known as 0-dimensional spaces.

Definition 3.3. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be *clopen-continuous* [28] if for each point x of X and each open set V containing $f(x)$, there exists a clopen set U containing x such that $f(U)\subset V$.

Remark 3.4. A space (X,τ) is ultra-regular if and only if every continuous function $f:(X,\tau)\to(Y,\sigma)$ is clopen-continuous.

Theorem 3.5. *For a function $f:(X,\tau)\to(Y,\sigma)$, the following statements are equivalent:*

(a) *$f:(X,\tau)\to(Y,\sigma)$ is slightly continuous;*
(b) *$f:(X,\tau)\to(Y,\sigma_u)$ is clopen-continuous;*
(c) *$f:(X,\tau)\to(Y,\sigma_u)$ is continuous;*
(d) *$f:(X,\tau_u)\to(Y,\sigma_u)$ is continuous.*

Proof. (a)⇒(b). Let $x\in X$ and V be an open set of (Y,σ_u) containing $f(x)$. There exists a clopen set W of (Y,σ) such that $f(x)\subset W\subset V$. Since f is slightly continuous, there exists a clopen set U containing x such that $f(U)\subset W$ and hence $f(U)\subset V$. This shows that $f(X,\tau)\to(Y,\sigma_u)$ is clopen-continuous.

(b)⇒(c). This is obvious.

(c)⇒(a). Let $x\in X$ and V be a clopen set of (Y,σ) containing $f(x)$. Then V is an open set of (Y,σ_u) and there exists $U\in\tau$ containing x such that $f(U)\subset V$. Therefore, $f:(X,\tau)\to(Y,\sigma)$ is slightly continuous.

(b)⇒(d). Let $x\in X$ and V any open set of (Y,σ_u) containing $f(x)$. By (b) there exists a clopen subset U of (X,τ) containing x such that $f(U)\subset V$. Since U is open in (X,τ_u), $f(X,\tau_u)\to(Y,\sigma_u)$ is continuous.

(d)⇒(c). Since $\tau_u\subset\tau$, the proof is obvious. □

Definition 3.6. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be *β-clopen-continuous* (resp., *pre-clopen-continuous, semi-clopen-continuous*) if for each point x of X and each open set V containing $f(x)$, there exists a β-clopen (resp., pre-clopen, semi-regular) set U containing x such that $f(U)\subset V$.

Theorem 3.7. *For a function $f:(X,\tau)\to(Y,\sigma)$, the following statements are equivalent:*

(a) *$f:(X,\tau)\to(Y,\sigma)$ is sl.β.c. (resp., slightly semi-continuous, faintly precontinuous);*
(b) *$f(X,\tau)\to(Y,\sigma_u)$ is β-clopen continuous (resp., semi-clopen continuous, pre-clopen continuous);*
(c) *$f(X,\tau)\to(Y,\sigma_u)$ is β-continuous (resp., semi-continuous, precontinuous);*
(d) *$f(X,\tau_u)\to(Y,\sigma_u)$ is β-continuous (resp., semi-continuous, precontinuous).*

Proof. The proof is similar to that of Theorem 3.5 and is thus omitted. □

Corollary 3.8 (see Pal and Bhattacharyya [7]). *A function $f:(X,\tau)\to(Y,\sigma)$ is faintly precontinuous if and only if $f^{-1}(V)\in\mathrm{PO}(X)$ for every δ^*-open set V of Y.*

4. Comparisons. In this section, we investigate the relationships between slightly β-continuous functions and other related functions. For this purpose, we will recall some definitions of functions.

Definition 4.1. A function $f : X \to Y$ is said to be *weakly β-continuous* [27] (resp., *weakly semi-continuous* [6], *almost weakly continuous* [16], or *quasi precontinuous* [25]) if for each point $x \in X$ and each open set V containing $f(x)$ there exists $U \in \mathrm{SPO}(X)$ (resp., $U \in \mathrm{SO}(X)$, $U \in \mathrm{PO}(X)$) containing x such that $f(U) \subset \mathrm{Cl}(V)$.

Definition 4.2. A function $f : X \to Y$ is said to be *contra-β-continuous* [13] (resp., *contra-precontinuous*) [13] if $f^{-1}(F) \in \mathrm{SPO}(X)$ (resp., $f^{-1}(F) \in \mathrm{PO}(X)$) for each closed set F of Y.

Definition 4.3. A function $f : X \to Y$ is said to be *β-quasi-irresolute* [14] if for each point $x \in X$ and each $V \in \mathrm{SO}(Y)$ containing $f(x)$ there exists $U \in \mathrm{SPO}(X,x)$ such that $f(U) \subset \mathrm{Cl}(V)$.

A function is said to be *β-irresolute* [18] if the preimages of β-open sets are β-open. It is obvious that a function $f : X \to Y$ is β-irresolute if and only if for each point $x \in X$ and each $V \in \mathrm{SPO}(Y, f(x))$ there exists $U \in \mathrm{SPO}(X,x)$ such that $f(U) \subset V$. We give an interesting characterization of β-quasi-irresolute functions and make clear the fact that β-irresolute functions are β-quasi-irresolute. A function $f : X \to Y$ is β-quasi-irresolute if and only if for each point $x \in X$ and each $V \in \mathrm{SPO}(Y, f(x))$ there exists $U \in \mathrm{SPO}(X,x)$ such that $f(U) \subset \mathrm{Cl}(V)$. This follows from the fact that for each β-open set V of Y, $\mathrm{Cl}(V) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V)))$ and $\mathrm{Cl}(V) \in \mathrm{SO}(Y)$.

From the above definitions we obtain the following diagram:

$$\begin{tikzcd}
 & \beta\text{-quasi-irresolute} \arrow[d, Rightarrow] & \\
\beta\text{-continuous} \arrow[r, Rightarrow] & \text{weakly } \beta\text{-continuous} \arrow[d, Rightarrow] & \text{weakly semi-continuous} \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
\text{contra-}\beta\text{-continuous} \arrow[r, Rightarrow] & \text{slightly } \beta\text{-continuous} & \text{slightly semi-continuous} \arrow[l, Rightarrow] \\
\text{contra-precontinuous} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \text{faintly precontinuous} \arrow[u, Rightarrow] & \text{slightly continuous} \arrow[l, Rightarrow] \arrow[u, Rightarrow] \\
 & \text{almost weakly continuous} \arrow[u, Rightarrow] &
\end{tikzcd}$$

Remark 4.4. Slight semi-continuity and faint precontinuity are independent of each other as Examples 4.5 and 4.6 show.

Example 4.5. Let $X = \{a,b,c\}$, τ the indiscrete topology, and $\sigma = \{\emptyset, X, \{a\}, \{b,c\}\}$. The identity function $f : (X,\tau) \to (X,\sigma)$ is precontinuous and faintly precontinuous. But it is not slightly semi-continuous since $f^{-1}(\{a\})$ is not semi-open in (X,τ).

Example 4.6. Let $X = \{a,b,c\}$, $\tau = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$, and $\sigma = \{\emptyset,\{a\},\{b,c\},X\}$. Then the identity $f : (X,\tau) \to (X,\sigma)$ is slightly semi-continuous by [23, Example 2.1] but not faintly precontinuous as $f^{-1}(\{a\})$ is not preclosed in (X,τ).

Remark 4.7. Contra-β-continuity and β-continuity are independent of each other as Examples 4.8 and 4.9 show.

Example 4.8. The identity function on the real line with the usual topology is continuous and hence β-continuous. But it is not contra-β-continuous since the preimage of any singleton is not β-open.

Example 4.9. Let $X = \{a,b\}$ be the Sierpinski space by setting $\tau = \{\emptyset,\{a\},X\}$ and $\sigma = \{\emptyset,\{b\},X\}$. The identity function $f : (X,\tau) \to (Y,\sigma)$ is contra-continuous by [10, Example 2.5] and hence contra-β-continuous but not β-continuous.

Definition 4.10. A topological space X is said to be

(a) *extremally disconnected* (briefly E.D.) if the closure of each open set of X is open in X,

(b) *a PS-space* [4] if every preopen set of X is semi-open in X,

(c) *locally indiscrete* [20] if every open set of X is closed in X.

Theorem 4.11. *For a function $f : X \to Y$, the following properties hold:*

(a) *If f is sl.β.c. and X is E.D., then f is faintly precontinuous.*

(b) *If f is sl.β.c. and X is a PS-space, then f is slightly semi-continuous.*

(c) *If f is sl.β.c. and X is an E.D. and PS-space, then f is slightly continuous.*

Proof. (a) Let $x \in X$ and $V \in \mathrm{CO}(Y,f(x))$. Now, put $U = f^{-1}(V)$. Since X is E.D., we have $U \in \mathrm{PO}(X,x)$ by [4, Theorem 5.1] and $f(U) \subset V$. Therefore, f is faintly precontinuous.

(b) Since X is a PS-space, every β-open set of X is semi-open by [4, Theorem 2.1] and the result follows easily.

(c) Let $V \in \mathrm{CO}(Y)$. Then by (a) and (b), $f^{-1}(V)$ is semi-regular and pre-clopen in X. Since $f^{-1}(V)$ is semi-closed and preopen, we have $\mathrm{Int}(\mathrm{Cl}(f^{-1}(V))) = f^{-1}(V)$. Since $f^{-1}(V)$ is semi-open and preclosed, we have $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) = f^{-1}(V)$. Therefore, $f^{-1}(V) \in \mathrm{CO}(X)$ and f is slightly continuous. □

Remark 4.12. We may define a function $f : X \to Y$ to be *slightly α-continuous* if $f^{-1}(V)$ is α-open in X for every clopen set V of Y. However, it is known in [22, Lemma 3.1] that a subset is α-open if and only if it is semi-open and preopen. Therefore, by the proof for Theorem 4.11(c) each α-open and α-closed set is clopen. Hence, slight α-continuity is equivalent to slight continuity.

Theorem 4.13. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following properties hold:*

(a) *If f is sl.β.c. and (Y,σ) is E.D., then f is β-quasi-irresolute.*

(b) *If f is sl.β.c. and (Y,σ) is ultra regular, then f is β-continuous.*

(c) *If f is sl.β.c. and (X,τ) is a PS-space and (Y,σ) is E.D., then f is weakly semi-continuous.*

(d) *If f is sl.β.c. and (Y,σ) is locally indiscrete, then f is β-continuous and contra β-continuous.*

Proof. (a) Let $x \in X$ and $V \in \mathrm{SO}(Y)$ containing $f(x)$. Then we have $\mathrm{Cl}(V) = \mathrm{Cl}(\mathrm{Int}(V))$ and hence $\mathrm{Cl}(V)$ is clopen in (Y,σ) since (Y,σ) is E.D. Since f is sl.β.c., there exists $U \in \mathrm{SPO}(X,x)$ such that $f(U) \subset \mathrm{Cl}(V)$. Therefore, f is β-quasi-irresolute.

(b) Since (Y,σ) is ultra regular, $\sigma_u = \sigma$ and by Theorem 3.7 the proof is obvious.

(c) Let $x \in X$ and V any open set containing $f(x)$. Then we have $\mathrm{Cl}(V) \in \mathrm{CO}(Y)$ since (Y,σ) is E.D. Since f is sl.β.c., there exists $U \in \mathrm{SPO}(X,x)$ such that $f(U) \subset \mathrm{Cl}(V)$. Since (X,τ) is a PS-space, $U \in \mathrm{SO}(X)$ by [4, Theorem 2.1], hence f is weakly semi-continuous.

(d) Let V be any open set of (Y,σ). Since (Y,σ) is locally indiscrete, V is clopen and hence $f^{-1}(V)$ is β-open and β-closed in (X,τ). Therefore, f is β-continuous and contra β-continuous. □

Theorem 4.14. *For a function $f : X \to Y$, the following properties hold:*

(a) *If f is sl.β.c., X is E.D. and Y is locally indiscrete, then f is contra-precontinuous.*

(b) *If f is sl.β.c. and X and Y are E.D., then f is almost weakly continuous.*

Proof. (a) Let F be any closed set of Y. By Theorem 4.13(d), f is contra-β-continuous and $f^{-1}(F) \in \mathrm{SPO}(X)$. Since X is E.D., $f^{-1}(F) \in \mathrm{PO}(X)$ and hence f is contra-precontinuous.

(b) Let $x \in X$ and V any open set containing $f(x)$. Then we have $\mathrm{Cl}(V) \in \mathrm{CO}(Y)$ since Y is E.D. Since f is sl.β.c., there exists $U \in \mathrm{SPO}(X,x)$ such that $f(U) \subset \mathrm{Cl}(V)$. Since X is E.D., $U \in \mathrm{PO}(X)$, hence f is almost weakly continuous by [26, Theorem 3.1]. □

5. Properties.

The composition of two slightly β-continuous functions need not be slightly β-continuous as shown by the following example due to Pal and Bhattacharyya [7].

Example 5.1. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\sigma = \{\emptyset, X\}$, and $\theta = \{\emptyset, X, \{a\}, \{b,c\}\}$. Let $f : (X,\tau) \to (X,\sigma)$ be the identity function and $g : (X,\sigma) \to (X,\theta)$ a function defined by $g(a) = b$, $g(b) = c$, and $g(c) = a$. Then f and g are faintly precontinuous by [7, Example 4] and hence sl.β.c. However, the composition $g \circ f$ is not sl.β.c.

If $f : X \to Y$ is an open continuous function, then f is β-irresolute and also the image $f(U)$ of each β-open set of X is β-open in Y.

Theorem 5.2. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then*

(a) *if f is sl.β.c. and g is slightly continuous, then $g \circ f$ is sl.β.c.,*

(b) *if f is β-irresolute and g is sl.β.c., then $g \circ f$ is sl.β.c.,*

(c) *let f be an open continuous surjection. Then g is sl.β.c. if and only if $g \circ f$ is sl.β.c.*

Proof. (a) Let $W \in \mathrm{CO}(Z)$. By the slight continuity of g, $g^{-1}(W) \in \mathrm{CO}(Y)$ and hence $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W) \in \mathrm{SPO}(X)$ since f is sl.β.c. This shows that $g \circ f$ is sl.β.c.

(b) Let $W \in \mathrm{CO}(Z)$. By the slight β-continuity of g, $g^{-1}(W) \in \mathrm{SPO}(Y)$ and hence $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W) \in \mathrm{SPO}(X)$ since f is β-irresolute. This shows that $g \circ f$ is sl.β.c.

(c) Let g be sl.β.c. Then, by (b) $g \circ f$ is sl.β.c. Conversely, let $g \circ f$ be sl.β.c. and $W \in \mathrm{CO}(Z)$. Then $(g \circ f)^{-1}(W) \in \mathrm{SPO}(X)$. Since f is an open continuous surjection, $f((g \circ f)^{-1}(W)) = g^{-1}(W) \in \mathrm{SPO}(Y)$. This shows that g is sl.β.c. □

Lemma 5.3 (see Abd El-Monsef et al. [1]). *Let X be a topological space and A, U subsets of X. Then*

(a) *if U is α-open in X and $A \in \mathrm{SPO}(X)$, then $A \cap U \in \mathrm{SPO}(U)$,*
(b) *if $A \in \mathrm{SPO}(U)$ and $U \in \mathrm{SPO}(X)$, then $A \in \mathrm{SPO}(X)$.*

Theorem 5.4. *Let $\{U_\gamma : \gamma \in \Gamma\}$ be any α-open cover of a topological space X. A function $f : X \to Y$ is sl.β.c. if and only if the restriction $f \mid U_\gamma : U_\gamma \to Y$ is sl.β.c. for each $\gamma \in \Gamma$.*

Proof

Necessity. Let γ be an arbitrarily fixed index and U_γ an α-open set of X. Let $x \in U_\gamma$ and $V \in \mathrm{CO}(Y)$ containing $(f \mid U_\gamma)(x) = f(x)$. Since f is sl.β.c., there exists $U \in \mathrm{SPO}(X)$ containing x such that $f(U) \subset V$. Since U_γ is α-open in X, by Lemma 5.3 $x \in U \cap U_\gamma \in \mathrm{SPO}(U_\gamma)$ and $(f \mid U_\gamma)(U \cap U_\gamma) = f(U \cap U_\gamma) \subset f(U) \subset V$. This shows that $f \mid U_\gamma$ is sl.β.c.

Sufficiency. Let $x \in X$ and $V \in \mathrm{CO}(Y)$ containing $f(x)$. There exists a $\gamma \in \Gamma$ such that $x \in U_\gamma$. Since $f \mid U_\gamma : U_\gamma \to Y$ is sl.β.c., there exists $U \in \mathrm{SPO}(U_\gamma)$ containing x such that $(f \mid U_\gamma)(U) \subset V$. By Lemma 5.3, $U \in \mathrm{SPO}(X)$ and $f(U) \subset V$. Therefore, f is sl.β.c. □

Theorem 5.5. *A function $f : X \to Y$ is sl.β.c. if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is sl.β.c.*

Proof. Suppose that g is sl.β.c. Let F be a clopen set of Y. Then $X \times F$ is a clopen set of $X \times Y$. Since g is sl.β.c., $g^{-1}(X \times F) = f^{-1}(F) \in \mathrm{SPO}(X)$. Therefore, f is sl.β.c. □

Let $\{X_\lambda : \lambda \in \Lambda\}$ and $\{Y_\lambda : \lambda \in \Lambda\}$ be two families of topological spaces with the same index set Λ. The product space of $\{X_\lambda : \lambda \in \Lambda\}$ is denoted by $\Pi\{X_\lambda : \lambda \in \Lambda\}$ (or simply ΠX_λ). Let $f_\lambda : X_\lambda \to Y_\lambda$ be a function for each $\lambda \in \Lambda$. The product function $f : \Pi X_\lambda \to \Pi Y_\lambda$ is defined by $f(\{x_\lambda\}) = \{f_\lambda(x_\lambda)\}$ for each $\{x_\lambda\} \in \Pi X_\lambda$.

Theorem 5.6. *If a function $f : X \to \Pi Y_\lambda$ is sl.β.c., then $P_\lambda \circ f : X \to Y_\lambda$ is sl.β.c. for each $\lambda \in \Lambda$, where P_λ is the projection of ΠY_λ onto Y_λ.*

Proof. Let V_λ be any clopen set of Y_λ. Then $P_\lambda^{-1}(V_\lambda)$ is clopen in ΠY_λ and hence $(P_\lambda \circ f)^{-1}(V_\lambda) = f^{-1}(P_\lambda^{-1}(V_\lambda))$ is β-open in X. Therefore, $P_\lambda \circ f$ is sl.β.c. □

Theorem 5.7. *If a function $f : \Pi X_\lambda \to \Pi Y_\lambda$ is sl.β.c., then $f_\lambda : X_\lambda \to Y_\lambda$ is sl.β.c. for each $\lambda \in \Lambda$.*

Proof. Let V_λ be any clopen set of Y_λ. Then, $P_\lambda^{-1}(V_\lambda)$ is clopen in ΠY_λ and $f^{-1}(P_\lambda^{-1}(V_\lambda)) = f_\lambda^{-1}(V_\lambda) \times \Pi\{X_\alpha : \alpha \in \Lambda - \{\lambda\}\}$. Since f is sl.β.c., $f^{-1}(P_\lambda^{-1}(V_\lambda))$ is β-open in ΠX_λ. Since the projection P_λ of ΠX_λ onto X_λ is open continuous, $f_\lambda^{-1}(V_\lambda)$ is β-open in X_λ and hence f_λ is sl.β.c. □

Definition 5.8. A topological space X is said to be

(a) *β-Hausdorff* [18] (resp., *ultra Hausdorff* [30]) if every two distinct points of X can be separated by disjoint β-open (resp., clopen) sets,

(b) *β-regular* [2] (resp., *ultra regular* [12]) if each pair of a point and a closed set not containing the point can be separated by disjoint β-open (resp., clopen) sets,

(c) *β-normal* [18] (resp., *ultra normal* [30]) if every two disjoint closed sets of X can be separated by β-open (resp., clopen) sets.

Theorem 5.9. *Let $f : X \to Y$ be a sl.β.c. injection. Then*

(a) *if Y is ultra Hausdorff, then X is β-Hausdorff,*

(b) *if Y is ultra regular and f is open or closed, then X is β-regular,*

(c) *if Y is ultra normal and f is closed, then X is β-normal.*

Proof. (a) Let x_1, x_2 be two distinct points of X. Then since f is injective and Y is ultra Hausdorff, there exist $V_1, V_2 \in \mathrm{CO}(Y)$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$, and $V_1 \cap V_2 = \emptyset$. By Theorem 3.2, $x_i \in f^{-1}(V_i) \in \mathrm{SPO}(X)$ for $i = 1,2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus X is β-Hausdorff.

(b) (i) Suppose that f is open. Let $x \in X$ and U be an open set containing x. Then $f(U)$ is an open set of Y containing $f(x)$. Since Y is ultra regular, there exists a clopen set V such that $f(x) \in V \subset f(U)$. Since f is a sl.β.c. injection, by Theorem 3.1 $x \in f^{-1}(V) \subset U$ and $f^{-1}(V)$ is β-clopen in X. Therefore, X is β-regular. (ii) Suppose that f is closed. Let $x \in X$ and F be any closed set of X not containing x. Since f is injective and closed, $f(x) \notin f(F)$ and $f(F)$ is closed in Y. By the ultra regularity of Y, there exists a clopen set V such that $f(x) \in V \subset Y - f(F)$. Therefore, $x \in f^{-1}(V)$ and $F \subset X - f^{-1}(V)$. By Theorem 3.2, $f^{-1}(V)$ is a β-clopen set in X. Thus, X is β-regular.

(c) Let F_1, F_2 be disjoint closed subsets of X. Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y. Since Y is ultra normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets V_1 and V_2. Therefore, we obtain $F_i \subset f^{-1}(V_i)$ and $f^{-1}(V_i) \in \mathrm{SPO}(X)$ for $i = 1,2$ from Theorem 3.2. Moreover, $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus X is β-normal. □

A subset A of a topological space X is said to be *semi pre β-closed* if for each $x \in X - A$ there exists a β-clopen set U containing x such that $U \cap A = \emptyset$.

Theorem 5.10. *If $f : X \to Y$ is sl.β.c. and Y is ultra Hausdorff, then*

(a) *the graph $G(f)$ of f is semi pre β-closed in the product space $X \times Y$,*

(b) *the set $\{(x_1, x_2) : f(x_1) = f(x_2)\}$ is semi pre β-closed in the product space $X \times X$.*

Proof. (a) Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exist clopen sets V and W such that $y \in V$, $f(x) \in W$, and $V \cap W = \emptyset$. Since f is sl.β.c., there exists a β-clopen set U containing x such that $f(U) \subset W$. Therefore, we obtain $V \cap f(U) = \emptyset$ and hence $(U \times V) \cap G(f) = \emptyset$ and $U \times V$ is a β-clopen set of $X \times Y$. This shows that $G(f)$ is semi pre β-closed in $X \times Y$.

(b) Set $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$. Let $(x_1, x_2) \notin A$, then $f(x_1) \neq f(x_2)$. Since Y is ultra Hausdorff, there exist $V_1, V_2 \in \mathrm{CO}(Y)$ containing $f(x_1)$, $f(x_2)$, respectively, such that $V_1 \cap V_2 = \emptyset$. Since f is sl.β.c., there exist β-clopen sets U_1, U_2 of X such that $x_i \in U_i$ and $f(U_i) \subset V_i$ for $i = 1,2$. Thus, $(x_1, x_2) \in U_1 \times U_2$ and $(U_1 \times U_2) \cap A = \emptyset$. Moreover, $U_1 \times U_2$ is β-clopen in $X \times X$ and A is semi pre β-closed in $X \times X$. □

A topological space X is said to be *β-connected* [27] if X cannot be expressed as the union of two disjoint nonempty β-open sets.

Theorem 5.11. *If $f : X \to Y$ is a sl.β.c. surjection and X is β-connected, then Y is connected.*

Proof. Assume that Y is not connected. Then there exist nonempty open sets V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. Therefore, V_1 and V_2 are clopen sets of Y. Since f is sl.β.c., $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are β-open sets in X. Moreover, we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Since f is surjective, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty. Therefore, X is not β-connected. This is a contradiction and hence Y is connected. $\square$

Corollary 5.12 (see Popa and Noiri [27]). *If $f : X \to Y$ is a weakly β-continuous surjection and X is β-connected, then Y is connected.*

Corollary 5.13. *If $f : X \to Y$ is a contra β-continuous surjection and X is β-connected, then Y is connected.*

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Takashi Noiri: Department of Mathematics, Yatsushiro College of Technology, Yatsushiro, Kumamoto, 866-8501, Japan

E-mail address: noiri@as.yatsushiro-nct.ac.jp