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ON SOME PROPERTIES OF THE LÜROTH-TYPE ALTERNATING SERIES REPRESENTATIONS FOR REAL NUMBERS

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ABSTRACT. We investigate some properties connected with the alternating Lüroth-type series representations for real numbers, in terms of the integer digits involved. In particular, we establish the analogous concept of the asymptotic density and the distribution of the maximum of the first n denominators, by applying appropriate limit theorems.

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1. Introduction. Let x be any number in the interval $I := (0,1)$. Then by using a general alternating series algorithm introduced by Knopfmacher and Knopfmacher [6, 7], analogous to a positive one of Oppenheim [8], we may prove that x has a unique finite or infinite representation in the form

$$\begin{aligned} x &= \frac{1}{\alpha_1} - \frac{1}{(\alpha_1+1)\alpha_1}\frac{1}{\alpha_2} + \frac{1}{(\alpha_1+1)\alpha_1(\alpha_2+1)\alpha_2}\frac{1}{\alpha_3} - \cdots \\ &= ((\alpha_1,\alpha_2,\dots,\alpha_n,\dots)), \end{aligned} \tag{1.1}$$

where $\alpha_n \geq 1$, $n \geq 1$.

Representation (1.1) is called *Lüroth-type* alternating expansion, while the positive integers $\alpha_1,\alpha_2,\dots,\alpha_n,\dots$ are called the *digits* (or the *denominators*) of the above mentioned expansion. It is obvious that the digits α_n are functions $\alpha_n(x)$ of x; therefore, it can be easily seen that the α_n's may be considered as random variables defined almost surely on I with respect to any probability measure on the σ-algebra B_I (in particular, with respect to the Lebesgue measure λ).

A lot of research has been carried out into the ergodic properties of the denominators in the Lüroth expansions of real numbers in $(0,1)$. In particular, it was studied independently by Šalát [9] and Jager and de Vroedt [3] not only the stochastic behaviour of the digits d_n (which are independent random variables on a probability space S, where the basic set is the unit interval $(0,1)$ and the probability is the Lebesgue measure) but some other important metric properties concerning the sequence $\{d_n\}_n$.

Similar further results were derived later by Kalpazidou, Knopfmacher, and Knopfmacher (see [4, 5]), for the alternating Lüroth-type series.

The aim of the present paper is to give some sharper properties for the alternating Lüroth-type expansions related with the order of magnitude of the digits α_n. For the ordinary Lüroth expansions, an analogous problem has been investigated by Šalát [9].

2. Preliminaries. Let

$$\begin{aligned} I_n &\equiv I_n(k_1,\dots k_n) \\ &= \{x \in I \mid \alpha_1(x) = k_1,\dots,\alpha_n(x) = k_n\}, \quad \text{for any } k_1,k_2,\dots,k_n \in \mathbb{N}^*, \end{aligned} \tag{2.1}$$

be the set of all $x \in I$ which have a unique expansion of the form (1.1) such that the digits $\alpha_1(x),\dots,\alpha_n(x)$ have the concrete values $k_1,\dots,k_n$.

Then according to a result of [5] concerning the stochastic behaviour of the α_n's, we have the following proposition.

Proposition 2.1. *The digits $\alpha_n(\cdot)$, $n \in \mathbb{N}^*$, are stochastically independent and identically distributed random variables with respect to Lebesgue measure λ, with*

$$\lambda(\alpha_n = k) = \frac{1}{k(k+1)}, \quad k \in \mathbb{N}^*. \tag{2.2}$$

Evidently the mean value of the digits α_n is given by

$$E(\alpha_n) = \sum_{k=1}^{+\infty} k\lambda(\alpha_n = k) = \sum_{k=1}^{+\infty} \frac{1}{k+1} = +\infty, \tag{2.3}$$

which means that the usual limit theorems do not apply.

Then, according to an interesting result which is a kind of converse to the strong law of large numbers due to Chow and Robbins [1], it may be obtained that, for a.a. x,

$$\limsup \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)}{n} = +\infty. \tag{2.4}$$

It is obvious that if we take the functions $u(\alpha_n)$ of α_n for which $\sum_{k=1}^{+\infty} u(\alpha_n)/k(k+1) < +\infty$, then we can apply the usual theorems, obtaining strong laws and asymptotic normality.

In Section 3, we investigate some sharper results for the digits α_n of the alternating Lüroth-type series following the spirit of Šalát [9], while in Section 4, we investigate the asymptotic behaviour of $M_n = M_n(x) = \max(\alpha_1,\dots,\alpha_n)$ by using Proposition 2.1.

3. Some remarks on the digits of the alternating Lüroth series. Let $\{c_n\}_n$ be an arbitrarily chosen sequence of real numbers and X_n the indicator variable of the fact $\{x \mid \alpha_n(x) > c_n(x)\}$, that is,

$$X_n = \begin{cases} 1, & \text{if } \alpha_n > c_n, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

By applying Proposition 2.1 we obtain that X_n are independent random variables and; moreover,

$$\begin{aligned} P(X_n = 1) &= \sum_{k=[c_n]+1}^{+\infty} \frac{1}{k(k+1)} = \sum_{k=[c_n]+1}^{+\infty} \left(\frac{1}{k} - \frac{1}{(k+1)}\right) = \frac{1}{[c_n]+1}, \\ P(X_n = 0) &= 1 - P(X_n = 1) = 1 - \frac{1}{[c_n]+1}, \end{aligned} \tag{3.2}$$

if we assume that $c_n \geq 0$. (This assumption does not restrict the generality of our investigation, since from the general alternating series algorithm we have $\alpha_n \geq 1$, for all n.) At first we prove the following theorems.

Theorem 3.1. *The series $\sum X_n$ converges a.e. if and only if*

$$\sum \frac{1}{c_n+1} < +\infty. \tag{3.3}$$

Moreover, if the mean value

$$E_N = \sum_{n=1}^{N} \frac{1}{[c_n]+1} \longrightarrow +\infty, \tag{3.4}$$

then by setting

$$Z_N = \sum_{n=1}^{N} X_n, \qquad V_N = \sum_{n=1}^{N} \frac{1}{[c_n]+1} \cdot \left(1 - \frac{1}{[c_n]+1}\right), \tag{3.5}$$

we take

$$\lambda\left(Z_N - E_N < z\sqrt{V_N}\right) \longrightarrow \int_{-\infty}^{Z} \frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2}\, dt \tag{3.6}$$

as $N \to +\infty$, in case $V_N \to +\infty$.

Proof. The first part of the above theorem is a consequence of the Borel-Cantelli lemma (see [2]). Furthermore, since

$$E(X_n) = \frac{1}{[c_n]+1}, \qquad \operatorname{Var}(X_n) = \frac{1}{[c_n]+1} \cdot \left(1 - \frac{1}{[c_n]+1}\right), \tag{3.7}$$

we may apply the central limit theorem under Lindeberg's conditions, that is,

$$\lambda\left(-\infty < \frac{Z_N - E_N}{\sqrt{V_N}} < z\right) \xrightarrow{N \to +\infty} \int_{-\infty}^{z} f(t)\, dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2}\, dt \tag{3.8}$$

and the proof is complete. □

Relation (3.6) implies that Z_N converges in probability to E_N if both E_N and V_N tend to $+\infty$. Although in the general case much more important information may be derived, we take in a special case the following theorem.

Theorem 3.2. *Assume that*

$$\lim_{N \to +\infty} \frac{E_N}{N} = c \tag{3.9}$$

exists and is positive, then for a.a. x in $(0,1)$

$$\lim_{N \to +\infty} \frac{(X_1 + X_2 + \cdots + X_N)}{N} = c. \tag{3.10}$$

The statement holds also in case $c = 0$.

Proof. We define the random variables K_n by the relation $K_n = X_n - 1/([c_n]+1)$. Consequently, we have

$$E(K_n) = E(X_n) - \frac{1}{[c_n]+1} = 0,$$
$$V(K_n) = \operatorname{Var}(X_n) = \frac{1}{[c_n]+1}\cdot\left(1-\frac{1}{[c_n]+1}\right). \tag{3.11}$$

By using the Kolmogorov's theorem [2] we obtain that, for a.a. x in $(0,1)$,

$$\lim_{N\to+\infty}\frac{(K_1+\cdots+K_N)}{N} = 0, \tag{3.12}$$

which gives the proof of our theorem. □

Theorem 3.2 is related to the concept of asymptotic density, which in the case of alternating Lüroth-type expansions is defined as follows. Let $\{b_n\}$ be an increasing sequence of positive integers and let $K(M)$ be the number of elements of the sequence $\{b_n\}$ for which $b_t \le M$. If the limit of $K(M)/M$ as $M\to+\infty$ exists, then we may say that the sequence $\{b_n\}$ has asymptotic density. Theorem 3.2 provides the criterion for the sequence $\{n/\alpha_n > c_n\}$ to have asymptotic density, for a.a. x. This means that applying Theorem 3.2 with $\alpha_n = 1$, for all $n \ge 1$, then with $\alpha_n = 2$, for all $n \ge 1$ we get successively the densities of $\{n \mid \alpha_n = 1\}$, $\{n \mid \alpha_n = 2\}$, and so on. Note that if $c_n \to +\infty$, then by Theorem 3.2, c exists and equals zero.

Using Proposition 2.1 and Borel-Cantelli lemma, we can have a sharper result about the behaviour of the sequence $\{\alpha_n\}$ according to the following theorem.

Theorem 3.3. *Let $\{c_n\}_n$, $\{d_n\}_n$, with $0 < c_n \le d_n$, be two sequences of real numbers which tend to $+\infty$. Moreover, assume that*

$$\limsup\frac{(c_n+1)}{d_n} = u < 1 \tag{3.13}$$

and that

$$\sum_{n=1}^{+\infty}\frac{1}{d_n(c_n+1)} = +\infty. \tag{3.14}$$

Then for a.a. x, the inequalities $d_n < \alpha_n \le d_n(1+1/(c_n+1))$ hold for many infinite values of n.

Proof. It is known that

$$\lambda(u < \alpha_n \le w) = \lambda(u < \alpha_n) - \lambda(w < \alpha_n). \tag{3.15}$$

Then from (3.2) we have

$$\begin{aligned}\lambda\left(d_n < \alpha_n \le d_n + \frac{d_n}{c_n+1}\right) &= \frac{1}{[d_n]+1} - \frac{1}{[d_n+d_n/(c_n+1)]+1}\\ &\ge \frac{1}{d_n+1} - \frac{1}{d_n+d_n/(c_n+1)}\\ &\ge \frac{1-(c_n+1)/d_n}{3d_n(c_n+1)} \ge \frac{a}{d_n(c_n+1)},\end{aligned} \tag{3.16}$$

where a is a suitable constant, which in view of (3.13), for n sufficiently large, can be chosen arbitrarily close to $1-u$, hence $a>0$. Now by using Proposition 2.1, the Borel-Cantelli lemma is applicable. So (3.14) and (3.16) imply the statement of Theorem 3.3, and the proof is complete. □

Theorem 3.3 states that if for a sequence d_n tending to $+\infty$, there is a sequence c_n such that (3.13) and (3.14) hold, then for a.a. x, infinitely often $\alpha_n \sim d_n$.

This raises the problem whether $M_n = \max(\alpha_1, \alpha_2, \dots, \alpha_n)$ follows an asymptotic law. We will deal with this problem in the next section.

4. The distribution of the maximum of the first n digits. If M_n is the maximum of the first n digits, then we take the following theorem.

Theorem 4.1. *For any fixed $y>0$,*

$$\lim_{n\to\infty} \lambda\left(\frac{M_n}{n} \le y\right) = \exp\left(-\frac{1}{y+1}\right). \tag{4.1}$$

Proof. We define the events $A_i = \{x \mid \alpha_i/n \le y\}$, $1 \le i \le n$. It is obvious that

$$\left\{x \,\middle|\, \frac{M_n}{n} \le y\right\} = \bigcap_{i=1}^{n} A_i, \tag{4.2}$$

and therefore by using Proposition 2.1 and (3.2), we have

$$\lambda\left(\frac{M_n}{n} \le y\right) = \prod_{i=1}^{n} \lambda(A_i) = \prod_{i=1}^{n} \lambda\left(\frac{\alpha_i}{n} \le y\right) = \prod_{i=1}^{n}\left[1-\lambda\left(\frac{\alpha_i}{n} > y\right)\right]. \tag{4.3}$$

But

$$\lambda\left(\frac{\alpha_i}{n} > y\right) = \lambda(\alpha_i > ny) = \sum_{k=[ny]+1}^{+\infty} \frac{1}{k(k+1)} = \frac{1}{[ny]+1}. \tag{4.4}$$

So

$$\lambda\left(\frac{M_n}{n} \le y\right) = \prod_{i=1}^{n}\left[1-\frac{1}{[ny]+1}\right] = \left(1-\frac{1}{[ny]+1}\right)^{n}. \tag{4.5}$$

Using a well-known characteristic limit relation, the proof is complete. □

Hence we can obtain the following corollary.

Corollary 4.2. *Let K_n be a random variable defined on the probability space (I, B_I, λ), and assume that K_n converges in probability to 1. Then*

$$\lim_{n\to+\infty} \lambda\left(\frac{K_n}{M_n} < y\right) = 1-e^{-y/(y+1)}. \tag{4.6}$$

Proof. We may write

$$\frac{K_n}{M_n} = \frac{K_n}{n} \cdot \frac{n}{M_n} = \frac{n}{M_n} + \frac{n}{M_n}\left(\frac{K_n}{n}-1\right). \tag{4.7}$$

Using (4.7) we have only to show that the second term tends in probability to 0, since then using a well-known result of Cramer we may obtain the statement of Corollary 4.2.

More precisely we have to show that, for any positive real number u,

$$\lim_{n\to+\infty} \lambda\left(\left|\frac{n}{M_n}\cdot\left(\frac{K_n}{n}-1\right)\right| \geq u\right) = 0. \tag{4.8}$$

If we apply this relation, for any fixed U, then we obtain

$$\lambda\left(\left|\frac{n}{M_n}\cdot\left(\frac{K_n}{n}-1\right)\right| \geq u\right) \leq \lambda\left(\frac{n}{M_n} > U\right) + \lambda\left(\left|\frac{K_n}{n}-1\right| \geq \frac{u}{U}\right). \tag{4.9}$$

From (4.9) we get that the first and the second terms are smaller than any prescribed real number by using Theorem 4.1 and the assumption on K_n, respectively. So the proof is complete. □

From the occurrence of the exponential distribution in Theorem 4.1 and Corollary 4.2 it can be derived that the number of terms of the sequence $\{\alpha_n\}$, which are of the same order as M_n, follows a Poisson distribution. This is given in the following theorem.

Theorem 4.3. *Let Y_n denote the number of terms in the sequence $\{\alpha_n\}$ for which $\alpha_n > y$. Then its asymptotic probability function is given by*

$$\lim_{n\to+\infty} \lambda(Y_n = K) = \frac{e^{-1(y+1)}}{K!(y+1)^K}. \tag{4.10}$$

Proof. Using Proposition 2.1 and relation (3.2) we have that the random variables Z_N follow a binomial distribution with parameters n and $1/([ny]+1)$. Since $n/([ny]+1) \to 1/(y+1)$, as $n \to \infty$, by the result known as "Poisson approximation to the binomial distribution" (see [2]), we obtain that the distribution of Y_n is Poisson with parameter $1/(y+1)$. So the proof is complete. □

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